Preface

The importance of **Electromagnetic Field Theory** is well known in various engineering fields. Overwhelming response to our books on various subjects inspired us to write this book. The book is structured to cover the key aspects of the subject **Electromagnetic Field Theory**.

The book uses plain, lucid language to explain fundamentals of this subject. The book provides logical method of explaining various complicated concepts and stepwise methods to explain the important topics. Each chapter is well supported with necessary illustrations, practical examples and solved problems. All chapters in this book are arranged in a proper sequence that permits each topic to build upon earlier studies. All care has been taken to make students comfortable in understanding the basic concepts of the subject.

The book not only covers the entire scope of the subject but explains the philosophy of the subject. This makes the understanding of this subject more clear and makes it more interesting. The book will be very useful not only to the students but also to the subject teachers. The students have to omit nothing and possibly have to cover nothing more.

We wish to express our profound thanks to all those who helped in making this book a reality. Much needed moral support and encouragement is provided on numerous occasions by our whole family. We wish to thank the **Publisher** and the entire team of family **Technical Publications** who have taken immense pain to get this book in time with quality printing.

Any suggestion for the improvement of the book will be acknowledged and well appreciated.

Authors

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Dedicated to Arjun, Apurva, Gururaj and Pradnya

Syllabus (Electromagnetic Field Theory)

Unit-I (Chapter - 1)

Co-ordinate Systems and Transformation : Cartesian co-ordinates, Circular cylindrical co-ordinates, Spherical co-ordinates.
Vector Calculus : Differential length, Area and volume, Line surface and volume integrals, Del operator, Gradient of a scalar, Divergence of a vector and divergence theorem, Curl of a vector and Stoke's theorem, Laplacian of a scalar.

Unit-II (Chapters - 2, 3, 4, 5, 6)

Electrostatics : Electrostatic fields, Coulombs law and field intensity, Electric field due to charge distribution, Electric flux density, Gauss's law - Maxwell's equation, Electric dipole and flux lines, Energy density in electrostatic fields.
Electric Field in Material Space : Properties of materials, Convection and conduction currents, Conductors, Polarization in dielectrics, Dielectric constants, Continuity equation and relaxation time, Boundary condition.
Electrostatic Boundary Value Problems : Poisson's and Laplace's equations, General procedures for solving Poisson's or Laplace's equations, Resistance and capacitance, Method of images.

Unit-III (Chapters - 7, 8)

Magnetostatics : Magnetostatic fields, Biot-Savart's law, Ampere's circuit law, Maxwell's equation, Application of ampere's law, Magnetic flux density - Maxwell's equation, Maxwell's equation for static fields, magnetic scalar and vector potential.
Magnetic Forces, Materials and Devices : Forces due to magnetic field, Magnetic torque and moment, A magnetic dipole, Magnetization in materials, Magnetic boundary conditions, Inductors and inductances, Magnetic energy.

Unit-IV (Chapters - 9, 10)

Waves and Applications : Maxwell's equation, Faraday's law, Transformer and motional electromotive forces, Displacement current, Maxwell's equation in final form.
Electromagnetic Wave Propagation : Wave propagation in lossy dielectrics, Plane waves in lossless dielectrics, Plane wave in free space, Plane waves in good conductors, Power and the Poynting vector, Reflection of a plane wave in a normal incidence.

Unit-V (Chapter - 11)

Transmission Lines : Transmission line parameters, Transmission line equations, Input impedance, Standing wave ratio and power, The Smith chart, Some applications of transmission lines.

Table of Contents :

Features of Book

- Use of clear, plain and lucid language making the understanding very easy.
- Use of informative, self explanatory diagrams, plots and graphs.
- Excellent theory well supported with the practical examples and illustrations.
- Important concepts are highlighted using Key Points throughout the book.
- Large number of solved examples.
- Approach of the book resembles classroom teaching.
- Book provides detailed insight into the subject.
- Stepwise explanation to mathematical derivations for easier understanding.

Electromagnetic Field Theory

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1 Vector Analysis

1.1 Introduction

Electromagnetics is a branch of physics or electrical engineering which is used to study the electric and magnetic phenomena. The electric and magnetic fields are closely related to each other.

Let us see, what is a field ? Consider a magnet. It has its own effect in a region surrounding it. The effect can be experienced by placing another magnet near the first magnet. Such an effect can be defined by a particular physical function. In the region surrounding the magnet, there exists a particular value for that physical function, at every point, describing the effect of magnet. So field can be defined as the region in which, at each point there exists a corresponding value of some physical function.

Thus field is a function that specifies a quantity everywhere in a region or a space. If at each point of a region or space, there is a corresponding value of some physical function then the region is called a field. If the field produced is due to magnetic effects, it is called **magnetic field**. There are two types of electric charges, positive and negative. Such an electric charge produces a field around it which is called an **electric field**. Moving charges produce a current and current carrying conductor produces a magnetic field. In such a case, electric and magnetic fields are related to each other. Such a field is called **electromagnetic field**. The comprehensive study of characteristics of electric, magnetic and combined fields, is nothing but the **engineering electromagnetics**. Such fields may be time varying or time independent.

It is seen that distribution of a quantity in a space is defined by a field. Hence to quantify the field, three dimensional representation plays an important role. Such a three dimensional representation can be made easy by the use of vector analysis. The problems involving various mathematical operations related to the fields distributed in three dimensional space can be conveniently handled with the help of vector analysis. A complete pictorial representation and clear understanding of the fields and the laws governing such fields, is possible with the help of vector analysis. Thus a good knowledge of vector analysis is an essential prerequisite for the understanding of engineering

electromagnetics. The vector analysis is a mathematical shorthand tool with which electromagnetic concepts can be most conveniently expressed.

This chapter gives the basic vector analysis required to understand engineering electromagnetics. The notations used in this chapter are followed throughout this book, to explain the subject.

1.2 Scalars and Vectors

The various quantities involved in the study of engineering electromagnetics can be classified as,

1. Scalars and 2. Vectors

1.2.1 Scalar

The scalar is a quantity whose value may be represented by a single real number, which may be positive or negative. The direction is not at all required in describing a scalar. Thus,

A **scalar** is a quantity which is wholly characterized by its magnitude.

The various examples of scalar quantity are temperature, mass, volume, density, speed, electric charge etc.

1.2.2 Vector

A quantity which has both, a magnitude and a specific direction in space is called a **vector**. In electromagnetics vectors defined in two and three dimensional spaces are required but vectors may be defined in n-dimensional space. Thus,

A **vector** is a quantity which is characterized by both, a magnitude and a direction.

The various examples of vector quantity are force, velocity, displacement, electric field intensity, magnetic field intensity, acceleration etc.

1.2.3 Scalar Field

A field is a region in which a particular physical function has a value at each and every point in that region. The distribution of a scalar quantity with a definite position in a space is called **scalar field**. For example the temperature of atmosphere. It has a definite value in the atmosphere but no need of direction to specify it hence it is a scalar field. The height of surface of earth above sea level is a scalar field. Few other examples of scalar field are sound intensity in an auditorium. light intensity in a room, atmospheric pressure in a given region etc.

1.2.4 Vector Field

If a quantity which is specified in a region to define a field is a vector then the corresponding field is called a **vector field.** For example the gravitational force on a mass

in a space is a vector field. This force has a value at various points in a space and always has a specific direction.

The other examples of vector field are the velocity of particles in a moving fluid, wind velocity of atmosphere, voltage gradient in a cable, displacement of a flying bird in a space, magnetic field existing from north to south field etc.

1.3 Representation of a Vector

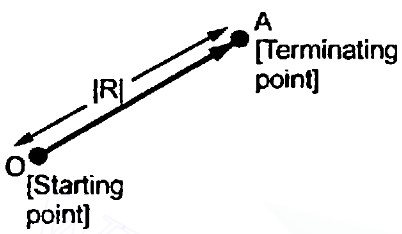


Fig. 1.1 Representation of a vector

In two dimensions, a vector can be represented by a straight line with an arrow in a plane. This is shown in the Fig. 1.1. The length of the segment is the magnitude of a vector while the arrow indicates the direction of the vector in a given co-ordinate system. The vector shown in the Fig. 1.1 is symbolically denoted as $\overline{OA}$. The point O is its starting point while A is its terminating point. Its length is called its magnitude, which is R for the vector OA shown. It is represented as $|\overline{OA}| = R$. It is the distance between the starting point and terminating point of a vector.

Key Point: *The vector hereafter will be indicated by bold letter with a bar over it.*

1.3.1 Unit Vector

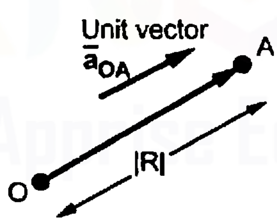


Fig. 1.2 Unit vector

A unit vector has a function to indicate the direction. Its magnitude is always unity, irrespective of the direction which it indicates and the co-ordinate system under consideration. Thus for any vector, to indicate its direction a unit vector can be used. Consider a unit vector $\bar{a}_{OA}$ in the direction of $\overline{OA}$ as shown in the Fig. 1.2. This vector indicates the direction of $\overline{OA}$ but its magnitude is unity.

So vector $\overline{OA}$ can be represented completely as its magnitude R and the direction as indicated by unit vector along its direction.

$$\therefore \quad \boxed{\overline{OA} = |\overline{OA}|\, \bar{a}_{OA} = R\, \bar{a}_{OA}}$$

where $\bar{a}_{OA}$ = Unit vector along the direction OA and $|\bar{a}_{OA}| = 1$

Key Point: *Hereafter, letter $\bar{a}$ is used to indicate the unit vector and its suffix indicates the direction of the unit vector. Thus $\bar{a}_x$ indicates the unit vector along x axis direction.*

In case if a vector is known then the unit vector along that vector can be obtained by dividing the vector by its magnitude. Thus unit vector can be expressed as,

$$\text{Unit vector } \bar{a}_{OA} = \frac{\overline{OA}}{|\overline{OA}|}$$

The idea and use of unit vector will be more clear at the time of discussion of various co-ordinate systems, later in the chapter.

1.4 Vector Algebra

The various mathematical operations such as addition, subtraction, multiplication etc. can be performed with the vectors. In this section the following mathematical operations with the vectors are discussed.

1. Scaling 2. Addition 3. Subtraction

1.4.1 Scaling of Vector

This is nothing but, **multiplication by a scalar** to a vector. Such a multiplication changes the magnitude (length) of a vector but not its direction, **when the scalar is positive.**

Let α = Scalar with which vector is to be multiplied

Then if $\alpha > 1$ then the magnitude of a vector increases but direction remains same, when multiplied. This is shown in the Fig. 1.3 (a). If $\alpha < 1$ then the magnitude of a vector decreases but direction remains same, when multiplied. This is shown in the Fig. 1.3 (b).

If $\alpha = -1$ then the **magnitude remains same but direction of the vector reverses,** when multiplied. This is shown in the Fig. 1.3 (c).

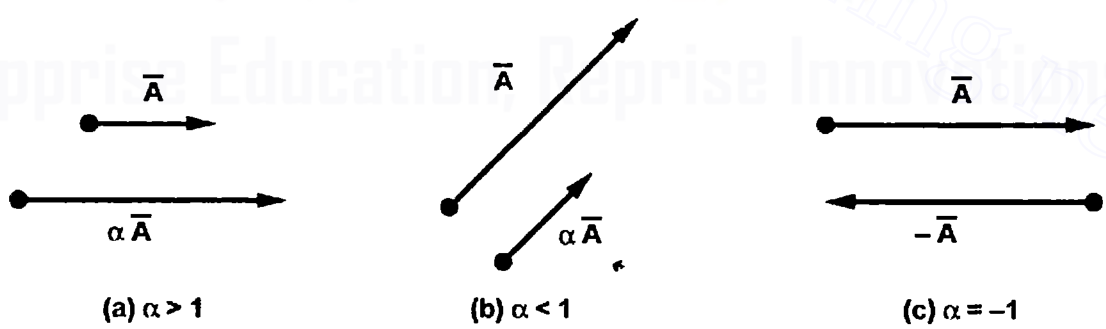


Fig. 1.3 Multiplication by a scalar

Key Point: *Thus if α is negative, the magnitude of vector changes by α times while the direction becomes exactly opposite to the original vector, after multiplication.*

1.4.2 Addition of Vectors

Consider two coplanar vectors as shown in the Fig. 1.4. The vectors which lie in the same plane are called **coplanar vectors.**

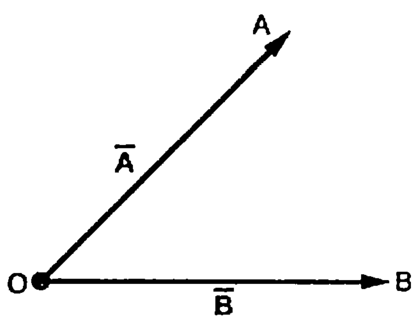


Fig. 1.4 Coplanar vectors

Let us find the sum of these two vectors $\bar{A}$ and $\bar{B}$, shown in the Fig. 1.4.

The procedure is to move one of the two vectors parallel to itself at the tip of the other vector. Thus move $\bar{A}$, parallel to itself at the tip of $\bar{B}$.

Then join tip of $\bar{A}$ moved, to the origin. This vector represents resultant which is the addition of the two vectors $\bar{A}$ and $\bar{B}$. This is shown in the Fig. 1.5.

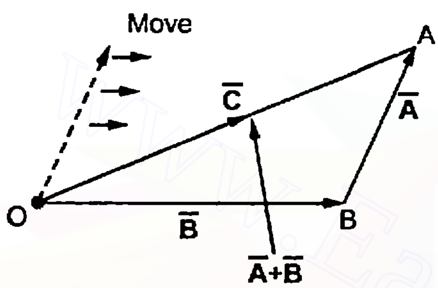


Fig. 1.5 Addition of vectors

Let us denote this resultant as $\bar{C}$ then

$$\bar{C} = \bar{A} + \bar{B}$$

It must be remembered that the direction of $\bar{C}$ is from origin O to the tip of the vector moved.

Another point which can be noticed that if $\bar{B}$ is moved parallel to itself at the tip of $\bar{A}$, we get the same resultant $\bar{C}$. Thus, the order of the addition is not important. The addition of vectors obeys the commutative law i.e. $\bar{A} + \bar{B} = \bar{B} + \bar{A}$.

Another method of performing the addition of vectors is the **parallelogram rule.** Complete the parallelogram as shown in the Fig. 1.6. Then the diagonal of the parallelogram represents the addition of the two vectors.

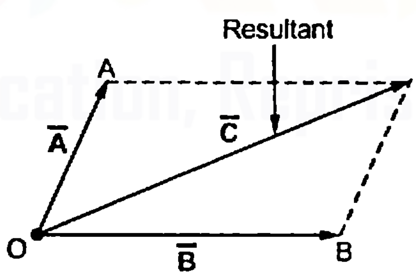
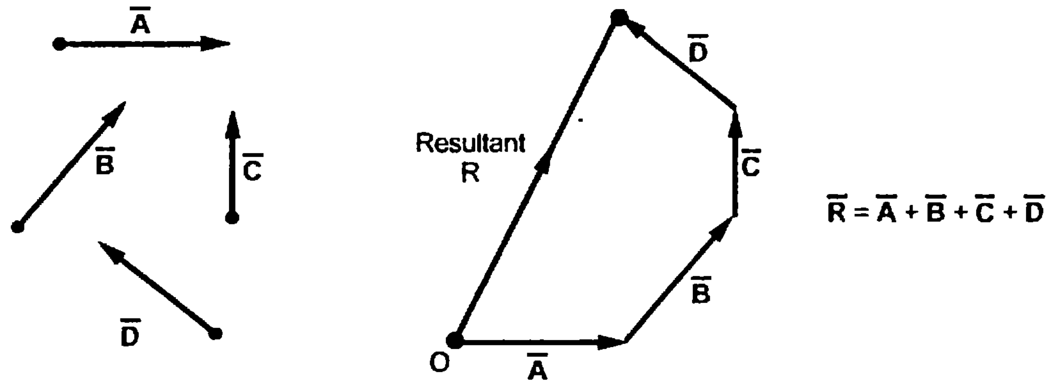


Fig. 1.6 Parallelogram rule for addition

By using any of these two methods not only two but any number of vectors can be added to obtain the resultant. For example, consider four vectors as shown in the Fig. 1.7(a). These can be added by shifting these vectors one by one to the tip of other vectors to complete the polygon. The vector joining origin O to the tip of the last shifted vector represents the sum, as shown in the Fig. 1.7 (b). This method is called **head to tail rule** of addition of vectors.

Once the co-ordinate systems are defined, then the vectors can be expressed in terms of the components along the axes of the co-ordinate system. Then by adding the corresponding components of the vectors, the components of the resultant vector which is



(a) Four vectors (b) Sum of the four vectors

Fig. 1.7

the addition of the vectors, can be obtained. This method is explained after the co-ordinate systems are discussed.

The following basic laws of algebra are obeyed by the vectors $\bar{A}$, $\bar{B}$ and $\bar{C}$:

Law	Addition	Multiplication by scalar
Commutative	$\bar{A} + \bar{B} = \bar{B} + \bar{A}$	$\alpha\bar{A} = \bar{A}\,\alpha$
Associative	$\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$	$\beta\,(\alpha\bar{A}) = (\beta\,\alpha)\bar{A}$
Distributive	$\alpha\,(\bar{A} + \bar{B}) = \alpha\bar{A} + \alpha\bar{B}$	$(\alpha + \beta)\bar{A} = \alpha\bar{A} + \beta\bar{A}$

Table 1.1

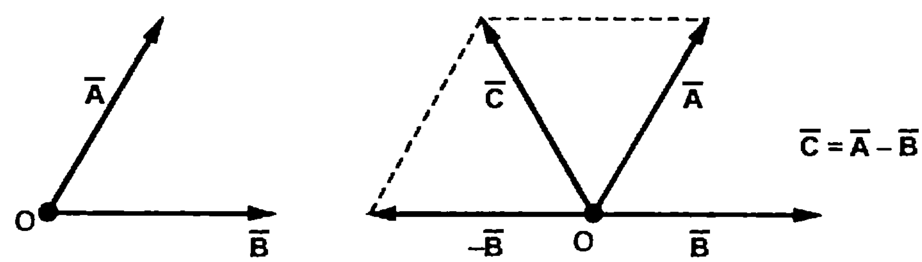
In this table α and β are the scalars i.e. constants.

1.4.3 Subtraction of Vectors

The subtraction of vectors can be obtained from the rules of addition. If $\bar{B}$ is to be subtracted from $\bar{A}$ then based on addition it can be represented as,

$$\bar{C} = \bar{A} + (-\bar{B})$$

Thus reverse the sign of $\bar{B}$ i.e. reverse its direction by multiplying it with −1 and then add it to $\bar{A}$ to obtain the subtraction. This is shown in the Fig. 1.8 (a) and (b).



(a) Vectors (b) Subtraction of vectors

Fig. 1.8

1.4.3.1 Identical Vectors

Two vectors are said to be **identical** if there difference is zero. Thus $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are identical if $\overline{\mathbf{A}} - \overline{\mathbf{B}} = 0$ i.e. $\overline{\mathbf{A}} = \overline{\mathbf{B}}$. Such two vectors are also called **equal vectors.**

1.5 The Co-ordinate Systems

To describe a vector accurately and to express a vector in terms of its components, it is necessary to have some reference directions. Such directions are represented in terms of various co-ordinate systems. There are various coordinate systems available in mathematics, out of which three co-ordinate systems are used in this book, which are

1. Cartesian or rectangular co-ordinate system
2. Cylindrical co-ordinate system
3. Spherical co-ordinate system

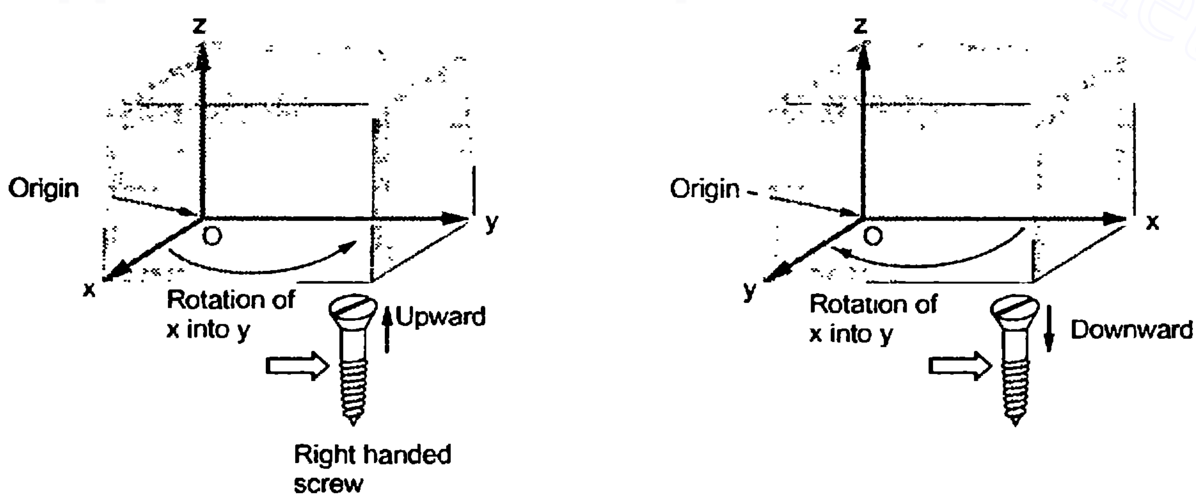
Let us discuss these systems in detail.

1.6 Cartesian Co-ordinate System

This is also called **rectangular co-ordinate system.** This system has three co-ordinate axes represented as x, y and z which are mutually at right angles to each other. These three axes intersect at a common point called **origin** of the system. There are two types of such system called

1. Right handed system and 2. Left handed system.

The right handed system means if x axis is rotated towards y axis through a smaller angle, then this rotation causes the upward movement of right handed screw in the z axis direction. This is shown in the Fig. 1.9 (a). In this system, if right hand is used then thumb indicates x axis, the forefinger indicates y axis and middle finger indicates z axis, when three fingers are held mutually perpendicular to each other.



(a) Right handed system **(b) Left handed system**

Fig. 1.9

In left handed system x and y axes are interchanged compared to right handed system. This means the rotation of x axis into y axis through smaller angle causes the downward movement of right handed screw in the z axis direction. This is shown in the Fig. 1.9 (b).

Key Point: *The right handed system is very commonly used and followed in this book.*

In cartesian co-ordinate system x = 0 plane indicates two dimensional y-z plane, y = 0 plane indicates two dimensional x-z plane and z = 0 plane indicates two dimensional x-y plane.

1.6.1 Representing a Point in Rectangular Co-ordinate System

A point in rectangular co-ordinate system is located by three co-ordinates namely x, y and z co-ordinates. The point can be reached by moving from origin, the distance x in x direction then the distance y in y direction and finally the distance z in z direction. Consider a point P having co-ordinates x_1, y_1 and z_1. It is represented as $P(x_1, y_1, z_1)$. It can be shown as in the Fig. 1.10 (a). The co-ordinates x_1, y_1 and z_1 can be positive or negative. The point $Q(3, -1, 2)$ can be shown in this system as in the Fig. 1.10 (b).

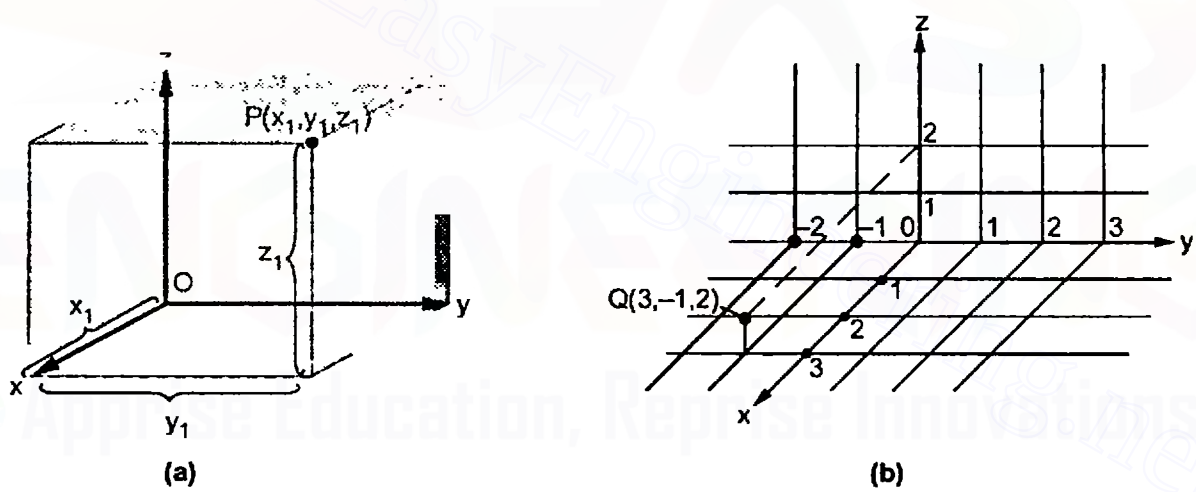


Fig. 1.10 Representing a point in cartesian system

Another method to define a point is to consider three surfaces namely x = constant, y = constant and z = constant planes. The common intersection point of these three surfaces is the point to be defined and the constants indicate the coordinates of that point. For example, consider point Q which is intersection of three planes namely x = 3 plane, y = -1 plane and z = 2 plane. The planes x = constant, y = constant and z = constant are shown in the Fig. 1.11. The constants may be positive or negative.

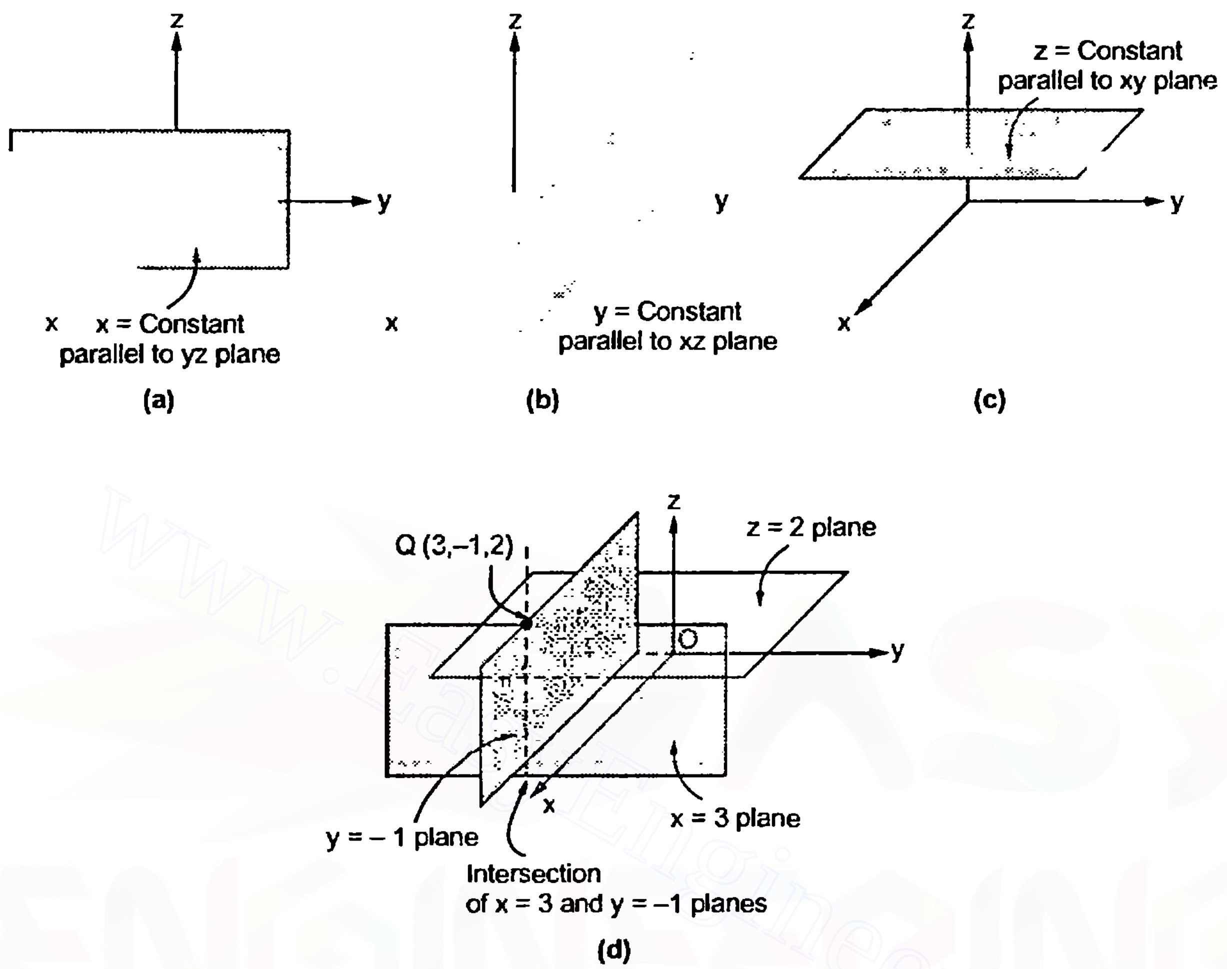


Fig. 1.11

1.6.2 Base Vectors

The **base vectors** are the unit vectors which are strictly oriented along the directions of the co-ordinate axes of the given co-ordinate system.

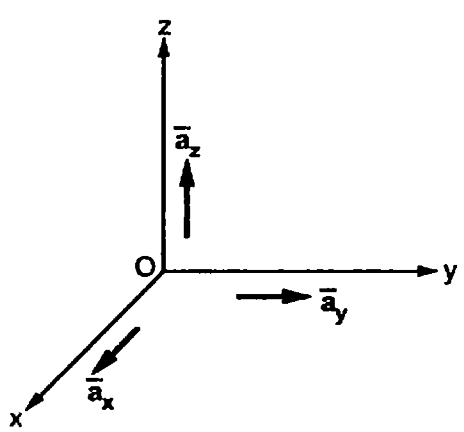


Fig. 1.12 Unit vectors in cartesian system

Thus for cartesian co-ordinate system, the three base vectors are the unit vectors oriented in x, y and z axis of the system. So $\bar{a}_x, \bar{a}_y$ and $\bar{a}_z$ are the base vectors of cartesian co-ordinate system. These are shown in the Fig. 1.12.

So any point on x-axis having co-ordinates $(x_1, 0, 0)$ can be represented by a vector joining origin to this point and denoted as $x_1 \bar{a}_x$.

The base vectors are very important in representing a vector in terms of its components, along the three co-ordinate axes.

1.6.3 Position and Distance Vectors

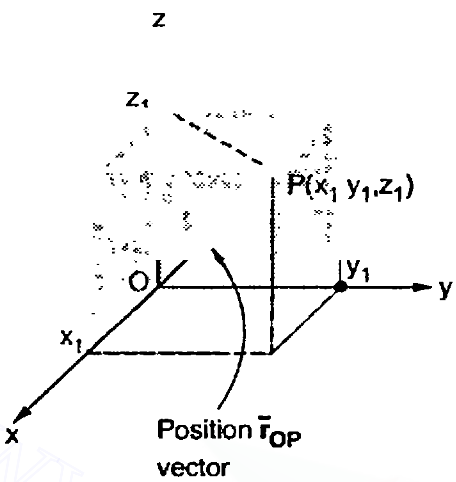


Fig. 1.13 Position vector

Consider a point $P(x_1, y_1, z_1)$ in cartesian co-ordinate system as shown in the Fig. 1.13. Then the **position vector** of point P is represented by the distance of point P from the origin, **directed from origin to point P.** This is also called **radius vector.** This is also shown in the Fig. 1.13.

Now the three components of this position vector $\bar{r}_{OP}$ are three vectors oriented along the three co-ordinate axes with the magnitudes x_1, y_1 and z_1. Thus the position vector of point P can be represented as,

$$\boxed{\bar{r}_{OP} = x_1\,\bar{a}_x + y_1\,\bar{a}_y + z_1\,\bar{a}_z} \quad \text{... (1)}$$

The magnitude of this vector interms of three mutually perpendicular components is given by,

$$\boxed{|\bar{r}_{OP}| = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}} \quad \text{... (2)}$$

Thus if point P has co-ordinates (1, 2, 3) then its position vector is,

$$\bar{r}_{OP} = 1\ \bar{a}_x + 2\ \bar{a}_y + 3\ \bar{a}_z$$

and $|\bar{r}_{OP}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14} = 3.7416$

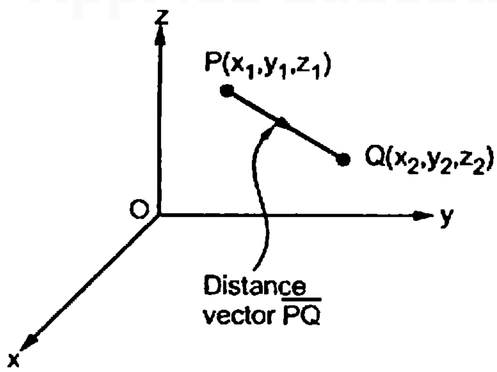


Fig. 1.14 Distance vector

Many a times the position vector is denoted by the vector representing that point itself i.e. for point P the position vector is $\bar{P}$, for point Q it is $\bar{Q}$ and so on. The same method is used hereafter in this book. Note the difference between a point and a position vector.

Now consider the two points in a cartesian coordinate system, P and Q with the co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. The points are shown in the Fig. 1.14. The individual position vectors of the points are,

$$\bar{P} = x_1\,\bar{a}_x + y_1\,\bar{a}_y + z_1\,\bar{a}_z$$

$$\bar{Q} = x_2\,\bar{a}_x + y_2\,\bar{a}_y + z_2\,\bar{a}_z$$

Then the distance or the displacement from P to Q is represented by a **distance vector** $\overline{PQ}$ and is given by,

$$\boxed{\overline{PQ} = \overline{Q}-\overline{P}=[x_2-x_1]\,\bar{a}_x+[y_2-y_1]\,\bar{a}_y+[z_2-z_1]\,\bar{a}_z} \quad \text{... (3)}$$

This is also called **separation vector.**

The magnitude of this vector is given by,

$$\boxed{|\overline{PQ}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}} \quad \text{.... (4)}$$

This is nothing but the length of the vector PQ. The equation (4) is called **distance formula** which gives the distance between the two points representing tips of the vectors.

Using the basic concept of unit vector, we can find unit vector along the direction PQ as,

$$\boxed{\bar{a}_{PQ} = \text{Unit vector along PQ} = \frac{\overline{PQ}}{|\overline{PQ}|}} \quad \text{... (5)}$$

Once the position vectors are known then various mathematical operations can be easily performed interms of the components of the various vectors.

Let us summarize procedure to obtain **distance vector** and **unit vector.**

Step 1 : Identify the direction of distance vector i.e. starting point (x_1, y_1, z_1) and terminating point (x_2, y_2, z_2).

Step 2 : Subtract the respective co-ordinates of starting point from terminating point. These are three components of distance vector i.e. $(x_2-x_1)\,\bar{a}_x$, $(y_2-y_1)\,\bar{a}_y$ and $(z_2-z_1)\,\bar{a}_z$

Step 3 : Adding the three components distance vector can be obtained.

Step 4 : Calculate the magnitude of the distance vector using equation (4).

Step 5 : Unit vector along the distance vector can be obtained by using equation (5).

Example 1.1 : *Obtain the unit vector in the direction from the origin towards the point* $P(3,-3,-2)$.

Solution : The origin O (0, 0, 0) while P (3, – 3, – 2) hence the distance vector $\overline{OP}$ is,

$$\overline{OP} = (3-0)\bar{a}_x+(-3-0)\bar{a}_y+(-2-0)\bar{a}_z = 3\bar{a}_x-3\bar{a}_y-2\bar{a}_z$$

$$\therefore \quad |\overline{OP}| = \sqrt{(3)^2+(-3)^2+(-2)^2} = 4.6904$$

Hence the unit vector along the direction OP is,

$$\bar{a}_{OP} = \frac{\overline{OP}}{|\overline{OP}|} = \frac{3\bar{a}_x-3\bar{a}_y-2\bar{a}_z}{4.6904}$$

$$= \mathbf{0.6396\,\bar{a}_x - 0.6396\,\bar{a}_y - 0.4264\,\bar{a}_z}$$

Example 1.2 : *Two points $A(2,2,1)$ and $B(3,-4,2)$ are given in the cartesian system. Obtain the vector from A to B and a unit vector directed from A to B.*

Solution : The starting point is A and terminating point is B.

Now $\bar{A} = 2\,\bar{a}_x + 2\bar{a}_y + \bar{a}_z$ and $\bar{B} = 3\bar{a}_x - 4\bar{a}_y + 2\bar{a}_z$

$\therefore \quad \overline{AB} = \bar{B} - \bar{A} = (3-2)\bar{a}_x + (-4-2)\bar{a}_y + (2-1)\bar{a}_z$

$\therefore \quad \overline{AB} = \bar{a}_x - 6\,\bar{a}_y + \bar{a}_z$

This is the vector directed from A to B.

Now $|\overline{AB}| = \sqrt{(1)^2 + (-6)^2 + (1)^2} = 6.1644$

Thus unit vector directed from A to B is,

$$\bar{a}_{AB} = \frac{\overline{AB}}{|\overline{AB}|} = \frac{\bar{a}_x - 6\bar{a}_y + \bar{a}_z}{6.1644}$$

$$= \mathbf{0.1622}\,\bar{a}_x - \mathbf{0.9733}\,\bar{a}_y + \mathbf{0.1622}\,\bar{a}_z$$

It can be cross checked that magnitude of this unit vector is unity i.e.

$\sqrt{(0.1622)^2 + (-0.9733)^2 + (0.1622)^2} = 1.$

1.6.4 Differential Elements in Cartesian Co-ordinate System

Consider a point $P(x, y, z)$ in the rectangular co-ordinate system. Let us increase each co-ordinate by a differential amount. A new point P′ will be obtained having co-ordinates $(x+dx, y+dy, z+dz)$.

Thus, dx = Differential length in x direction

dy = Differential length in y direction

dz = Differential length in z direction

Hence differential vector length also called **elementary vector length** can be represented as,

$$\boxed{\overline{dl} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z} \quad \text{... (6)}$$

This is the vector joining original point P to new point P′.

Now point P is the intersection of three planes while point P′ is the intersection of three new planes which are slightly displaced from original three planes. These six planes together define a differential volume which is a rectangular parallelepiped as shown in the Fig. 1.15. The diagonal of this parallelepiped is the differential vector length.

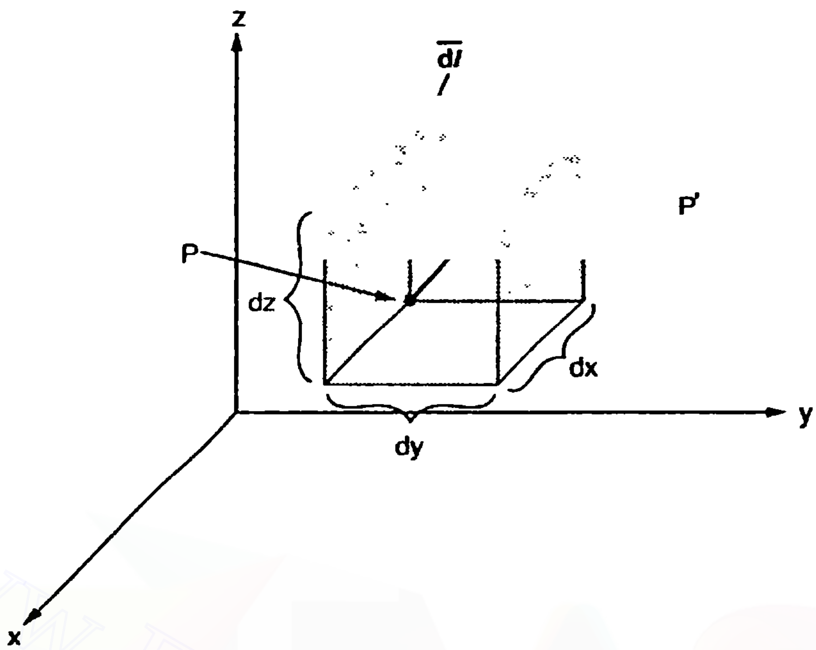


Fig. 1.15 Differential elements and different length in cartesian system

The distance of P′ from P is given by magnitude of the differential vector length,

$$\boxed{\left|\overline{dl}\right| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}} \quad \text{... (7)}$$

Hence the **differential volume** of the rectangular parallelepiped is given by,

$$\boxed{dv = dx\ dy\ dz} \quad \text{...(8)}$$

Note that $\overline{dl}$ is a vector but dv is a scalar.

Let us define **differential surface areas.** The differential surface element $\overline{dS}$ is represented as,

$$\boxed{d\overline{S} = dS\ \bar{a}_n} \quad \text{...(9)}$$

where dS = Differential surface area of the element

$\bar{a}_n$ = Unit vector normal to the surface dS

Thus various differential surface elements in cartesian co-ordinate system are shown in the Fig. 1.16.

The vector representation of these elements is given as,

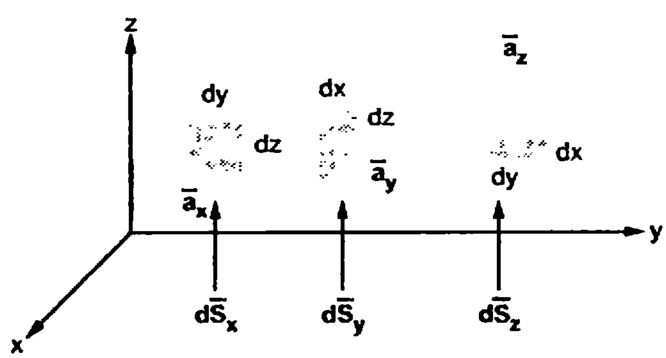


Fig. 1.16 Differential surface elements in cartesian system

$d\bar{S}_x$ = Differential vector surface area normal to x direction

$$= dydz\ \bar{a}_x \quad \text{... (10)}$$

$d\bar{S}_y$ = Differential vector surface area normal to y direction

$$= dxdz\ \bar{a}_y \quad \text{... (11)}$$

$d\bar{S}_z$ = Differential vector surface area normal to z direction

$$= dxdy\ \bar{a}_z \quad \text{... (12)}$$

The differential elements play very important role in the study of engineering electromagnetics.

Example 1.3 : *Given three points in cartesian co-ordinate system as $A(3,-2,1)$, $B(-3,-3,5)$, $C(2,6,-4)$.*

Find : i) The vector from A to C.

ii) The unit vector from B to A.

iii) The distance from B to C.

iv) The vector from A to the midpoint of the straight line joining B to C.

Solution : The position vectors for the given points are,

$$\bar{A} = 3\bar{a}_x - 2\bar{a}_y + \bar{a}_z\ ,\quad \bar{B} = -3\bar{a}_x - 3\bar{a}_y + 5\bar{a}_z\ ,\quad \bar{C} = 2\bar{a}_x + 6\bar{a}_y - 4\bar{a}_z$$

i) The vector from A to C is,

$$\overline{AC} = \bar{C} - \bar{A} = [2-3]\bar{a}_x + [6-(-2)]\bar{a}_y + [-4-1]\bar{a}_z$$

$$= -\bar{a}_x + 8\bar{a}_y - 5\bar{a}_z$$

ii) For unit vector from B to A, obtain distance vector $\overline{BA}$ first.

$\therefore \quad \overline{BA} = \bar{A} - \bar{B}$ ... as starting is B and terminating is A

$$= [3-(-3)]\bar{a}_x + [(-2)-(-3)]\bar{a}_y + [1-5]\bar{a}_z$$

$$= 6\bar{a}_x + \bar{a}_y - 4\bar{a}_z$$

$$\therefore \quad |\overline{BA}| = \sqrt{(6)^2 + (1)^2 + (-4)^2} = 7.2801$$

$$\therefore \bar{a}_{BA} = \frac{\overline{BA}}{|\overline{BA}|} = \frac{6\bar{a}_x + \bar{a}_y - 4\bar{a}_z}{7.2801} = 0.8241\ \bar{a}_x + 0.1373\ \bar{a}_y - 0.5494\ \bar{a}_z$$

iii) For distance between B and C, obtain $\overline{BC}$

$$\overline{BC} = \bar{C} - \bar{B} = [2-(-3)]\bar{a}_x + [6-(-3)]\bar{a}_y + [(-4)-(5)]\bar{a}_z = 5\bar{a}_x + 9\bar{a}_y - 9\bar{a}_z$$

$$\therefore \quad \text{Distance BC} = \sqrt{(5)^2 + (9)^2 + (-9)^2} = 13.6747$$

iv) Let $B(x_1, y_1, z_1)$ and $C(x_2, y_2, z_2)$ then the co-ordinates of midpoint of BC are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

$\therefore$ Midpoint of BC $= \left(\frac{-3+2}{2}, \frac{-3+6}{2}, \frac{5-4}{2}\right) = (-0.5, 1.5, 0.5)$

Hence the vector from A to this midpoint is

$$= [-0.5-3]\bar{a}_x + [1.5-(-2)]\bar{a}_y + [0.5-1]\,\bar{a}_z = -3.5\ \bar{a}_x + 3.5\ \bar{a}_y - 0.5\ \bar{a}_z$$

1.7 Cylindrical Co-ordinate System

The circular cylindrical co-ordinate system is the three dimensional version of polar co-ordinate system. The surfaces used to define the cylindrical co-ordinate system are,

1. Plane of constant z which is parallel to xy plane.
2. A cylinder of radius r with z axis as the axis of the cylinder.
3. A half plane perpendicular to xy plane and at an angle ϕ with respect to xz plane. The angle ϕ is called **azimuthal angle.**

The ranges of the variables are,

$$\boxed{\begin{aligned} &0 \le r \le \infty \\ &0 \le \phi \le 2\pi \\ &-\infty < z \le \infty \end{aligned}}$$

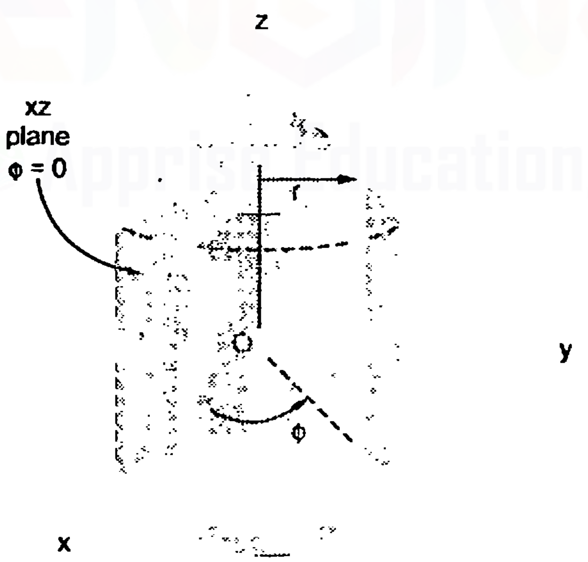
... (1)

... (2)

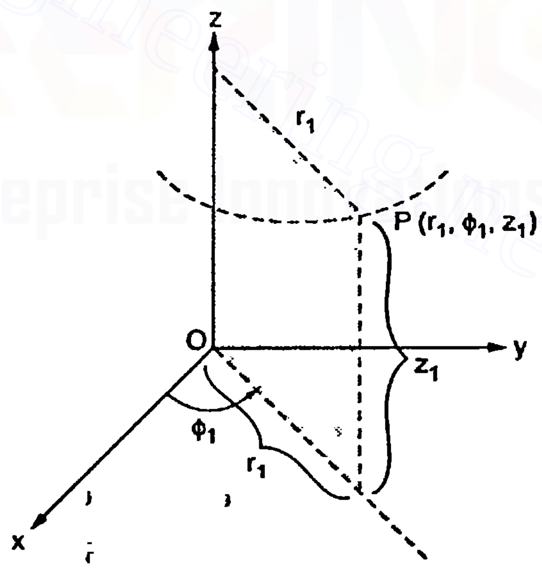
... (3)

The point P in cylindrical co-ordinate system has three co-ordinates r, ϕ and z whose values lie in the respective ranges given by the equations (1), (2) and (3).

The point $P(r, \phi_1, z_1)$ can be shown as in the Fig. 1.17(b).



(a) Cylindrical co-ordinate system



(b) Point P (r_1, ϕ_1, z_1)

Fig. 1.17

Key Point: *Note that angle ϕ is expressed in radians and for ϕ, anticlockwise measurement is treated positive while clockwise measurement is treated negative.*

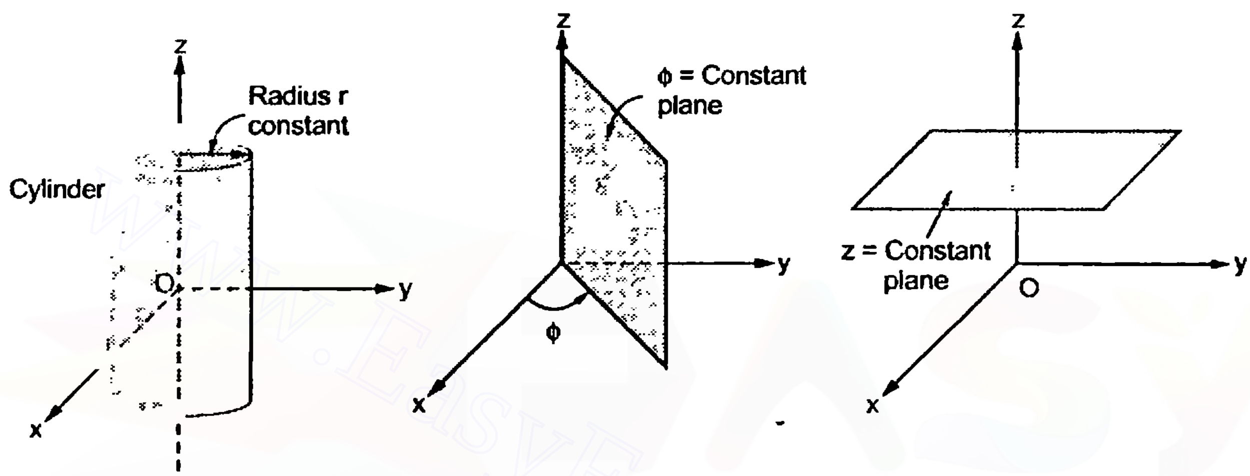
The point P can be defined as the intersection of three surfaces in cylindrical co-ordinate system. These three surfaces are,

r = Constant which is a circular cylinder with z axis as its axis.

ϕ = Constant plane which is a vertical plane perpendicular to xy plane making angle ϕ with respect to xz plane.

z = Constant plane is a plane parallel to xy plane.

These surfaces are shown in the Fig. 1.18.



(a) r = Constant (b) ϕ = Constant (c) z = Constant

Fig. 1.18

The intersection of any two surfaces out of the above three surfaces is either a line or a circle and intersection of three surfaces defines a point P.

The intersection of z = constant and r = constant is a circle. The intersection of ϕ = constant and r = constant is a line. The point P which is intersection of all three surfaces is shown in the Fig. 1.19.

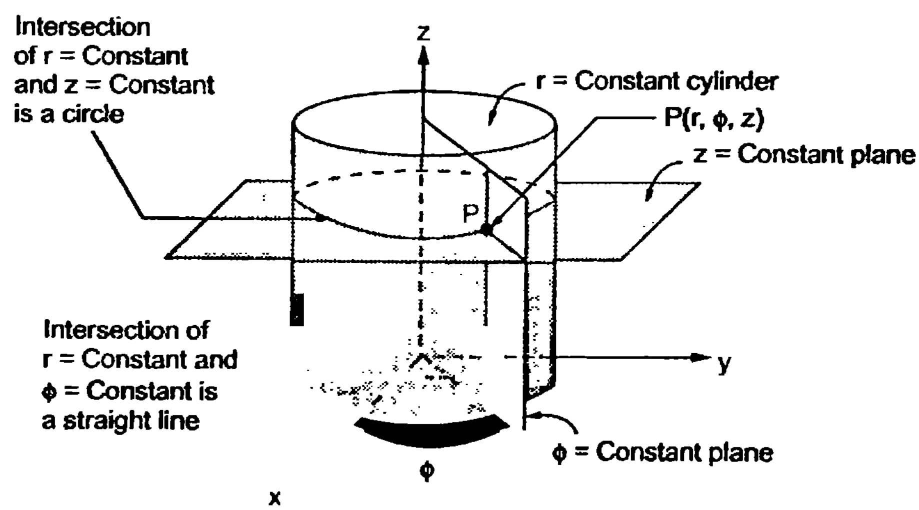


Fig. 1.19 Representing point P in cylindrical system

1.7.1 Base Vectors

Similar to cartesian coordinate system, there are three unit vectors in the r, ϕ and z directions denoted as $\bar{a}_r, \bar{a}_\phi$ and $\bar{a}_z$.

These unit vectors are shown in the Fig. 1.20.

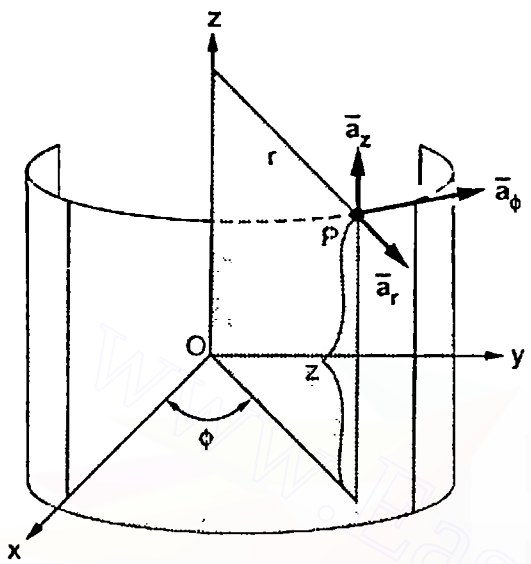


Fig. 1.20 Unit vectors in cylindrical system

These are mutually perpendicular to each other.

The $\bar{a}_r$ lies in a plane parallel to the xy plane and is perpendicular to the surface of the cylinder at a given point, coming radially outward.

The unit vector $\bar{a}_\phi$ lies also in a plane parallel to the xy plane but it is tangent to the cylinder and pointing in a direction of increasing ϕ at the given point.

The unit vector $\bar{a}_z$ is parallel to z axis and directed towards increasing z.

Hence vector of point P can be represented as,

$$\boxed{\bar{P} = P_r \bar{a}_r + P_\phi \bar{a}_\phi + P_z \bar{a}_z} \quad \text{... (4)}$$

where P_r is radius r, P_ϕ is angle ϕ and P_z is z co-ordinate of point P.

Key Point: *In cartesian co-ordinate system, the unit vectors are not dependent on the co-ordinates. But in cylindrical co-ordinate system $\bar{a}_r$ and $\bar{a}_\phi$ are functions of ϕ co-ordinate as their directions change as ϕ changes. Hence in integration or differentiation with respect to ϕ components in $\bar{a}_r$ and $\bar{a}_\phi$ should not be treated constants.*

1.7.2 Differential Elements in Cylindrical Co-ordinate System

Consider a point $P(r, \phi, z)$ in a cylindrical co-ordinate system. Let each co-ordinate is increased by the differential amount. The differential increments in r, ϕ, z are dr, dϕ and dz respectively.

Now there are two cylinders of radius r and r + dr. There are two radial planes at the angles ϕ and $\phi + d\phi$. And there are two horizontal planes at the heights z and z + dz. All these surfaces enclose a small volume as shown in the Fig. 1.21.

The differential lengths in r and z directions are dr and dz respectively. In ϕ direction, dϕ is the change in angle ϕ and is not the differential length. Due to this change dϕ, there exists a differential arc length in ϕ direction. This differential length, due to dϕ, in ϕ direction is r dϕ as shown in the Fig. 1.21.

Thus the differential lengths are,

$$dr = \text{Differential length in r direction} \quad \text{... (5)}$$

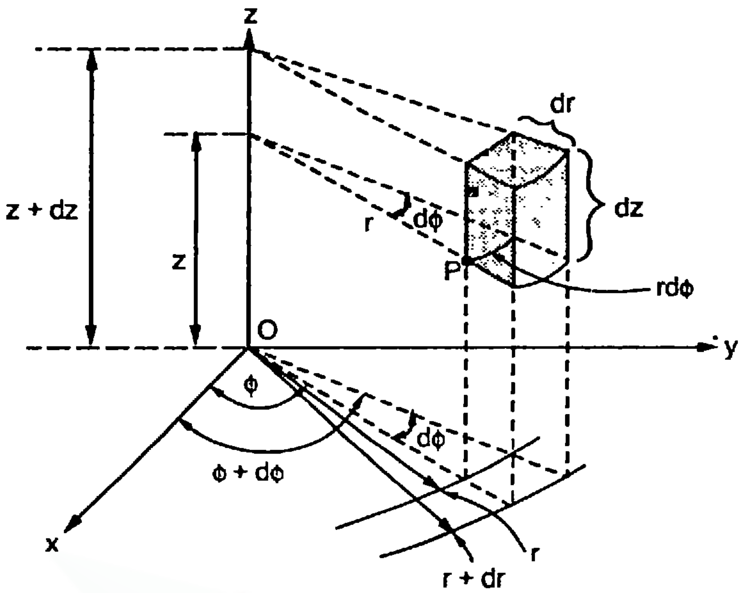


Fig. 1.21 Differential volume in cylindrical co-ordinate system

$r\,d\phi$ = Differential length in ϕ direction ... (6)

dz = Differential length in z direction ... (7)

Hence the **differential vector length** in cylindrical co-ordinate system is given by,

$$\boxed{\overline{dl} = dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z} \quad \text{... (8)}$$

The magnitude of the differential length vector is given by,

$$\boxed{|\overline{dl}| = \sqrt{(dr)^2 + (r\,d\phi)^2 + (dz)^2}} \quad \text{... (9)}$$

Hence the **differential volume** of the differential element formed is given by,

$$\boxed{dv = r\,dr\,d\phi dz} \quad \text{... (10)}$$

The **differential surface areas** in the three directions are shown in the Fig. 1.22.

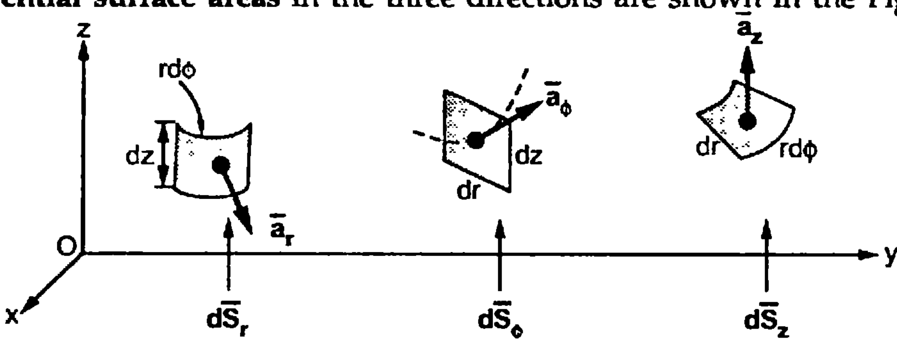


Fig. 1.22 Differential surface elements in cylindrical system

The vector representation of these differential surface areas are given by,

$d\bar{S}_r$ = Differential vector surface area normal to r direction

$= r\,d\phi\,dz\,\bar{a}_r$ ... (11)

$d\bar{S}_\phi$ = Differential vector surface area normal to ϕ direction

$$= dr\, dz\, \bar{a}_\phi \qquad \dots (12)$$

$d\bar{S}_z$ = Differential vector surface area normal to z direction

$$= r\, dr\, d\phi\, \bar{a}_z \qquad \dots (13)$$

1.7.3 Relationship between Cartesian and Cylindrical Systems

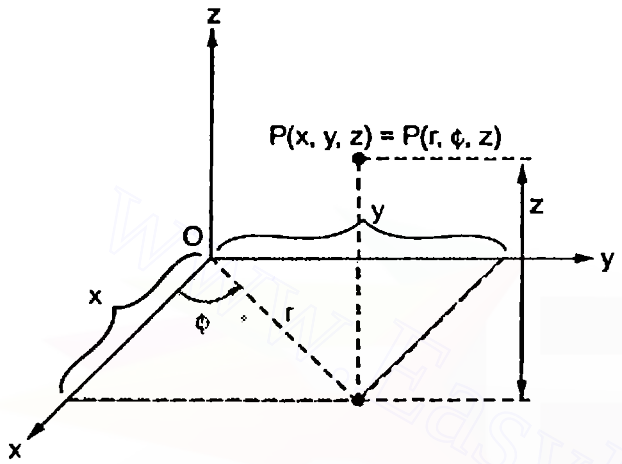


Fig. 1.23 Relationship between cartesian and cylindrical systems

Consider a point P whose cartesian co-ordinates are x, y and z while the cylindrical co-ordinates are r, ϕ and z, as shown in the Fig. 1.23.

Looking at the xy plane we can write,

$x = r \cos \phi$ and $y = r \sin \phi$

The z remains same in both the systems.

Hence transformation from cylindrical to cartesian can be obtained from the equations,

$$\boxed{x = r \cos \phi, \quad y = r \sin \phi \quad \text{and} \quad z = z} \qquad \dots (14)$$

It can be seen that, r can be expressed interms of x and y as,

$$r = \sqrt{x^2 + y^2}$$

While $$\tan \phi = \frac{y}{x}$$

Thus the transformation from cartesian to cylindrical can be obtained from the equations,

$$\boxed{r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \frac{y}{x} \quad \text{and} \quad z = z} \qquad \dots (15)$$

Note : While using the equations (15) note that r is positive or zero, hence positive sign of square root must be considered. **While calculating ϕ make sure the signs of x and** y. If both are positive, ϕ is positive in the first quadrant. If x is negative and y is positive then the point is in the second quadrant hence ϕ must be within $+90°$ and $+180°$ i.e. within $-180°$ and $-270°$. Thus for $x = -2$ and $y = 1$ we get $\phi = \tan^{-1}\left[\frac{1}{-2}\right] = -26.56°$ but it should be taken as $-26.56°+180°$ i.e. 154.43°. Hence when x is negative, it is necessary to add 180° to the ϕ calculated using $\tan^{-1}$ function, to obtain accurate ϕ corresponding to the point.

When y is negative and x is positive then ϕ is in fourth quadrant i.e. within 0° and −90° i.e. 270° and 360°. Similarly when x is negative and y is also negative the point is in third quadrant and accordingly ϕ must be between −90° to −180° i.e. +180° and +270°. So 180° must be subtracted from the ϕ calculated by $\tan^{-1}$ function, to get accurate ϕ when both x and y are negative. Thus if $x = y = -3$ then $\phi = \tan^{-1}\left[\frac{-3}{-3}\right] = +45°$ but actually it is $45° - 180° = -135°$ i.e. $-135° + 360° = +225°$.

Example 1.4 : *Consider a cylinder of length L and radius R. Obtain its volume by integration.*

Solution : The cylinder is shown in the Fig. 1.24.

Consider a differential volume in the cylindrical co-ordinate system as,

$$dv = r\,dr\,d\phi\,dz$$

For the given cylinder r varies from 0 to R, z varies from 0 to L while ϕ varies from 0 to 2π radians. These are the limits of integration.

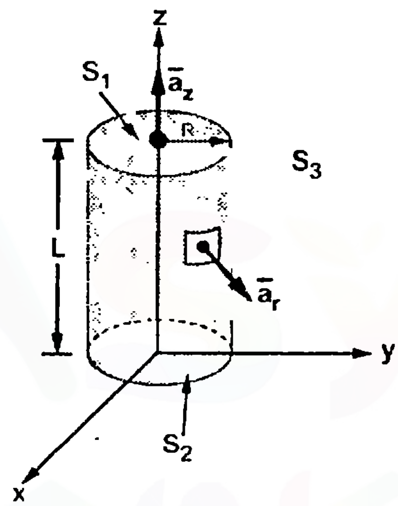


Fig. 1.24

$$\therefore \text{ Volume of cylinder} = \int_0^L \int_0^{2\pi} \int_0^R r\,dr\,d\phi\,dz$$

$$= \int_0^L \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^R d\phi\,dz$$

$$= \frac{R^2}{2}\int_0^L [\phi]_0^{2\pi}\,dz$$

$$= 2\pi\frac{R^2}{2}[z]_0^L = \pi R^2 L$$

Example 1.5 : *Calculate the total surface area of the cylinder having length L and radius R by the method of integration.*

Solution : Consider the upper surface area, the normal to which is $\bar{a}_z$. So the differential surface area normal to z direction is $r\,d\phi\,dr$. Consider the Fig. 1.24.

$$\therefore \quad S_1 = \int_0^{2\pi}\int_0^R r\,d\phi\,dr$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^R d\phi = \frac{R^2}{2}\times[\phi]_0^{2\pi} = \pi R^2$$

The bottom surface area S_2 is same as S_1 i.e. πR^2. For remaining surface area consider the differential surface area normal to r direction which is $r\,d\phi\,dz$.

$$S_3 = \int_0^L \int_0^{2\pi} r\, d\phi\, dz \quad \text{but} \quad r = R \text{ is constant}$$

$$= \int_0^L \int_0^{2\pi} R\, d\phi\, dz = R[\phi]_0^{2\pi} [z]_0^L = 2\pi RL$$

$$\text{Total surface area} = S_1 + S_2 + S_3 = \pi R^2 + \pi R^2 + 2\pi RL$$

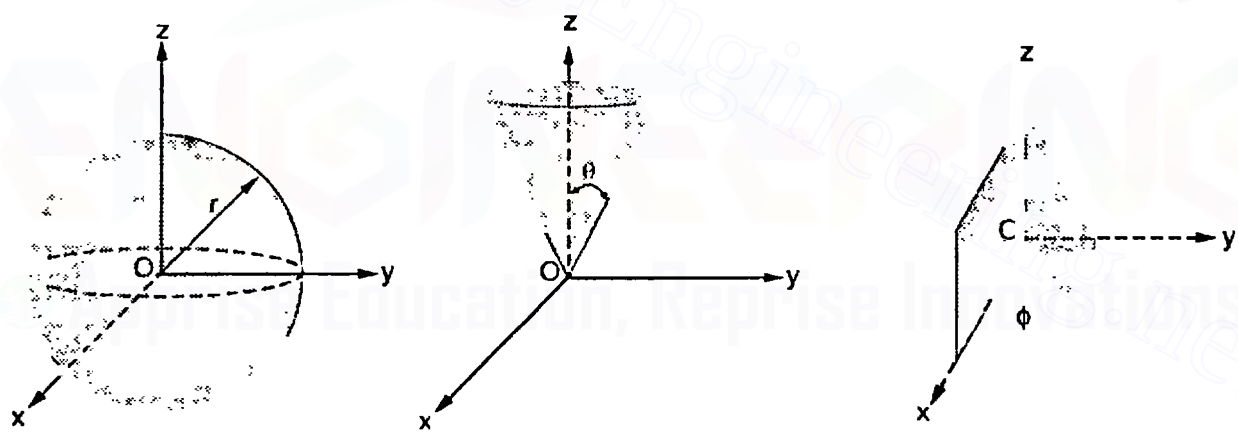
$$= 2\pi R(R+L)$$

1.8 Spherical Co-ordinate System

The surfaces which are used to define the spherical co-ordinate system on the three cartesian axes are,

1. Sphere of radius r, origin as the centre of the sphere.
2. A right circular cone with its apex at the origin and its axis as z axis. Its half angle is θ. It rotates about z axis and θ varies from 0 to 180°.
3. A half plane perpendicular to xy plane containing z axis, making an angle ϕ with the xz plane.

Thus the three co-ordinates of a point P in the spherical co-ordinate system are (r, θ, ϕ). These surfaces are shown in the Fig. 1.25.



(a) Sphere of radius r with centre as origin

(b) Right circular cone with apex at origin

(c) Half plane perpendicular to xy plane

Fig. 1.25

The ranges of the variables are,

$$\boxed{\begin{aligned} &0 \le r < \infty && \dots (1)\\ &0 \le \phi \le 2\pi && \dots (2)\\ &0 \le \theta \le \pi \text{ as half angle} && \dots (3)\end{aligned}}$$

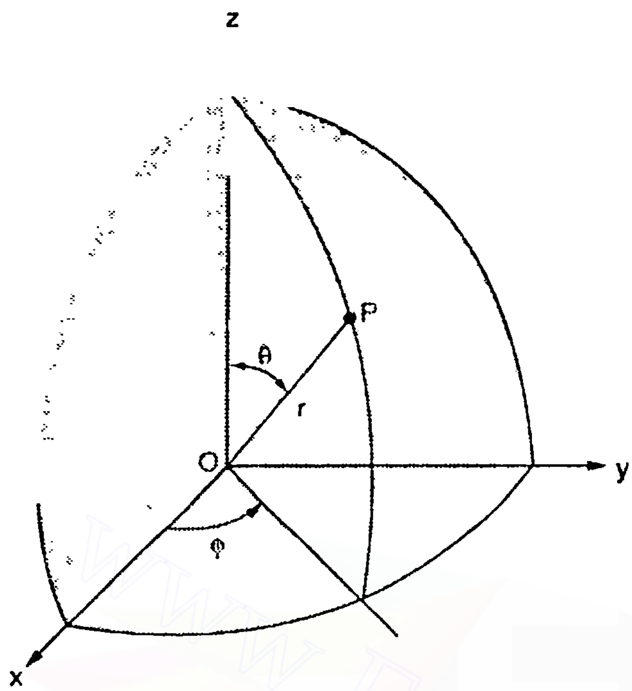


Fig. 1.26 Representing point P in spherical co-ordinate system

The point $P(r, \theta, \phi)$ can be represented in the spherical co-ordinate system as shown in the Fig. 1.26. The angles θ and ϕ are measured in radians.

The point P can be defined as the intersection of three surfaces in spherical co-ordinate system. These three surfaces are,

r = Constant which is a sphere with centre as origin.

θ = Constant which is right circular cone with apex as origin and axis as z axis.

ϕ = Constant is a plane perpendicular to xy plane.

The surfaces are already shown in the Fig. 1.25. The intersection of the sphere i.e. r = Constant surface and right circular cone i.e. θ = Constant surface is a horizontal circle as shown in the Fig. 1.27. As seen from the Fig. 1.27, the radius of this circle is $r \sin \theta$

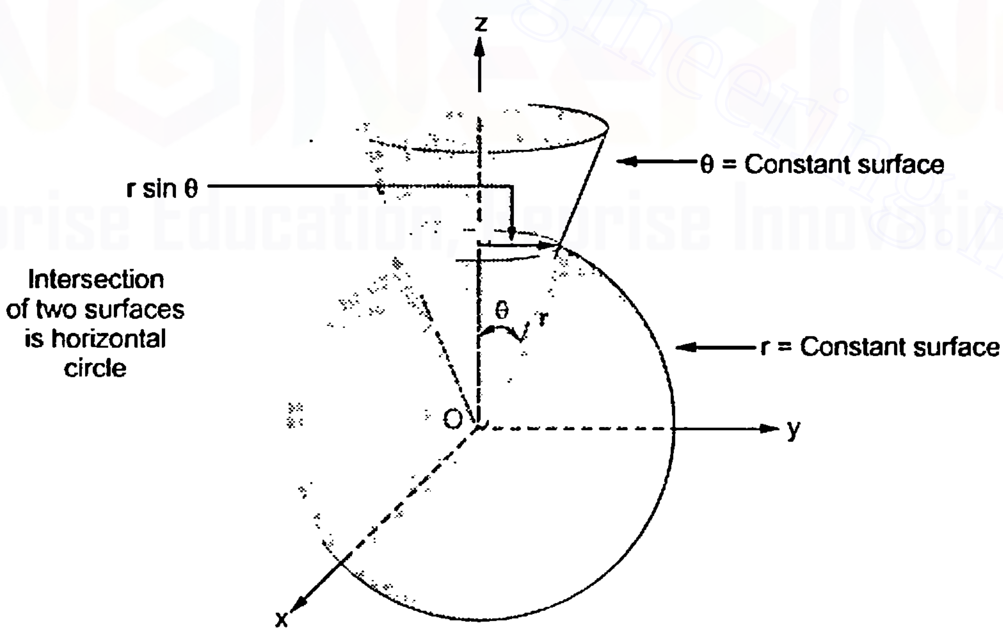


Fig. 1.27

Now consider intersection of ϕ = constant plane with the intersection of r = constant and θ = constant planes as shown in the Fig. 1.28. This defines a point P.

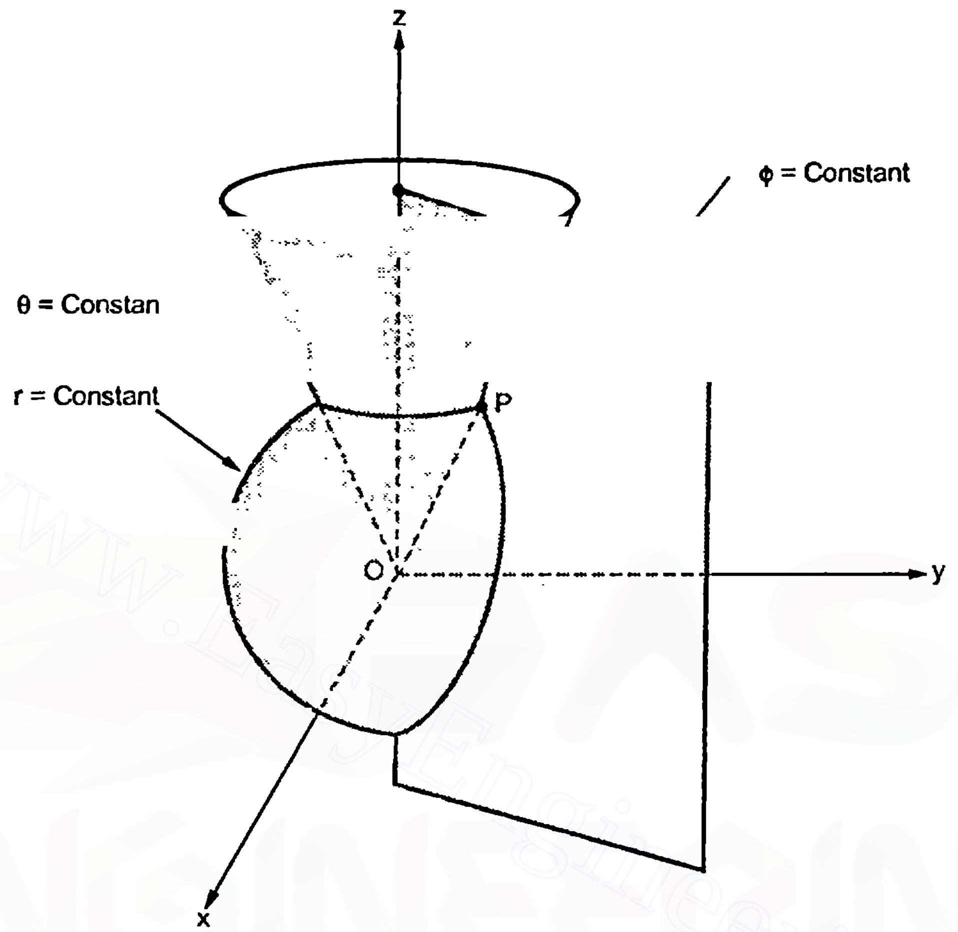


Fig. 1.28 Representing point P in spherical co-ordinate system

1.8.1 Base Vectors

Similar to other two co-ordinate systems, there are three unit vectors in the r, θ and ϕ directions denoted as $\bar{a}_r$, $\bar{a}_\theta$ and $\bar{a}_\phi$. These unit vectors are mutually perpendicular to each other and are shown in the Fig. 1.29. The unit vector $\bar{a}_r$ is directed from the centre of the sphere i.e. origin to the given point P. It is directed radially outward, normal to the sphere. It lies in the cone θ = constant and plane ϕ = constant.

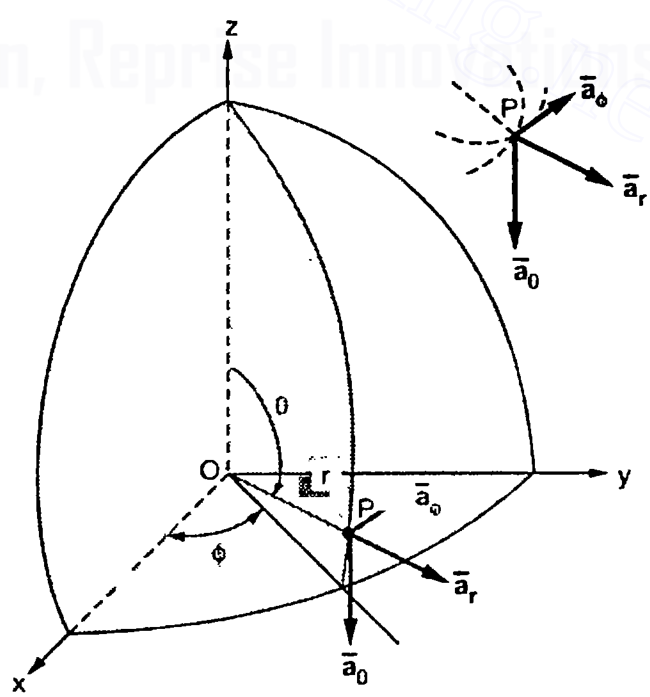


Fig. 1.29 Unit vectors in spherical co-ordinate systems

The unit vector $\bar{a}_\theta$ is tangent to the sphere and oriented in the direction of increasing θ. It is normal to the conical surface.

The third unit vector $\bar{a}_\phi$ is tangent to the sphere and also tangent to the conical surface. It is oriented in the direction of increasing ϕ. It is same as defined in the cylindrical co-ordinate system.

Hence vector of point P can be represented as,

$$\boxed{\bar{P} = P_r\,\bar{a}_r + P_\theta\,\bar{a}_\theta + P_\phi\,\bar{a}_\phi} \qquad \text{... (4)}$$

where P_r is the radius r and P_θ, P_ϕ are the two angle components of point P.

1.8.2 Differential Elements in Spherical Co-ordinate System

Consider a point $P(r, \theta, \phi)$ in a spherical co-ordinate system. Let each co-ordinate is increased by the differential amount. The differential increments in r, θ, ϕ are dr, dθ and dϕ.

Now there are two spheres of radius r and r+dr. There are two cones with half angles θ and θ+dθ. There are two planes at the angles ϕ and ϕ+dϕ measured from xz plane. All these surfaces enclose a small volume as shown in the Fig. 1.30.

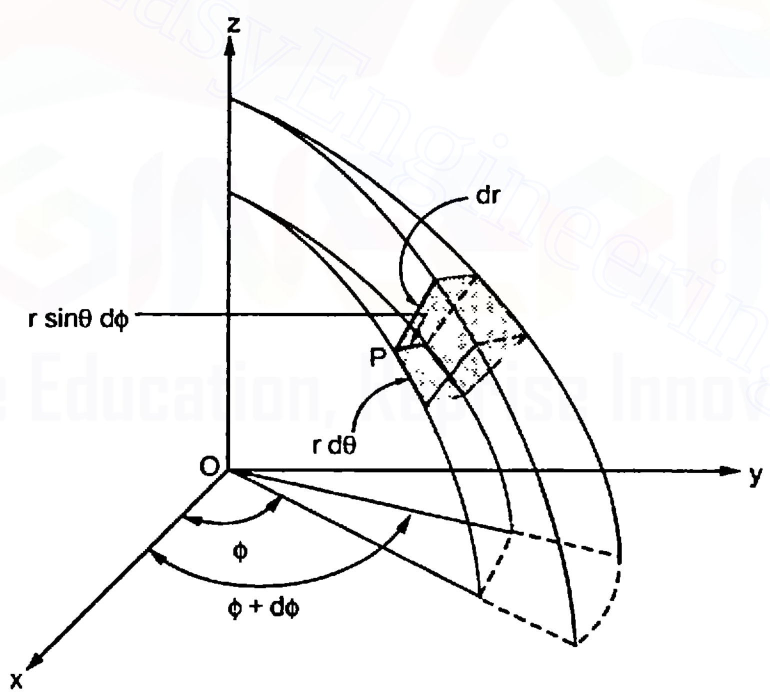


Fig. 1.30 Differential volume in spherical co-ordinate system

The differential length in r direction is dr. The differential length in ϕ direction is $r\sin\theta\, d\phi$. The differential length in θ direction is $r\,d\theta$. Thus,

$$dr = \text{Differential length in r direction} \qquad \text{... (5)}$$

$$r\,d\theta = \text{Differential length in } \theta \text{ direction} \qquad \text{... (6)}$$

$$r\sin\theta\, d\phi = \text{Differential length in } \phi \text{ direction} \qquad \text{... (7)}$$

Hence the **differential vector length** in spherical coordinate system is given by,

$$\boxed{\overline{dl} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi} \qquad \text{... (8)}$$

The magnitude of the differential length vector is given by,

$$\boxed{\left|\overline{dl}\right| = \sqrt{(dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2}} \qquad \text{... (9)}$$

Hence the **differential volume** of the differential element formed, in spherical co-ordinate system is given by,

$$\boxed{dv = r^2 \sin\theta\,dr\,d\theta\,d\phi} \qquad \text{... (10)}$$

The **differential surface areas** in the three directions are shown in the Fig. 1.31.

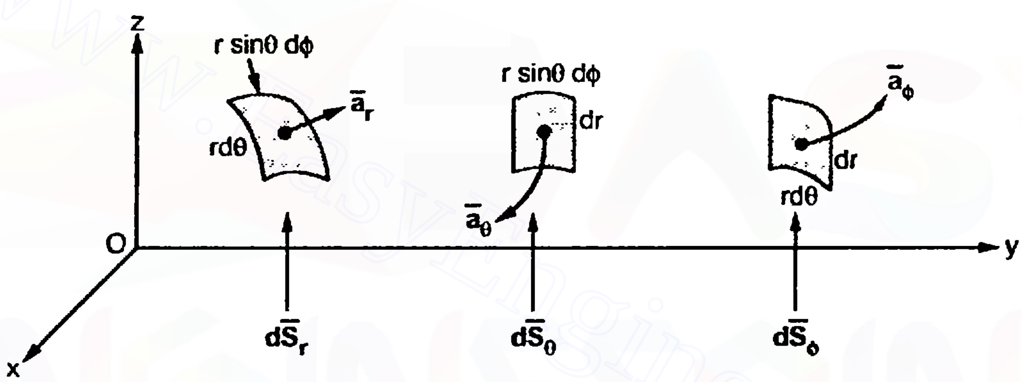


Fig. 1.31 Differential surface elements in spherical co-ordinate system

The vector representation of these differential surface areas are given by,

$d\bar{S}_r$ = Differential vector surface area normal to r direction

$$= r^2 \sin\theta\,d\theta\,d\phi \qquad \text{... (11)}$$

$d\bar{S}_\theta$ = Differential vector surface area normal to θ direction

$$= r\sin\theta\,dr\,d\phi \qquad \text{... (12)}$$

$d\bar{S}_\phi$ = Differential vector surface area normal to ϕ direction

$$= r\,dr\,d\theta \qquad \text{... (13)}$$

1.8.3 Relationship between Cartesian and Spherical Systems

Consider a point P whose cartesian coordinates are x, y and z while the spherical co-ordinates are r, θ and ϕ as shown in the Fig. 1.32.

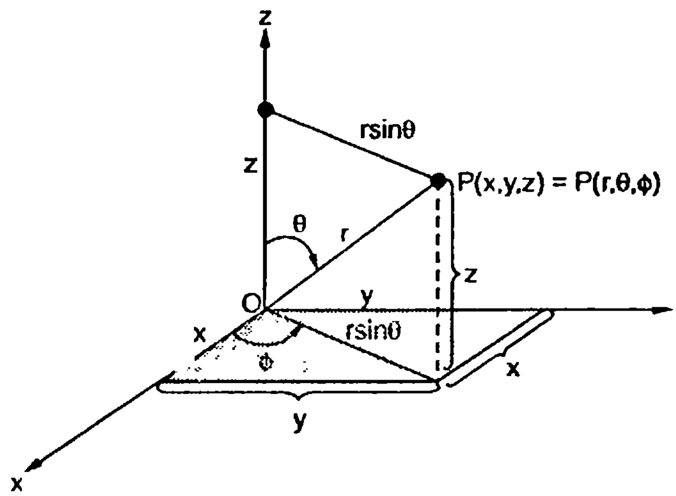


Fig. 1.32 Relationship between cartesian and spherical systems

Looking at the xy plane we can write,

$$x = r\sin\theta\cos\phi \quad \text{and} \quad y = r\sin\theta\sin\phi$$

While $\quad z = r\cos\theta$

Hence the transformation from spherical to cartesian can be obtained from the equations,

$$\boxed{x = r\sin\theta\cos\phi, \; y = r\sin\theta\sin\phi \; \text{ and } \; z = r\cos\theta} \quad \text{... (14)}$$

Now r can be expressed as,

$$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$

$$= r^2\sin^2\theta[\sin^2\phi+\cos^2\phi] + r^2\cos^2\theta$$

$$= r^2[\sin^2\theta+\cos^2\theta] = r^2$$

$$r = \sqrt{x^2+y^2+z^2}$$

While $\tan\phi = \dfrac{y}{x}$ and $\cos\theta = \dfrac{z}{r}$

As r is known, θ can be obtained.

Thus the transformation from cartesian to spherical coordinate system can be obtained from the equations,

$$\boxed{r = \sqrt{x^2+y^2+z^2}, \; \theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right] \; \text{and} \; \phi = \tan^{-1}\frac{y}{x}} \quad \text{... (15)}$$

Remember that r is positive and varies from 0 to ∞, θ varies from 0 to π radians i.e. 0° to 180° and ϕ varies from 0 to 2π radians i.e. 0° to 360°.

Key Point: *While using above formulae, care must be taken to place the angles θ and ϕ in the correct quadrants according to the signs of x, y and z.*

Example 1.6 : *Calculate the volume of a sphere of radius R using integration.*

Solution : The differential volume of a sphere is,

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

The limits for r are 0 to R, as sphere is of radius R.

The θ varies from 0 to π while ϕ varies from 0 to 2π.

$$\therefore \quad v = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^R \sin\theta \, d\theta \, d\phi = \frac{R^3}{3}\int_0^{2\pi}[-\cos\theta]_0^{\pi} \, d\phi$$

$$= \frac{R^3}{3}[-\cos\pi-(-\cos 0)]\int_0^{2\pi} d\phi = \frac{R^3}{3}[-(-1)-(-1)][\phi]_0^{2\pi}$$

$$= \frac{R^3}{3}\times 2\times 2\pi = \frac{4}{3}\pi R^3$$

Example 1.7 : *Calculate the surface area of a sphere of radius R, by integration.*

Solution : Consider the differential surface area normal to the r direction which is,

$$dS_r = r^2 \sin\theta \, d\theta \, d\phi$$

Now the limits of ϕ are 0 to 2π while θ varies from 0 to π.

$$\therefore \quad S_r = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \, d\theta d\phi$$

But note that **radius of sphere is constant,** given as $r = R$.

$$S_r = R^2\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi = R^2[-\cos\theta]_0^{\pi} \, [\phi]_0^{2\pi}$$

$$= R^2\times[-\cos\pi-(-\cos 0)]\times 2\pi = R^2[-(-1)-(-1)]2\pi = 4\pi R^2$$

Example 1.8 : *Use spherical co-ordinates and integrate to find the area of the region $0 \le \phi \le \alpha$ on the spherical shell of radius a. What is the area if $\alpha = 2\pi$?*

Solution : Consider the spherical shell of radius a hence $r = a$ is constant.

Consider differential surface area normal to r direction which is radially outward.

$$dS_r = r^2 \sin\theta d\theta d\phi = a^2 \sin\theta d\theta d\phi \qquad \text{... as } r = a$$

But ϕ is varying between 0 to α while for spherical shell θ varies from 0 to π.

$$\therefore \quad S_r = a^2\int_0^{\alpha}\int_0^{\pi} \sin\theta \, d\theta d\phi = a^2[-\cos\theta]_0^{\pi}[\phi]_0^{\alpha}$$

$$= a^2 \cdot [-\cos\pi-(-\cos 0)]\alpha = 2a^2\alpha$$

So area of the region is $2a^2\alpha$.

If $\alpha = 2\pi$, the area of the region becomes $4\pi a^2$, as the shell becomes complete sphere of radius a when ϕ varies from 0 to 2π.

1.9 Vector Multiplication

Uptill now the addition, subtraction and multiplication by scalar to a vector is discussed. Let us discuss the multiplication of two or more vectors. The knowledge of vector multiplication allows us to transform the vectors from one coordinate system to other.

Consider two vectors $\bar{A}$ and $\bar{B}$. There are two types of products existing depending upon the result of the multiplication. These two types of products are,

1. Scalar or Dot product
2. Vector or Cross product

Let us discuss the characteristics of these two products.

1.10 Scalar or Dot Product of Vectors

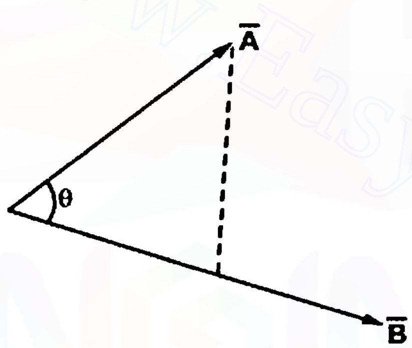


Fig. 1.33

The scalar or dot of the two vectors $\bar{A}$ and $\bar{B}$ is denoted as $\bar{A} \cdot \bar{B}$ and defined as the product of the magnitude of A, the magnitude of B and the cosine of the smaller angle between them.

It also can be defined as the product of magnitude of $\bar{B}$ and the projection of $\bar{A}$ onto $\bar{B}$ or vice versa.

Mathematically it is expressed as,

$$\boxed{\bar{A} \cdot \bar{B} = |\bar{A}|\,|\bar{B}| \cos \theta_{AB}} \quad \text{... (1)}$$

The result of such a dot product is scalar hence it is also called **scalar product.**

1.10.1 Properties of Dot Product

The various properties of the dot product are,

1. If the two vectors are parallel to each other i.e. $\theta = 0°$ then $\cos \theta_{AB} = 1$ thus

$$\bar{A} \cdot \bar{B} = |\bar{A}|\ |\bar{B}| \text{ for parallel vectors} \quad \text{... (2)}$$

2. If the two vectors are perpendicular to each other i.e. $\theta = 90°$ then $\cos \theta_{AB} = 0$ thus

$$\bar{A} \cdot \bar{B} = 0 \text{ for perpendicular vectors} \quad \text{... (3)}$$

In other words, if dot product of the two vectors is zero, the two vectors are perpendicular to each other.

3. The dot product obeys **commutative law,**

$$\therefore \quad \bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A} \quad \text{... (4)}$$

4. The dot product obeys **distributive law,**

$$\therefore \quad \bar{A} \cdot (\bar{B} + \bar{C}) = \bar{A} \cdot \bar{B} + \bar{A} \cdot \bar{C} \quad \text{... (5)}$$

5. If the dot product of vector with itself is performed, the result is square of the magnitude of that vector.

$$\bar{A} \cdot \bar{A} = |\bar{A}| \; |\bar{A}| \cos 0° = |\bar{A}|^2 \quad \text{... (6)}$$

6. Consider the unit vectors $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ in cartesian co-ordinate system. All these vectors are mutually **perpendicular** to each other. Hence the dot product of different unit vectors is zero.

$$\boxed{\bar{a}_x \cdot \bar{a}_y = \bar{a}_y \cdot \bar{a}_x = \bar{a}_x \cdot \bar{a}_z = \bar{a}_z \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_z = \bar{a}_z \cdot \bar{a}_y = 0} \quad \text{... (7)}$$

7. Any unit vector dotted with itself is unity,

$$\boxed{\bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_z = 1} \quad \text{... (8)}$$

8. Consider two vectors in cartesian co-ordinate system,

$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$ and $\bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$

Now $\bar{A} \cdot \bar{B} = (A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z) \cdot (B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z)$

This product has nine scalar terms as dot product obeys distributive law. But from the equation (7), six terms out of nine will be zero involving the dot products of different unit vectors. While the remaining three terms involve the unit vector dotted with itself, the result of which is unity.

$$\therefore \quad \bar{A} \cdot \bar{B} = A_x B_x (\bar{a}_x \cdot \bar{a}_x) + A_y B_y (\bar{a}_y \cdot \bar{a}_y) + A_z B_z (\bar{a}_z \cdot \bar{a}_z)$$

$$\therefore \quad \bar{A} \cdot \bar{B} = A_x B_x + A_y B_y + A_z B_z \quad \text{... (9)}$$

1.10.2 Applications of Dot Product

The applications of dot product are,

1. To determine the angle between the two vectors.

The angle can be determined as,

$$\boxed{\theta = \cos^{-1} \left\{ \frac{\bar{A} \cdot \bar{B}}{|\bar{A}| \; |\bar{B}|} \right\}}$$

2. To find the component of a vector in a given direction.

Consider a vector $\bar{P}$ and a unit vector $\bar{a}$ as shown in the Fig. 1.34. The **component** of vector $\bar{P}$ in the direction of unit vector $\bar{a}$ is $\bar{P} \cdot \bar{a}$. This is a scalar quantity. This is shown in the Fig. 1.34 (a).

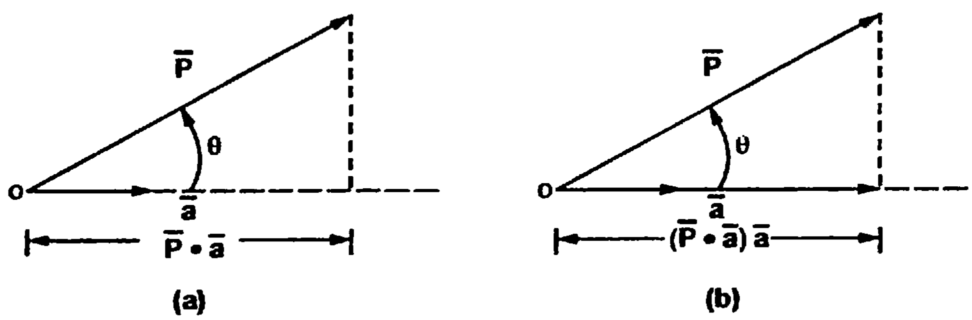


Fig. 1.34

$$\boxed{\bar{P} \bullet \bar{a} = |\bar{P}|\ |\bar{a}| \cos\theta = |\bar{P}| \cos\theta}$$

The sign of this component is positive if $0 \le \theta < 90°$ while the sign of this component is negative if $90° < \theta \le 180°$. If the **component vector** of $\bar{A}$ in the direction of unit vector $\bar{a}$ is required then multiply the component obtained by that unit vector, as shown in the Fig. 1.34(b). Thus $(\bar{P} \bullet \bar{a})\ \bar{a}$ **is the component vector of** $\bar{P}$ **in the direction of** $\bar{a}$.

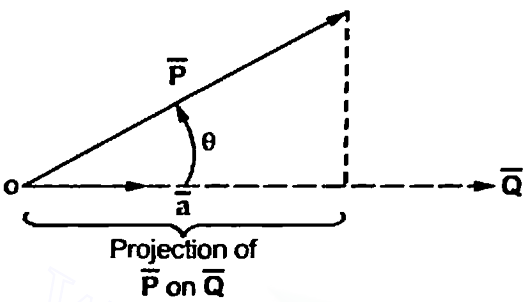


Fig. 1.34 (c)

Thus **component** of $\bar{P}$ in the direction of a_x is $\bar{P} \bullet \bar{a}_x$ i.e. P_x while the **component vector** of $\bar{P}$ in the direction of $\bar{a}_x$ is $P_x\ \bar{a}_x$.

This is the geometrical meaning of dot product, to find **projection** of $\bar{P}$ in the direction of unit vector $\bar{a}$.

If the projection of $\bar{P}$ on other vector $\bar{Q}$ is to be obtained then it is necessary to find unit vector in the direction of $\bar{Q}$ first i.e. $\bar{a}_Q$.

Then the projection of $\bar{P}$ on $\bar{Q}$ is given by $\bar{P} \bullet \bar{a}_Q$.

As $\bar{a}_Q = \dfrac{\bar{Q}}{|\bar{Q}|}$ then the projection of $\bar{P}$ on $\bar{Q}$ can be expressed as,

$$\bar{P} \bullet \frac{\bar{Q}}{|\bar{Q}|} = \frac{\bar{P} \bullet \bar{Q}}{|\bar{Q}|}.$$

3. Physically, work done by a constant force can be expressed as a dot product of two vectors.

Consider a constant force $\bar{F}$ acting on a body and it causes the displacement $\bar{d}$ of that body. Then the work done W is the product of the force and the component of the displacement in the direction of force which can be expressed as,

$$W = |\bar{F}|\, d \cos\theta = \bar{F} \bullet \bar{d}$$

But if the force applied varies along the path then the total work done is to be calculated by the integration of a dot product as,

$$W = \int \bar{F} \bullet \overline{dl}$$

Example 1.9 : *Given the two vectors,*

$\bar{A} = 2\bar{a}_x - 5\bar{a}_y - 4\,\bar{a}_z$ *and* $\bar{B} = 3\bar{a}_x + 5\bar{a}_y + 2\bar{a}_z$

Find the dot product and the angle between the two vectors.

Solution : The dot product is,

$$\bar{A} \bullet \bar{B} = A_x B_x + A_y B_y + A_z B_z$$
$$= (2 \times 3) + (-5)(5) + (-4)(2) = 6 - 25 - 8$$
$$= -27$$

As $\bar{A} \cdot \bar{B}$ is negative, it is expected that the angle between the two is greater than 90°

$$|\bar{A}| = \sqrt{(2)^2 + (-5)^2 + (-4)^2}$$

$$= \sqrt{45}$$

$$|\bar{B}| = \sqrt{(3)^2 + (5)^2 + (2)^2}$$

$$= \sqrt{38}$$

$$\therefore \qquad \theta = \cos^{-1}\left\{\frac{\bar{A} \cdot \bar{B}}{|\bar{A}|\ |\bar{B}|}\right\}$$

$$= \cos^{-1}\left\{\frac{-27}{\sqrt{45}\sqrt{38}}\right\}$$

$$= 130.762°$$

Example 1.10 : *Given vector field* $\bar{G} = (y-1)\bar{a}_x + 2x\bar{a}_y$. *Find this vector field at* $P(2, 3, 1)$ *and its projection on* $\bar{B} = 5\bar{a}_x - \bar{a}_y + 2\bar{a}_z$.

Solution : The field $\bar{G}$ at point P is,

$\bar{G}$ at P $= 2\bar{a}_x + 4\bar{a}_y$ ... Substituting co-ordinates of P in $\bar{G}$

To find its projection on $\bar{B}$, first find $\bar{a}_B$, the unit vector in the direction of $\bar{B}$.

$$\therefore \qquad \bar{a}_B = \frac{\bar{B}}{|\bar{B}|} = \frac{5\bar{a}_x - \bar{a}_y + 2\bar{a}_z}{\sqrt{(5)^2 + (-1)^2 + (2)^2}}$$

$$= 0.9128\ \bar{a}_x - 0.1825\ \bar{a}_y + 0.3651\ \bar{a}_z$$

Hence projection of $\bar{G}$ at P on the vector $\bar{B}$ is,

$$= (\bar{G} \text{ at P}) \cdot \bar{a}_B$$

$$= (2 \times 0.9128) + (4 \times -0.1825) + (0 \times 0.3651) = 1.0956$$

1.11 Vector or Cross Product of Vectors

Consider the two vectors $\bar{A}$ and $\bar{B}$ then the cross **product** is denoted as $\bar{A} \times \bar{B}$ and defined as the product of the magnitudes of $\bar{A}$ and $\bar{B}$ and the sine of the smaller angle between $\bar{A}$ and $\bar{B}$. But this product is a vector quantity and has a **direction perpendicular to the plane** containing the two vectors $\bar{A}$ and $\bar{B}$. But for any plane there are two perpendicular directions, upwards and downwards. To avoid the confusion, the direction of the cross product is along the perpendicular direction to the plane which is in the direction of advancement of a **right handed screw** when $\bar{A}$ is turned into $\bar{B}$. Thus right hand screw rule decides the direction of the cross product.

Mathematically the cross product is expressed as,

$$\boxed{\bar{A} \times \bar{B} = |\bar{A}| \; |\bar{B}| \sin \theta_{AB} \, \bar{a}_N} \quad \text{... (1)}$$

where $\bar{a}_N$ = Unit vector perpendicular to the plane of $\bar{A}$ and $\bar{B}$ in the direction decided by the right hand screw rule.

The concept of $\bar{a}_N$ is shown in the Fig. 1.35.

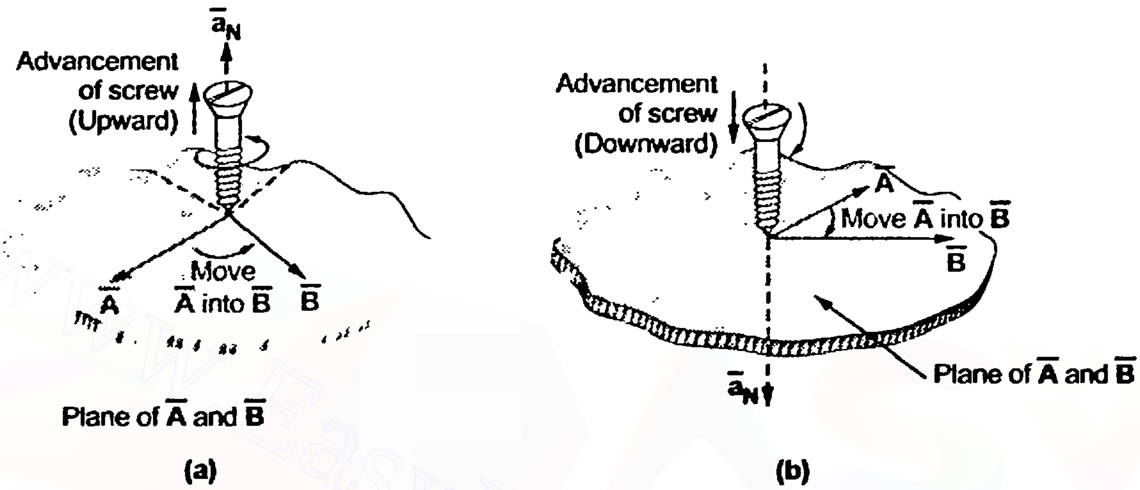


Fig. 1.35 Direction of cross product

1.11.1 Properties of Cross Product

The various properties of cross product are,

1. **The commutative law is not applicable** to the cross product. Thus,

$$\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A} \quad \text{... (2)}$$

Consider the two vectors as shown in the Fig. 1.36 (a). Then $\bar{A} \times \bar{B}$ gives unit vector $\bar{a}_N$ in the upward direction. But if $\bar{B} \times \bar{A}$ is obtained then direction of $\bar{a}_N$ must be determined by rotating $\bar{B}$ into $\bar{A}$ which results into downward direction. This is shown in the Fig. 1.36 (b).

Hence cross product is not commutative.

2. Reversing the order of the vectors $\bar{A}$ and $\bar{B}$, a unit vector $\bar{a}_N$ reverses its direction hence we can write,

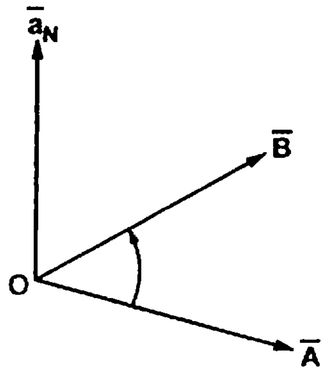
$$\bar{B} \times \bar{A} = -[\bar{A} \times \bar{B}] \quad \text{... (3)}$$

It is anticommutative in nature.

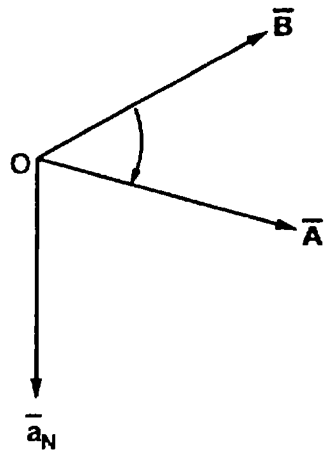
If order of cross product is changed, the magnitude remains same, but direction gets reversed.

3. The cross product is not associative. Thus,

$$\bar{A} \times (\bar{B} \times \bar{C}) \neq (\bar{A} \times \bar{B}) \times \bar{C} \quad \text{... (4)}$$



(a) $\bar{A} \times \bar{B}$



(b) $\bar{B} \times \bar{A}$

Fig. 1.36

4. With respect to addition the cross product is distributive. Thus,

$$\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C} \qquad \text{... (5)}$$

5. If the two vectors are **parallel** to each other i.e. they are in the same direction then $\theta = 0°$ and hence cross product of such two vectors is **zero.**

Thus if cross product of the two vectors is zero then those two vectors are parallel i.e. are in the same direction, assuming none of the two vectors itself is zero.

6. If the cross product of a vector $\bar{A}$ with itself is calculated, it is zero as $\theta = 0°$.

$$\therefore \quad \bar{A} \times \bar{A} = 0 \qquad \text{... (6)}$$

7. **Cross product of unit vectors :** Consider the unit vectors $\bar{a}_x, \bar{a}_y$ and $\bar{a}_z$ which are mutually perpendicular to each other, as shown in the Fig. 1.37.

Then,

$$\bar{a}_x \times \bar{a}_y = |\bar{a}_x| \; |\bar{a}_y| \; \sin(90°) \bar{a}_N$$

In this case, $\bar{a}_N = \bar{a}_z$

and $|\bar{a}_x| = |\bar{a}_y| = \sin(90°) = 1$

$$\therefore \quad \boxed{\begin{aligned} \bar{a}_x \times \bar{a}_y &= \bar{a}_z & \text{... (7)} \\ \bar{a}_y \times \bar{a}_z &= \bar{a}_x & \text{... (8)} \\ \bar{a}_z \times \bar{a}_x &= \bar{a}_y & \text{... (9)} \end{aligned}}$$

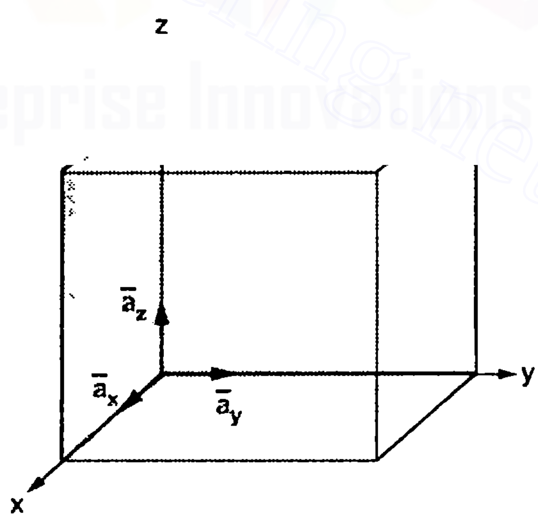
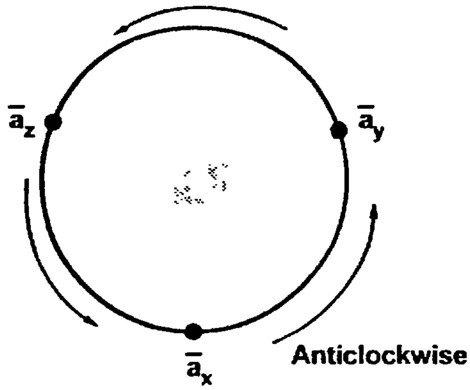


Fig. 1.37

But if the order of unit vectors is reversed, the result is negative of the remaining third unit vector. Thus,

$$\boxed{\bar{a}_y \times \bar{a}_x = -\bar{a}_z, \quad \bar{a}_z \times \bar{a}_y = -\bar{a}_x, \quad \bar{a}_x \times \bar{a}_z = -\bar{a}_y} \qquad \text{... (10)}$$

This can be remembered by a circle indicating cyclic permutations of cross products of unit vectors as shown in the Fig. 1.38.



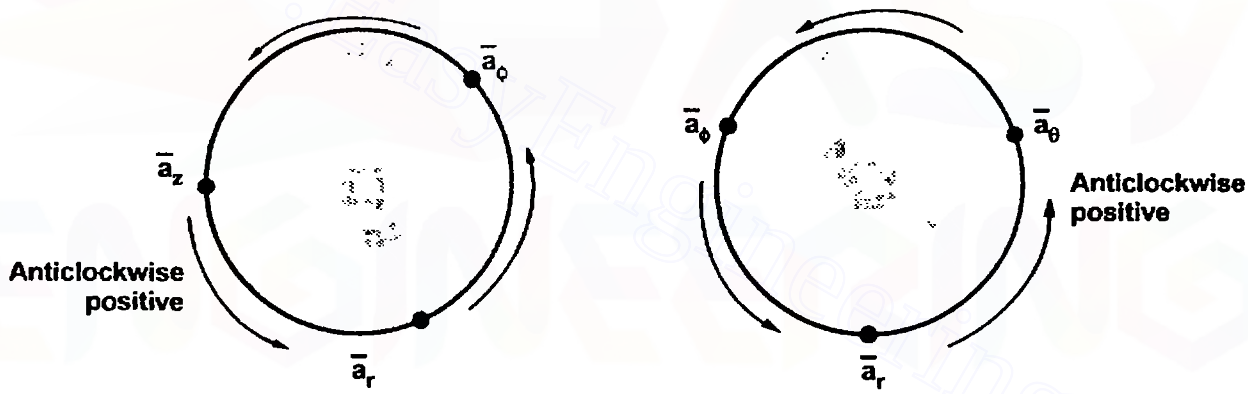
(a) Positive result (b) Negative result

Fig. 1.38

While as cross product of vector with itself is zero we can write,

$$\boxed{\bar{a}_x \times \bar{a}_x = \bar{a}_y \times \bar{a}_y = \bar{a}_z \times \bar{a}_z = 0} \quad \text{... (11)}$$

The result is applicable for the unit vectors in the remaining two co-ordinate systems.



(a) Cylindrical system (b) Spherical system

Fig. 1.39

From the Fig. 1.39 we can write,

$\bar{a}_r \times \bar{a}_\phi = \bar{a}_z,\ \bar{a}_\theta \times \bar{a}_\phi = \bar{a}_r$ and so on.

Key Point: *The clockwise direction gives negative result.*

8. **Cross product in determinant form :** Consider the two vectors in the cartesian system as,

$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$ and $\bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$

Then the cross product of the two vectors is,

$$\begin{aligned} \bar{A} \times \bar{B} = \; & A_x B_x (\bar{a}_x \times \bar{a}_x) + A_x B_y (\bar{a}_x \times \bar{a}_y) + A_x B_z (\bar{a}_x \times \bar{a}_z) \\ & + A_y B_x (\bar{a}_y \times \bar{a}_x) + A_y B_y (\bar{a}_y \times \bar{a}_y) + A_x B_z (\bar{a}_y \times \bar{a}_z) \\ & + A_z B_x (\bar{a}_z \times \bar{a}_x) + A_z B_y (\bar{a}_z \times \bar{a}_y) + A_z B_z (\bar{a}_z \times \bar{a}_z) \end{aligned}$$

$$= 0+A_x B_y \bar{a}_z - A_x B_z \bar{a}_y - A_y B_x \bar{a}_z + 0 + A_y B_z \bar{a}_x$$
$$+A_z B_x \bar{a}_y - A_z B_y \bar{a}_x + 0$$
$$= (A_y B_z - A_z B_y)\bar{a}_x + (A_z B_x - A_x B_z)\bar{a}_y + (A_x B_y - A_y B_x)\bar{a}_z$$

This result can be expressed in determinant form as,

$$\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \text{... (12(a))}$$

If $\bar{A}$ and $\bar{B}$ are in cylindrical system then

$$\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix} \qquad \text{... (12(b))}$$

If $\bar{A}$ and $\bar{B}$ are in spherical system then

$$\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\phi \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix} \qquad \text{... (12(c))}$$

1.11.2 Applications of Cross Product

The different applications of cross product are,

1. The cross product is the replacement to the right hand rule used in electrical engineering to determine the direction of force experienced by current carrying conductor placed in a magnetic field.

Thus if I is the current flowing through conductor while $\bar{L}$ is the vector length considered to indicate the direction of current through the conductor. The uniform magnetic flux density is denoted by vector $\bar{B}$. Then the force experienced by conductor is given by,

$$\bar{F} = I\bar{L}\times\bar{B}$$

2. Another physical quantity which can be represented by cross product is **moment of a force.** The moment of a force (or torque) acting on a rigid body, which can rotate about an axis perpendicular to a plane containing the force is defined to be the magnitude of the force multiplied by the perpendicular distance from the force to the axis. This is shown in the Fig. 1.40.

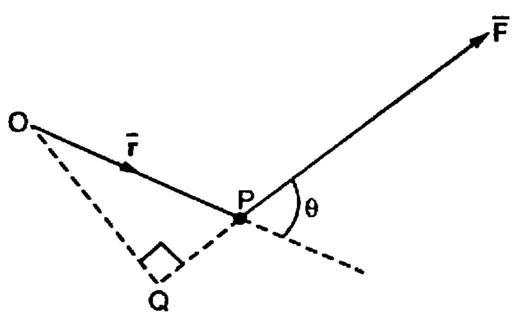


Fig. 1.40

The moment of force $\bar{F}$ about a point O is $\bar{M}$. Its magnitude is $|\bar{F}|\,|\bar{r}|\sin\theta$ where $|\bar{r}|\sin\theta$ is the perpendicular distance of $\bar{F}$ from O i.e. OQ.

$\therefore \bar{M} = \bar{r}\times\bar{F} = |\bar{r}|\,|\bar{F}|\sin\theta\,\bar{a}_N$ where $\bar{a}_N$ is the unit vector indicating direction of $\bar{M}$ which is perpendicular to the plane i.e. paper and coming out of paper according to right hand screw rule.

Example 1.11 : *Given the two coplanar vectors*

$\bar{A} = 3\bar{a}_x + 4\bar{a}_y - 5\bar{a}_z$ *and* $\bar{B} = -6\bar{a}_x + 2\bar{a}_y + 4\bar{a}_z$

Obtain the unit vector normal to the plane containing the vectors $\bar{A}$ *and* $\bar{B}$.

Solution : Note that the unit vector normal to the plane containing the vectors $\bar{A}$ and $\bar{B}$ is the unit vector in the direction of cross product of $\bar{A}$ and $\bar{B}$.

Now
$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 3 & 4 & -5 \\ -6 & 2 & 4 \end{vmatrix}$$

$$= \bar{a}_x \begin{vmatrix} 4 & -5 \\ 2 & 4 \end{vmatrix} - \bar{a}_y \begin{vmatrix} 3 & -5 \\ -6 & 4 \end{vmatrix} + \bar{a}_z \begin{vmatrix} 3 & 4 \\ -6 & 2 \end{vmatrix}$$

$$= 26\bar{a}_x + 18\bar{a}_y + 30\bar{a}_z$$

$$\therefore \quad \bar{a}_N = \frac{\bar{A} \times \bar{B}}{|\bar{A} \times \bar{B}|} = \frac{26\bar{a}_x - 18\bar{a}_y + 30\bar{a}_z}{\sqrt{(26)^2 + (18)^2 + (30)^2}}$$

$$= \mathbf{0.5964}\ \bar{a}_x + \mathbf{0.4129}\ \bar{a}_y + \mathbf{0.6882}\ \bar{a}_z$$

This is the unit vector normal to the plane containing $\bar{A}$ and $\bar{B}$.

1.12 Products of Three Vectors

Let $\bar{A}, \bar{B}$ and $\bar{C}$ are the three given vectors. Then the product of these three vectors is classified in two ways called,

1. Scalar triple product 2. Vector triple product.

1.12.1 Scalar Triple Product

The scalar triple product of the three vectors $\bar{A}, \bar{B}$ and $\bar{C}$ is mathematically defined as,

$$\boxed{\bar{A} \cdot (\bar{B} \times \bar{C}) = \bar{B} \cdot (\bar{C} \times \bar{A}) = \bar{C} \cdot (\bar{A} \times \bar{B})} \quad \ldots (1)$$

Thus if,
$$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$$
$$\bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$$
$$\bar{C} = C_x \bar{a}_x + C_y \bar{a}_y + C_z \bar{a}_z$$

Then the scalar triple product is obtained by the determinant,

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad \ldots (2)$$

The result of this product is a scalar and hence the product is called scalar triple product. The cyclic order a b c is important.

1.12.1.1 Characteristics of Scalar Triple Product

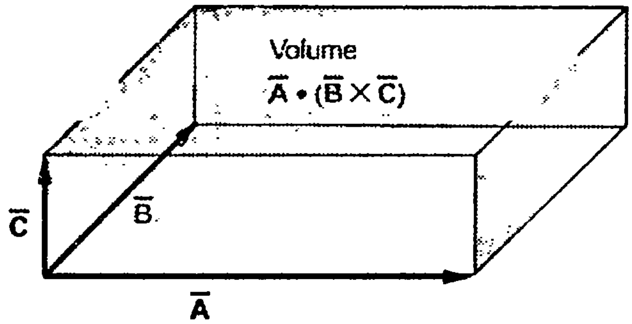


Fig. 1.41

1. The scalar triple product represents the volume of the parallelepiped with edges $\bar{A}, \bar{B}$ and $\bar{C}$, drawn from the same origin, as shown in the Fig. 1.41.

2. The scalar triple product depends only on the cyclic order 'a b c' and not on the position of the $\cdot$ and $\times$ in the product. If the cyclic order is broken by permuting two of the vectors, the sign is reversed.

$$\therefore \quad \bar{A} \cdot (\bar{B} \times \bar{C}) = -\bar{B} \cdot (\bar{A} \times \bar{C})$$

3. If two of the three vectors are equal then the result of the scalar triple product is zero.

$$\therefore \quad \bar{A} \cdot (\bar{A} \times \bar{C}) = 0$$

4. The scalar triple product is distributive.

1.12.2 Vector Triple Product

The vector triple product of the three vectors $\bar{A}, \bar{B}$ and $\bar{C}$ is mathematically defined as,

$$\boxed{\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})} \quad \dots (3)$$

The rule can be remembered as 'bac-cab' rule. The above rule can be easily proved by writing the cartensian components of each term in the equation. The position of the brackets is very important.

1.12.2.1 Characteristics of Vector Triple Product

1. It must be noted that in the vector triple product,

$$(\bar{A} \cdot \bar{B})\,\bar{C} \neq \bar{A}(\bar{B} \cdot \bar{C})$$

$$\text{but} \quad (\bar{A} \cdot \bar{B})\,\bar{C} = \bar{C}(\bar{A} \cdot \bar{B})$$

This is because $\bar{A} \cdot \bar{B}$ is a scalar and multiplication by scalar to a vector is commutative.

2. From the basic definition we can write,

$$\bar{B} \times (\bar{C} \times \bar{A}) = \bar{C}(\bar{B} \cdot \bar{A}) - \bar{A}(\bar{B} \cdot \bar{C}) \quad \dots (4)$$

$$\bar{C} \times (\bar{A} \times \bar{B}) = \bar{A}(\bar{C} \cdot \bar{B}) - \bar{B}(\bar{C} \cdot \bar{A}) \quad \dots (5)$$

But dot product is commutative hence $\bar{C} \cdot \bar{A} = \bar{A} \cdot \bar{C}$ and so on. Hence addition of (3), (4) and (5) is zero.

$$\therefore \bar{A} \times (\bar{B} \times \bar{C}) + \bar{B} \times (\bar{C} \times \bar{A}) + \bar{C} \times (\bar{A} \times \bar{B}) = 0 \quad \dots (6)$$

The result of the vector triple product is a vector.

Example 1.12 : *The three fields are given by,*

$$\bar{A} = 2\bar{a}_x - \bar{a}_z \ , \ \bar{B} = 2\bar{a}_x - \bar{a}_y + 2\bar{a}_z \ , \ \bar{C} = 2\bar{a}_x - 3\bar{a}_y + \bar{a}_z$$

Find the scalar and vector triple product.

Solution : The scalar triple product is,

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 14$$

The vector triple product is,

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$$

$$\bar{A} \cdot \bar{C} = (2)(2) + (0)(-3) + (-1)(1) = 3$$

$$\bar{A} \cdot \bar{B} = (2)(2) + (0)(-1) + (-1)(2) = 2$$

$$\therefore \quad \bar{A} \times (\bar{B} \times \bar{C}) = 3\bar{B} - 2\bar{C} = 3[2\bar{a}_x - \bar{a}_y + 2\bar{a}_z] - 2[2\bar{a}_x - 3\bar{a}_y + \bar{a}_z]$$

$$= 2\,\bar{a}_x + 3\,\bar{a}_y + 4\,\bar{a}_z$$

1.13 Transformation of Vectors

Getting familiar with the dot product and cross product, it is possible now to transform the vectors from one cordinate system to other coordinate system.

1.13.1 Transformation of Vectors from Cartesian to Cylindrical

Consider a vector $\bar{A}$ in cartesian coordinate system as,

$$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z \quad \text{... (1)}$$

While the same vector in cylindrical coordinate system can be represented as,

$$\bar{A} = A_r \bar{a}_r + A_\phi \bar{a}_\phi + A_z \bar{a}_z \quad \text{... (2)}$$

From the dot product it is known that the component of vector in the direction of any unit vector is its dot product with that unit vector. Hence the component of $\bar{A}$ in the direction $\bar{a}_r$ is the dot product of $\bar{A}$ with $\bar{a}_r$. This component is nothing but A_r.

$$\therefore \quad A_r = [\bar{A} \cdot \bar{a}_r] \quad \text{... (3)}$$

$$\therefore \quad A_r = [A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z] \cdot \bar{a}_r$$

$$\therefore \quad A_r = A_x \bar{a}_x \cdot \bar{a}_r + A_y \bar{a}_y \cdot \bar{a}_r + A_z \bar{a}_z \cdot \bar{a}_r \quad \text{... (4)}$$

The magnitudes of all unit vectors is unity hence it is necessary to find angle between the unit vectors to obtain the various dot products.

The Fig. 1.42 (a) shows three dimensional view of various unit vectors.

Consider a xy plane in which the angles between the unit vectors are shown, as in the Fig. 1.42 (b).

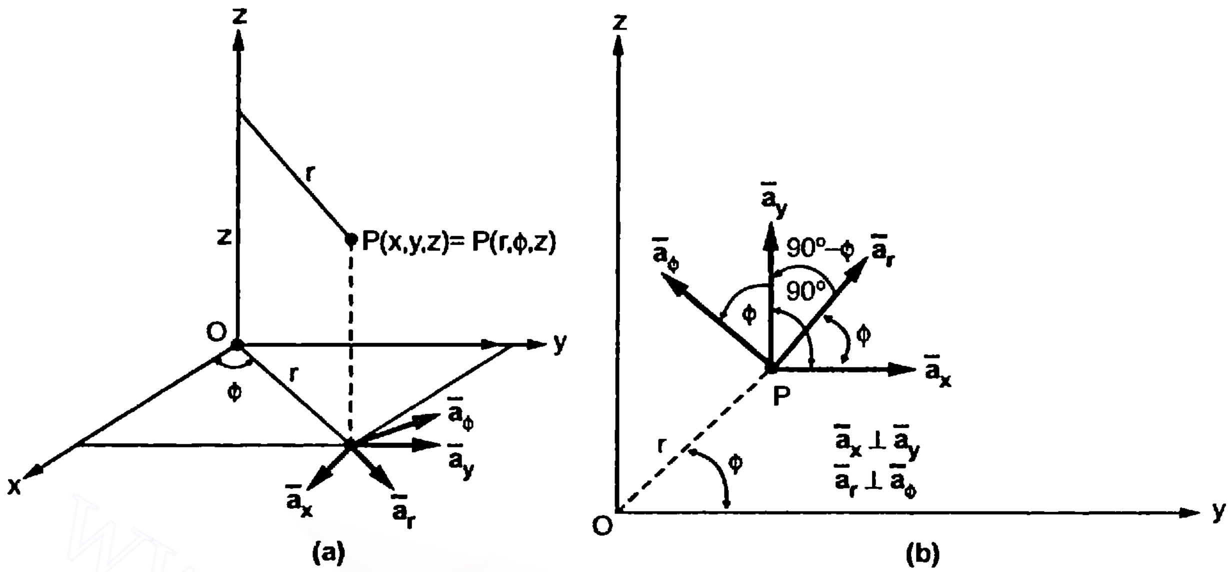


Fig. 1.42 Transformation of vectors

The angle between $\bar{a}_x$ and $\bar{a}_r$ is ϕ.
The angle between $\bar{a}_y$ and $\bar{a}_r$ is $90-\phi$.

The angle between $\bar{a}_x$ and $\bar{a}_\phi$ is $90+\phi$.

The angle between $\bar{a}_y$ and $\bar{a}_\phi$ is ϕ.

$$\therefore \quad \bar{a}_x \cdot \bar{a}_r = (1)(1)\cos(\phi) = \cos\phi \quad \text{... (5)}$$

$$\therefore \quad \bar{a}_x \cdot \bar{a}_\phi = (1)(1)\cos(90+\phi) = -\sin\phi \quad \text{... (6)}$$

$$\therefore \quad \bar{a}_y \cdot \bar{a}_r = (1)(1)\cos(90-\phi) = \sin\phi \quad \text{... (7)}$$

$$\therefore \quad \bar{a}_y \cdot \bar{a}_\phi = (1)(1)\cos(\phi) = \cos\phi \quad \text{... (8)}$$

and $\bar{a}_z \cdot \bar{a}_r = \bar{a}_z \cdot \bar{a}_\phi = 0$ as $\bar{a}_z$ is perpendicular to $\bar{a}_r$ and $\bar{a}_\phi$ ... (9)

and $\bar{a}_z \cdot \bar{a}_z = 1$ ... (10)

Substituting in equation (4) we get,

$$A_r = A_x \cos\phi + A_y \sin\phi \quad \text{... (11)}$$

Similarly finding A_ϕ as $[\bar{A} \cdot \bar{a}_\phi]$ and A_z as $[\bar{A} \cdot \bar{a}_z]$ we get,

$$A_\phi = -A_x \sin\phi + A_y \cos\phi \quad \text{... (12)}$$

and $A_z = A_z$ ... (13)

The results of dot product are summarized in the tabular form as,

Dot operator •	$\bar{a}_r$	$\bar{a}_\phi$	$\bar{a}_z$
$\bar{a}_x$	$\cos\phi$	$-\sin\phi$	0
$\bar{a}_y$	$\sin\phi$	$\cos\phi$	0
$\bar{a}_z$	0	0	1

Table 1.2

The results of transformations can be expressed in the matrix form as,

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

1.13.2 Transformation of Vectors from Cylindrical to Cartesian

Now it is necessary to find the transformation from cylindrical to cartesian hence assume $\overline{A}$ is known in cylindrical system. Thus component of $\overline{A}$ in $\bar{a}_x$ direction is given by,

$$\overline{\mathbf{A}}_x = [\overline{\mathbf{A}} \bullet \bar{a}_x] = [A_r \bar{a}_r + A_\phi \bar{a}_\phi + A_z \bar{a}_z] \bullet \bar{a}_x$$

$$\therefore \quad A_x = A_r \bar{a}_r \bullet \bar{a}_x + A_\phi \bar{a}_\phi \bullet \bar{a}_x + A_z \bar{a}_z \bullet \bar{a}_x \quad \text{... (14)}$$

As dot product is commutative $\bar{a}_r \bullet \bar{a}_x = \bar{a}_x \bullet \bar{a}_r = \cos\phi$ and so on. Hence referring Table 1.2 we can write the results directly as,

$$A_x = A_r \cos\phi - A_\phi \sin\phi \quad \text{... (15)}$$

$$A_y = A_r \sin\phi + A_\phi \cos\phi \quad \text{... (16)}$$

$$A_z = A_z \quad \text{... (17)}$$

The result can be summarized in the matrix form as,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

Example 1.13 : *Transform the vector field $\overline{W} = 10\bar{a}_x - 8\bar{a}_y + 6\bar{a}_z$ to cylindrical co-ordinate system, at point $P(10, -8, 6)$.*

Solution : From the given field $\overline{W}$,

$W_x = 10$, $W_y = -8$ and $W_z = 6$

Now $\quad W_r = \overline{W} \bullet \bar{a}_r = [10\bar{a}_x - 8\bar{a}_y + 6\bar{a}_z] \bullet \bar{a}_r$

$$= 10\bar{a}_x \bullet \bar{a}_r - 8\bar{a}_y \bullet \bar{a}_r + 6\bar{a}_z \bullet \bar{a}_r$$

$$= 10(\cos\phi) - 8(\sin\phi) + 6(0) \quad \text{... Refer Table 1.2}$$

For point P, $x = 10$ and $y = -8$

$$\therefore \quad \phi = \tan^{-1}\frac{y}{x} \quad \text{... Relation between cartesian and cylindrical}$$

$$= \tan^{-1}\left[\frac{-8}{10}\right] = -38.6598^\circ$$

As y is negative and x is positive, ϕ is in fourth quadrant. Hence ϕ calculated is correct.

$\therefore \quad \cos\phi = 0.7808 \quad \text{and} \quad \sin\phi = -0.6246$

$\therefore \quad W_r = 10\times(0.7808)-8\times(-0.6246)=12.804$

Now $\quad W_\phi = \overline{W}\cdot\bar{a}_\phi = 10\bar{a}_x\cdot\bar{a}_\phi - 8\bar{a}_y\cdot\bar{a}_\phi + 6\bar{a}_z\cdot\bar{a}_\phi$

$= 10(-\sin\phi)-8\cos\phi+0 = 0$

And $\quad W_z = \overline{W}\cdot\bar{a}_z = 10\bar{a}_x\cdot\bar{a}_z - 8\bar{a}_y\cdot\bar{a}_z + 6\bar{a}_z\cdot\bar{a}_z$

$= 10\times0-8\times0+6\times1=6$

$\therefore \quad \overline{W} = \mathbf{12.804\,\bar{a}_r + 6\bar{a}_z}$ in cylindrical system.

Example 1.14 : *Give the cartesian co-ordinates of the vector field $\bar{H}=20\bar{a}_r-10\bar{a}_\phi+3\bar{a}_z$, at point $P(x=5, y=2, z=-1)$.*

Solution : The given vector is in cylindrical system.

$\therefore \quad H_x = \overline{H}\cdot\bar{a}_x = 20\bar{a}_r\cdot\bar{a}_x - 10\bar{a}_\phi\cdot\bar{a}_x + 3\bar{a}_z\cdot\bar{a}_x$

$= 20\cos\phi-10(-\sin\phi)+0$ ... Refer Table 1.2

At point P, $x = 5$, $y = 2$ and $z = -1$

Now $\quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{5} = 21.8014°$

$\therefore \quad \cos\phi = 0.9284 \quad \text{and} \quad \sin\phi = 0.3714$

$\therefore \quad H_x = 20\times(0.9284)+10\times0.3714=22.282$

Then $\quad H_y = \overline{H}\cdot\bar{a}_y = 20\bar{a}_r\cdot\bar{a}_y - 10\bar{a}_\phi\cdot\bar{a}_y + 3\bar{a}_z\cdot\bar{a}_y$

$= 20\sin\phi-10\cos\phi+0$

$= 20\times(0.3714)-10\times(0.9284)=-1.856$

And $\quad H_z = \overline{H}\cdot\bar{a}_z = 20\bar{a}_r\cdot\bar{a}_z - 10\bar{a}_\phi\cdot\bar{a}_z + 3\bar{a}_z\cdot\bar{a}_z$

$= 20\times0-10\times0+3\times1=3$

$\therefore \quad \overline{H} = \mathbf{22.282\,\bar{a}_x - 1.856\,\bar{a}_y + 3\,\bar{a}_z}$ in cartesian system.

1.13.3 Transformation of Vectors from Cartesian to Spherical

Let the vector $\overline{A}$ expressed in the cartesian system as,

$$\overline{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$$

It is required to transform it into spherical system. The component of $\overline{A}$ in $\bar{a}_r$ direction is given by,

$$A_r = \overline{A}\cdot\bar{a}_r = [A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z]\cdot\bar{a}_r$$

$$= A_x\bar{a}_x\cdot\bar{a}_r + A_y\bar{a}_y\cdot\bar{a}_r + A_z\bar{a}_z\cdot\bar{a}_r \quad \text{... (18)}$$

Note : Though the radius representation r used in cylindrical and spherical systems is same, the directions $\bar{a}_r$ in both the systems are different. Infact many times r is represented as ρ in cylindrical system. But ρ is used to represent other quantity in this book hence r is used in cylindrical system. **Hence $\bar{a}_x \cdot \bar{a}_r$ will be different when $\bar{a}_r$ is of spherical system than the $\bar{a}_r$ of cylindrical system** and so on.

While $$A_\theta = \bar{A} \cdot \bar{a}_\theta = [A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z] \cdot \bar{a}_\theta$$

$$= A_x \bar{a}_x \cdot \bar{a}_\theta + A_y \bar{a}_y \cdot \bar{a}_\theta + A_z \bar{a}_z \cdot \bar{a}_\theta \quad \text{... (19)}$$

And $$A_\phi = \bar{A} \cdot \bar{a}_\phi = [A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z] \cdot \bar{a}_\phi$$

$$= A_x \bar{a}_x \cdot \bar{a}_\phi + A_y \bar{a}_y \cdot \bar{a}_\phi + A_z \bar{a}_z \cdot \bar{a}_\phi \quad \text{... (20)}$$

The dot products can be obtained by first taking the projection of spherical unit vector on the xy plane and then taking the projection onto the desired axis. Thus for $\bar{a}_x \cdot \bar{a}_r$, project $\bar{a}_r$ on the xy plane which is $\sin\theta$ and then project on the x axis which is $\sin\theta\cos\phi$

$$\therefore \quad \bar{a}_x \cdot \bar{a}_r = \bar{a}_r \cdot \bar{a}_x = \sin\theta\cos\phi$$

In the similar fashion the other dot products can be obtained. The results of the dot products are summarized in the Table 1.3.

Dot operator •	$\bar{a}_r$	$\bar{a}_\theta$	$\bar{a}_\phi$
$\bar{a}_x$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\bar{a}_y$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\bar{a}_z$	$\cos\theta$	$-\sin\theta$	0

Table 1.3

Using the results of Table 1.3, the results of vector transformation from cartesian to spherical can be summarized in the matrix form as,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

1.13.4 Transformation of Vectors from Spherical to Cartesian

To find the reverse transformation, assume that the $\bar{A}$ is known in spherical system as,

$$\bar{A} = A_r \bar{a}_r + A_\theta \bar{a}_\theta + A_\phi \bar{a}_\phi$$

Hence component of $\bar{A}$ in $\bar{a}_x, \bar{a}_y$ and $\bar{a}_z$ are given by $\bar{A} \cdot \bar{a}_x$, $\bar{A} \cdot \bar{a}_y$ and $\bar{A} \cdot \bar{a}_z$ respectively.

Thus we get the results as,

$$A_x = A_r \bar{a}_r \cdot \bar{a}_x + A_\theta \bar{a}_\theta \cdot \bar{a}_x + A_\phi \bar{a}_\phi \cdot \bar{a}_x \quad \text{... (21)}$$

$$A_y = A_r \bar{a}_r \cdot \bar{a}_y + A_\theta \bar{a}_\theta \cdot \bar{a}_y + A_\phi \bar{a}_\phi \cdot \bar{a}_y \quad \text{... (22)}$$

$$A_z = A_r \bar{a}_r \cdot \bar{a}_z + A_\theta \bar{a}_\theta \cdot \bar{a}_z + A_\phi \bar{a}_\phi \cdot \bar{a}_z \quad \text{... (23)}$$

Using the Table 1.3, the results can be expressed in the matrix form as,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

1.13.5 Distances in all Co-ordinate Systems

Consider two points A and B with the position vectors as,

$\overline{A} = x_1 \bar{a}_x + y_1 \bar{a}_y + z_1 \bar{a}_z$ and $\overline{B} = x_2 \bar{a}_x + y_2 \bar{a}_y + z_2 \bar{a}_z$

then the distance d between the two points in all the three co-ordinate systems are given by,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad \text{... Cartesian}$$

$$d = \sqrt{r_2^2 + r_1^2 - 2 r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2} \quad \text{... Cylindrical}$$

$$d = \sqrt{r_2^2 + r_1^2 - 2 r_1 r_2 \cos\theta_2 \cos\theta_1 - 2 r_1 r_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)} \quad \text{... Spherical}$$

These results may be used directly in electromagnetics wherever required.

Example 1.15 : *Obtain the spherical coordinates of 10 $\bar{a}_x$ at the point $P(x = -3, y = 2, z = 4)$.*

Solution : Given vector is in cartesian system say $\overline{F} = 10\bar{a}_x$.

Then $F_r = \overline{F} \cdot \bar{a}_r = 10\bar{a}_x \cdot \bar{a}_r$

$= 10 \sin\theta \cos\phi$ ... Refer Table 1.3

At point P, $x = -3$, $y = 2$, $z = 4$

Using the relationship between cartesian and spherical,

$x = r \sin\theta \cos\phi \quad y = r \sin\theta \sin\phi \quad z = r \cos\theta$

$$\therefore \quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{-3} = -33.69°$$

But x is negative and y is positive hence ϕ must be between +90° and +180°. So add 180° to the ϕ to get correct ϕ.

$$\therefore \quad \phi = -33.69° + 180° = +146.31°$$

$$\therefore \quad \cos\phi = -0.832 \text{ and } \sin\phi = 0.5547$$

And
$$\theta = \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \cos^{-1}\frac{4}{\sqrt{(-3)^2 + (2)^2 + (4)^2}} = 42.0311°$$

$\therefore \quad \cos\theta = 0.7428$ and $\sin\theta = 0.6695$

$\therefore \quad F_r = 10 \times 0.6695 \times (-0.832) = -5.5702$

$$F_\theta = \bar{F} \cdot \bar{a}_\theta = 10\,\bar{a}_x \cdot \bar{a}_\theta = 10\cos\theta\cos\phi$$

$$= 10 \times 0.7428 \times (-0.832) = -6.18$$

$$F_\phi = \bar{F} \cdot \bar{a}_\phi = 10\,\bar{a}_x \cdot \bar{a}_\phi = 10(-\sin\phi)$$

$$= 10 \times (-0.5547) = -5.547$$

$\therefore \quad \bar{F} = -5.5702\,\bar{a}_r - 6.18\,\bar{a}_\theta - 5.547\,\bar{a}_\phi$ in spherical system.

Example 1.16 : *Express $\bar{B} = r^2\,\bar{a}_r + \sin\theta\,\bar{a}_\phi$ in the cartesian co-ordinates. Hence obtain $\bar{B}$ at $P(1, 2, 3)$.*

Solution : Given $\bar{B}$ is in spherical system as there is $\sin\theta$ in it and its cartesian co-ordinates are to be obtained, Referring Table 1.3,

$\therefore \quad B_x = \bar{B} \cdot \bar{a}_x = r^2\,\bar{a}_r \cdot \bar{a}_x + \sin\theta\,\bar{a}_\phi \cdot \bar{a}_x$

$$= r^2 \sin\theta\cos\phi + \sin\theta(-\sin\phi) \quad \text{... (1)}$$

Then $\quad B_y = \bar{B} \cdot \bar{a}_y = r^2\,\bar{a}_r \cdot \bar{a}_y + \sin\theta\,\bar{a}_\phi \cdot \bar{a}_y$

$$= r^2 \sin\theta\sin\phi + \sin\theta\cos\phi \quad \text{... (2)}$$

And $\quad B_z = \bar{B} \cdot \bar{a}_z = r^2\,\bar{a}_r \cdot \bar{a}_z + \sin\theta\,\bar{a}_\phi \cdot \bar{a}_z$

$$= r^2 \cos\theta + \sin\theta(0) = r^2\cos\theta \quad \text{... (3)}$$

Now it is known that,

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \tan^{-1}\frac{y}{x} \quad \text{and} \quad \theta = \cos^{-1}\frac{z}{r}$$

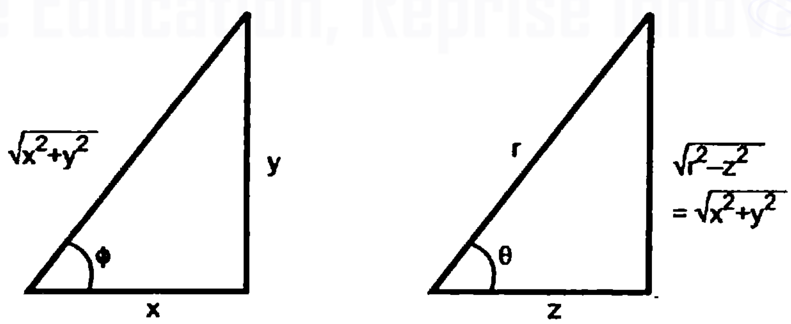


Fig. 1.43

From Fig. 1.43,

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{r} \quad \text{and} \quad \cos\theta = \frac{z}{r}$$

Using in equation (1), (2) and (3) we get,

$$B_x = r^2 \frac{\sqrt{x^2+y^2}}{r} \frac{x}{\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}}{r}\left(-\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= (r\,x) - \frac{y}{r} = \sqrt{x^2+y^2+z^2}\,(x) - \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$B_y = r^2 \frac{\sqrt{x^2+y^2}}{r} \frac{y}{\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}}{r} \frac{x}{\sqrt{x^2+y^2}}$$

$$= (r\,y) + \frac{x}{r} = \sqrt{x^2+y^2+z^2}\,(y) + \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$B_z = r^2 \times \frac{z}{r} = (r\,z) = \sqrt{x^2+y^2+z^2}\,(z)$$

$$\therefore \quad \bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$$

Thus $\bar{B}$ at P (1, 2, 3) is, $\bar{B} = 3.207\,\bar{a}_x + 7.7504\,\bar{a}_y + 11.2248\,\bar{a}_z$

1.13.6 Transformation of Vectors from Spherical to Cylindrical

Let the vector $\bar{A}$ is known in the spherical co-ordinates.

$$\therefore \quad \bar{A} = A_r\,\bar{a}_r + A_\theta\,\bar{a}_\theta + A_\phi\,\bar{a}_\phi$$

The components of $\bar{A}$ in cylindrical system are given by,

$$A_\rho = A_r\,\bar{a}_r \cdot \bar{a}_\rho + A_\theta\,\bar{a}_\theta \cdot \bar{a}_\rho + A_\phi\,\bar{a}_\phi \cdot \bar{a}_\rho$$

$$A_\phi = A_r\,\bar{a}_r \cdot \bar{a}_\phi + A_\theta\,\bar{a}_\theta \cdot \bar{a}_\phi + A_\phi\,\bar{a}_\phi \cdot \bar{a}_\phi$$

$$A_z = A_r\,\bar{a}_r \cdot \bar{a}_z + A_\theta\,\bar{a}_\theta \cdot \bar{a}_z + A_\phi\,\bar{a}_\phi \cdot \bar{a}_z$$

Now $\quad \bar{a}_r \cdot \bar{a}_\rho = \sin\theta, \quad \bar{a}_\theta \cdot \bar{a}_\rho = \cos\theta, \quad \bar{a}_\phi \cdot \bar{a}_\rho = 0$

$\bar{a}_r \cdot \bar{a}_\phi = 0, \quad \bar{a}_\theta \cdot \bar{a}_\phi = 0, \quad \bar{a}_\phi \cdot \bar{a}_\phi = 1$

$\bar{a}_r \cdot \bar{a}_z = \cos\theta, \quad \bar{a}_\theta \cdot \bar{a}_z = -\sin\theta, \quad \bar{a}_\phi \cdot \bar{a}_z = 0$

$$\therefore \quad \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \qquad \ldots (24)$$

1.13.7 Transformation of Vectors from Cylindrical to Spherical

Let the vector $\bar{A}$ is known in the cylindrical co-ordinates.

$$\therefore \quad \bar{A} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$$

The components of $\bar{A}$ in the spherical system are given by,

$$A_r = A_\rho \bar{a}_\rho \cdot \bar{a}_r + A_\phi \bar{a}_\phi \cdot \bar{a}_r + A_z \bar{a}_z \cdot \bar{a}_r$$

$$A_\theta = A_\rho \bar{a}_\rho \cdot \bar{a}_\theta + A_\phi \bar{a}_\phi \cdot \bar{a}_\theta + A_z \bar{a}_z \cdot \bar{a}_\theta$$

$$A_\phi = A_\rho \bar{a}_\rho \cdot \bar{a}_\phi + A_\phi \bar{a}_\phi \cdot \bar{a}_\phi + A_z \bar{a}_z \cdot \bar{a}_\phi$$

$$\therefore \quad \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \quad \text{... (25)}$$

Key Point: To avoid the confusion between $\bar{a}_r$ in cylindrical and spherical in the cylindrical system $\bar{a}_\rho$ is used.

Using equations (24) and (25), any vector can be converted from cylindrical to spherical or spherical to cylindrical system.

Example 1.17 : *Express vector $\bar{B}$ in cartesian and cylindrical systems.*

Given, $\bar{B} = \frac{10}{r}\bar{a}_r + r\cos\theta\,\bar{a}_\theta + \bar{a}_\phi$

Then find $\bar{B}$ at (– 3, 4, 0) and (5, π/2, – 2)

Solution :

$$\bar{B} = \frac{10}{r}\bar{a}_r + r\cos\theta\,\bar{a}_\theta + \bar{a}_\phi$$

$$\therefore \quad B_r = \frac{10}{r}, \quad B_\theta = r\cos\theta, \quad B_\phi = 1 \quad \text{... in spherical}$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \frac{10}{r} \\ r\cos\theta \\ 1 \end{bmatrix}$$

$$\therefore \quad B_x = \frac{10}{r}\sin\theta\cos\phi + r\cos^2\theta\cos\phi - \sin\phi \quad \text{... (1)}$$

$$\therefore \quad B_y = \frac{10}{r}\sin\theta\sin\phi + r\cos^2\theta\sin\phi + \cos\phi \quad \text{... (2)}$$

$$\therefore \quad B_z = \frac{10}{r}\cos\theta - r\sin\theta\cos\theta \quad \text{... (3)}$$

But $$r = \sqrt{x^2+y^2+z^2}, \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}, \quad \tan\phi = \frac{y}{x}$$

$\therefore \quad \sin\theta = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$

Using equations (1), (2) and (3), $\bar{B}$ in cartesian system is :

$$\bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z \quad \text{where,}$$

$$B_x = \frac{10\,x}{x^2+y^2+z^2} + \frac{xz^2}{\sqrt{(x^2+y^2)\,(x^2+y^2+z^2)}} - \frac{y}{\sqrt{x^2+y^2}} \qquad \ldots (4)$$

$$B_y = \frac{10\,y}{x^2+y^2+z^2} + \frac{yz^2}{\sqrt{(x^2+y^2)\,(x^2+y^2+z^2)}} + \frac{x}{\sqrt{x^2+y^2}} \qquad \ldots (5)$$

$$B_z = \frac{10\,z}{x^2+y^2+z^2} - \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \qquad \ldots (6)$$

At (– 3, 4, 0), $x = -3, \; y = 4, \; z = 0$

$\therefore \quad \bar{B} = -2\,\bar{a}_x + \bar{a}_y$ ... In cartesian

For transforming spherical to cylindrical use,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$\therefore \quad B_\rho = \sin\theta\, B_r + \cos\theta\, B_\theta = \frac{10\sin\theta}{r} + r\cos^2\theta$$

$$B_\phi = B_\phi = 1$$

$$B_z = \cos\theta\; B_r - \sin\theta\, B_\theta = \frac{10\cos\theta}{r} - r\sin\theta\cos\theta$$

Now $\quad \rho = r\sin\theta, \; z = r\cos\theta, \; \phi = \phi, \; r = \sqrt{\rho^2+z^2}, \; \theta = \tan^{-1}\dfrac{\rho}{z}$

And $\quad \tan\theta = \dfrac{\rho}{z}$ hence $\sin\theta = \dfrac{\rho}{\sqrt{\rho^2+z^2}}, \; \cos\theta = \dfrac{z}{\sqrt{\rho^2+z^2}}$

$$\therefore \quad \bar{B} = B_\rho\,\bar{a}_\rho + B_\phi\,\bar{a}_\phi + B_z\,\bar{a}_z \quad \text{where,}$$

$$B_\rho = \frac{10\rho}{\rho^2+z^2} + \frac{z^2}{\sqrt{\rho^2+z^2}}, \quad B_\phi = 1, \quad B_z = \frac{10\,z}{\rho^2+z^2} - \frac{\rho z}{\sqrt{\rho^2+z^2}}$$

At given point $\left(5, \dfrac{\pi}{2}, -2\right)$, $\rho = 5, \; \phi = \dfrac{\pi}{2}$ and $z = -2$

$$\therefore \quad B_\rho = \frac{10\times 5}{5^2+(-2)^2}+\frac{(-2)^2}{\sqrt{5^2+(-2)^2}} = 2.467, \quad B_\phi = 1$$

$$B_z = \frac{10\times(-2)}{5^2+(-2)^2}-\frac{5\times(-2)}{\sqrt{5^2+(-2)^2}} = 1.167$$

$$\therefore \quad \bar{B} = 2.467\,\bar{a}_\rho + \bar{a}_\phi + 1.167\,\bar{a}_z \qquad \text{... In cylindrical}$$

1.14 Types of Integral Related to Electromagnetic Theory

In electromagnetic theory a charge can exist in point form, line form, surface form or volume form. Hence while dealing with the analysis of such charge distributions, the various types of integrals are required. These types are,

1. Line integral
2. Surface integral
3. Volume integral.

1.14.1 Line Integral

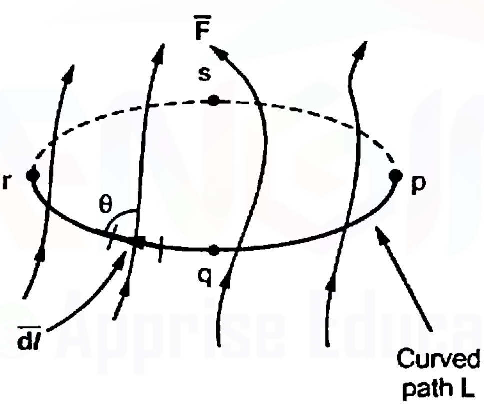


Fig. 1.44 Line integral

A line can exist as a straight line or it can be a distance travelled along a curve. Thus in general, from mathematical point of view, **a line is a curved path in a space.**

Consider a vector field $\bar{F}$ shown in the Fig. 1.44. The curved path shown in the field is p - r. This is called a path of integration and corresponding integral can be defined as,

$$\boxed{\int_L \bar{F}\cdot\overline{dl} = \int_p^r |F|\, dl \cos\theta} \qquad ...(1)$$

... Using definition of dot product

where $\quad dl$ = Elementary length

This is called **line integral** of $\bar{F}$ around the curved path L. It represents an integral of the tangential component of $\bar{F}$ along the path L.

The curved path can be of two types,

i) Open path as p - r shown in the Fig. 1.44.

ii) Closed path as p - q - r - s - p shown in the Fig. 1.44.

The closed path is also called a **contour.** The corresponding integral is called **contour integral, closed integral** or **circular integral** and mathematically defined as,

$$\boxed{\oint_L \bar{F} \bullet \overline{dl} = \text{Circular integral}} \quad ...(2)$$

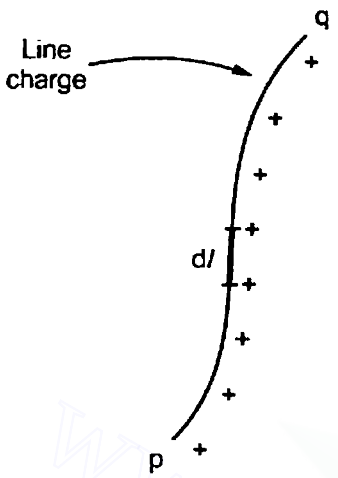


Fig. 1.45 Line charge

This integral represents **circulation of the vector field $\bar{F}$ around the closed path L.**

If there exists a charge along a line as shown in the Fig. 1.45, then the total charge is obtained by calculating a line integral.

$$\therefore \quad \boxed{Q = \int_L \rho_L \, dl} \quad ...(3)$$

where ρ_l = Line charge density i.e. charge per unit length (C/m)

Key Point: *In evaluating line integration, the $\bar{dl}$ direction is assumed to be always positive and limits of integration decide the sign of the integral.*

1.14.2 Surface Integral

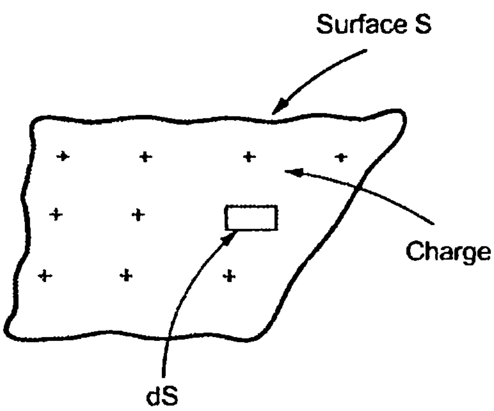
In electromagnetic theory a charge may exist in a distributed form. It may be spreaded over a surface as shown in the Fig. 1.46(a). Similarly a flux ϕ may pass through a surface as shown in the Fig. 1.46(b). While doing analysis of such cases an integral is required called **surface integral,** to be carried out over a surface related to a vector field.

For a charge distribution shown in the Fig. 1.46(a), we can write for the total charge existing on the surface as,

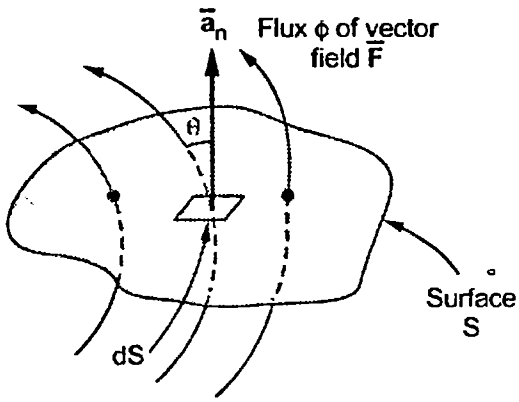
$$\boxed{Q = \int_S \rho_S \, dS} \quad ...(4)$$

where ρ_S = Surface charge density in C/m^2

dS = Elementary surface



(a) Surface charge



b) Flux crossing a surface

Fig. 1.46

Similarly for the Fig. 1.46(b), the total flux crossing the surface S can be expressed as,

$$\boxed{\phi = \int_S \bar{F} \cdot d\bar{S} = \int_S |\bar{F}|\, dS \cos\theta = \int_S \bar{F} \cdot \bar{a}_n\, dS} \quad \text{... (5)}$$

where $\bar{a}_n$ = Unit vector normal to the surface S

Both the equations (4) and (5) represent the **surface integrals** and mathematically it becomes a **double integration** while solving the problems.

If the surface is closed, then it defines a volume and corresponding surface integral is given by,

$$\boxed{\phi = \oint_S \bar{F} \cdot d\bar{S}} \quad \text{...(6)}$$

This represents the net outward flux of vector field $\bar{F}$ from surface S.

Key Point: *1. The closed surface defines a volume.*

2. The surface integral involves the double integration procedure mathematically.

1.14.3 Volume Integral

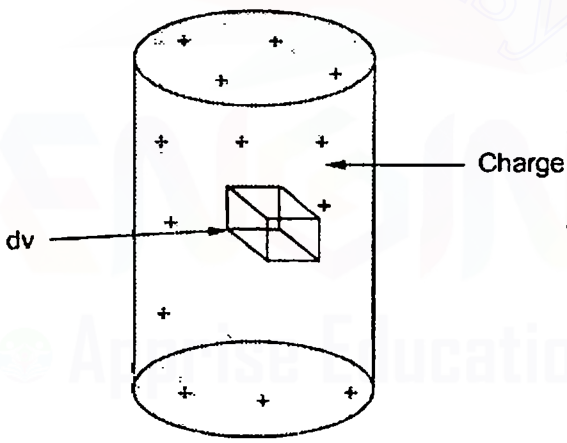


Fig. 1.47 Volume charge

If the charge distribution exists in a three dimensional volume form as shown in the Fig.1.47 then a **volume integral** is required to calculate the total charge.

Thus if ρ_v is the volume charge density over a volume v then the volume integral is defined as,

$$\boxed{Q = \int_v \rho_v \, dv} \quad \text{...(7)}$$

where dv = Elementary volume

Example 1.18 : *Calculate the circulation of vector field,*

$$\bar{F} = r^2 \cos\phi\, \bar{a}_r + z \sin\phi\, \bar{a}_z$$

around the path L defined by $0 \le r \le 3$, $0 \le \phi \le 45°$ and $z = 0$ as shown in the Fig. 1.48(a).

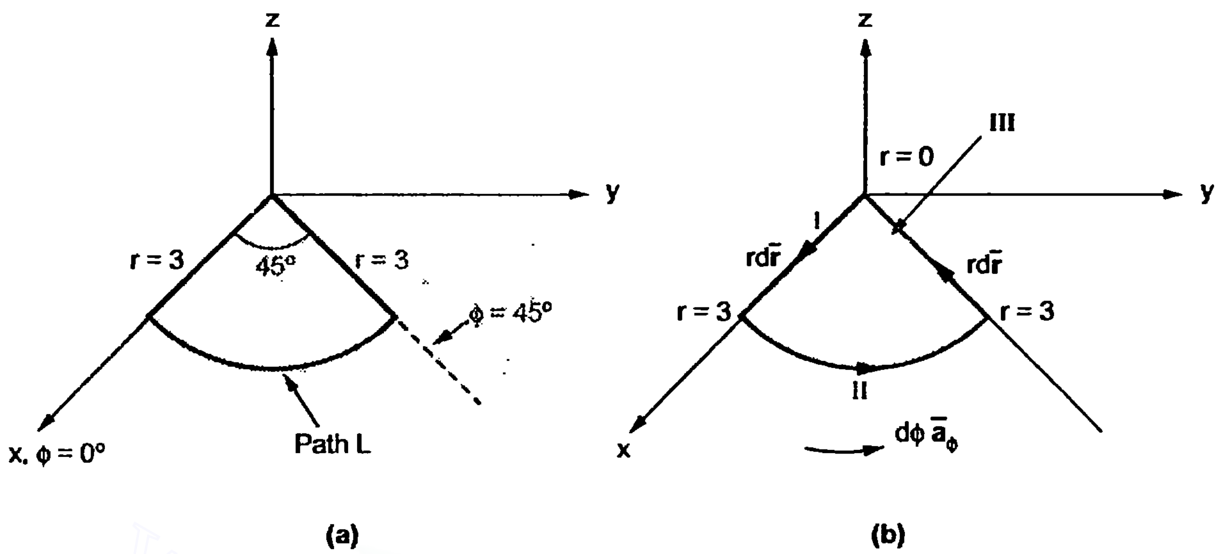


Fig. 1.48

Solution : Divide the given path L into three sections.

Section I : r varies from 0 to 3, $\phi = 0°$ and z = 0

$$\therefore \quad d\bar{l} = dr\,\bar{a}_r \qquad \text{... Along radial direction}$$

$$\therefore \quad \int_I \bar{F}\cdot d\bar{l} = \int_{r=0}^{3} (r^2 \cos\phi\ \bar{a}_r + z\sin\phi\,\bar{a}_z)\cdot dr\ \bar{a}_r$$

$$= \int_{r=0}^{3} r^2 \cos\phi\ dr \qquad \text{... } \bar{a}_r\cdot\bar{a}_r = 1,\ \ \bar{a}_z\cdot\bar{a}_r = 0$$

$$= \left[\frac{r^3}{3}\right]_0^3 \cos 0° = \left[\frac{27}{3}\right][1] = 9$$

Section II : r is constant 3, ϕ varies from 0 to 45°, z = 0

$$\therefore \quad d\bar{l} = d\phi\,\bar{a}_\phi \qquad \text{... Along } \phi \text{ direction}$$

$$\therefore \quad \int_{II} \bar{F}\cdot d\bar{l} = \int_{\phi=0}^{45°} (r^2 \cos\phi\ \bar{a}_r + z\sin\phi\,\bar{a}_z)\cdot d\phi\,\bar{a}_\phi$$

$$= 0 \qquad \text{... } \bar{a}_r\cdot\bar{a}_\phi = \bar{a}_z\cdot\bar{a}_\phi = 0$$

Section III : r varies from 3 to 0, $\phi = 45°$ and z = 0

$$d\bar{l} = dr\,\bar{a}_r$$

Note that $d\bar{l}$ is always positive, limits of integration from r = 3 to 0 taking care of direction.

$$\therefore \quad \int_{III} \bar{F}\cdot d\bar{l} = \int_{r=3}^{0} (r^2 \cos\phi\ \bar{a}_r + z\sin\phi\,\bar{a}_z)\cdot dr\ \bar{a}_r$$

$$= \int_{r=3}^{0} r^2 \cos\phi \; dr \qquad \ldots \bar{a}_r \cdot \bar{a}_r = 1,\; \bar{a}_z \cdot \bar{a}_r = 0$$

$$= \cos 45° \left[\frac{r^3}{3}\right]_3^0 = 0.7071 \left[\frac{-27}{3}\right] = -6.3639$$

$$\therefore \quad \oint_L \bar{F} \cdot d\bar{l} = 9 + 0 - 6.3639 = 2.636$$

1.15 Divergence

It is seen that $\oint_S \bar{F} \cdot d\bar{S}$ gives the flux flowing across the surface S. Then mathematically **divergence** is defined as the net outward flow of the flux per unit volume over a closed incremental surface. It is denoted as div $\bar{F}$ and given by,

$$\boxed{\text{div } \bar{F} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{F} \cdot d\bar{S}}{\Delta v} = \text{Divergence of } \bar{F}} \qquad \ldots(1)$$

where Δv = Differential volume element

Key Point: *Divergence of vector field $\bar{F}$ at a point P is the outward flux per unit volume as the volume shrinks about point P i.e. lim $\Delta v \to 0$ representing differential volume element at point P.*

Symbolically it is denoted as,

$$\nabla \cdot \bar{F} = \text{Divergence of } \bar{F} \qquad \ldots(2)$$

where ∇ = Vector operator $= \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z$

But $\bar{F} = F_x \bar{a}_x + F_y \bar{a}_y + F_z \bar{a}_z$

$$\therefore \quad \boxed{\nabla \cdot \bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{div } \bar{F}} \qquad \ldots(3)$$

This is divergence of $\bar{F}$ in **Cartesian system.**

Similarly divergence in other co-ordinate systems are,

$$\nabla \cdot \bar{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \quad \text{Cylindrical} \qquad \ldots(4)$$

$$\nabla \cdot \bar{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \quad \text{Spherical} \qquad \ldots(5)$$

Physically divergence at a point indicate how much that vector field diverges from that point.

Consider a solenoid i.e. electromagnet obtained by winding a coil around the core. When current passes through it, flux is produced around it. Such a flux completes a closed path through the solenoid hence solenoidal field does not diverge. Thus mathematically, **the vector field having its divergence zero is called solenoidal field.**

$$\therefore \qquad \boxed{\nabla \cdot \bar{A} = 0 \quad \text{for } \bar{A} \text{ to be solenoidal}}$$

Key Point: *The concept and physical significance of divergence is elaborated in great detail in the section 3.10 of chapter 3.*

1.16 Divergence Theorem

It is known that,

$$\nabla \cdot \bar{F} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{F} \cdot d\bar{S}}{\Delta v} \qquad \text{... Definition of divergence}$$

From this definition it can be written that,

$$\boxed{\oint_S \bar{F} \cdot d\bar{S} = \int_v (\nabla \cdot \bar{F})\, dv} \qquad \text{...(1)}$$

This eqution (1) is known as **divergence theorem** or **Gauss-Ostrogradsky theorem.**

The Divergence theorem **states** that,

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface.

The theorem can be applied to any vector field but partial derivatives of that vector field must exist. The divergence theorem as applied to the flux density. Both sides of the divergence theorem give the net charge enclosed by the closed surface i.e. net flux crossing the closed surface.

Key Point: *The divergence theorem converts the surface integral into a volume integral, provided that the closed surface encloses certain volume.*

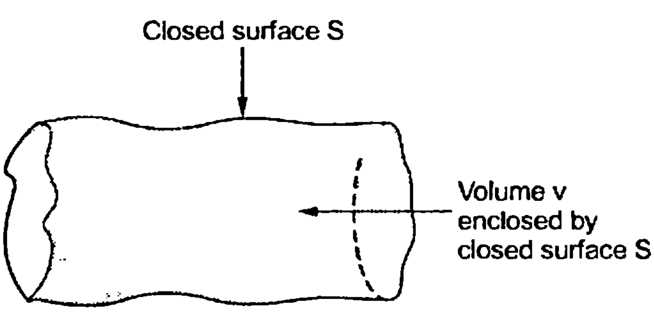


Fig. 1.49

This is advantageous in electromagnetic theory as volume integrals are more easy to evaluate than the surface integrals.

The Fig. 1.49 shows how closed surface S encloses a volume v for which divergence theorem is applicable.

Key Point: *The divergence theorem as applied with Gauss's law is included in the section 3.12 of chapter 3.*

Example 1.19 : *A particular vector field $\bar{F} = r^2 \cos^2 \phi \, \bar{a}_r + z \sin \phi \, \bar{a}_\phi$ is in cylindrical system. Find the flux emanating due to this field from the closed surface of the cylinder $0 \le z \le 1$, $r = 4$. Verify the divergence theorem.*

Solution : The outward flux is given by,

$$\phi = \oint_S \bar{F} \cdot d\bar{S} \text{ over a closed surface S}$$

The cylindrical surface is shown in the Fig. 1.50.

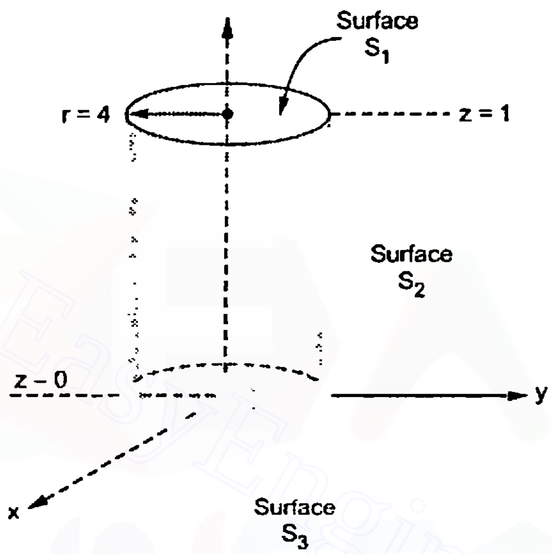


Fig. 1.50

The total surface is made up of,

1. Top surface S_1 for which z = 1, r varies from 0 to 4 and ϕ varies from 0 to 2π.
2. Lateral surface for which z varies from 0 to 1, ϕ from 0 to 2π and r = 4.
3. Bottom surface S_3 for which z = 0, r varies from 0 to 4 and ϕ varies from 0 to 2π.

For S_1, $\quad d\bar{S} = r \, dr \, d\phi \, \bar{a}_z$

For S_2, $\quad d\bar{S} = r \, dz \, d\phi \, \bar{a}_r$

For S_3, $\quad d\bar{S} = r \, dr \, d\phi \, (-\bar{a}_z)$

$$\therefore \quad \oint_{S_1} \bar{F} \cdot d\bar{S} = \oint_{S_1} (r^2 \cos^2 \phi \bar{a}_r + z \sin \phi \bar{a}_\phi) \cdot (r \, dr \, d\phi \bar{a}_z)$$

$$= 0 \qquad \ldots \bar{a}_r \cdot \bar{a}_z = \bar{a}_\phi \cdot \bar{a}_z = 0$$

$$\oint_{S_3} \bar{F} \cdot d\bar{S} = \oint_{S_3} (r^2 \cos^2 \phi \bar{a}_r + z \sin \phi \bar{a}_\phi) \cdot [r \, dr \, d\phi \, (-\bar{a}_z)]$$

$$= 0 \qquad \ldots \bar{a}_r \cdot \bar{a}_z = \bar{a}_\phi \cdot \bar{a}_z = 0$$

$$\oint_{S_2} \bar{F} \cdot d\bar{S} = \oint_{S_2} (r^2 \cos^2 \phi \bar{a}_r + z \sin \phi \bar{a}_\phi) \cdot (r \, dz \, d\phi \bar{a}_r)$$

$$= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} r^2 \cos^2 \phi \, r \, dz \, d\phi \qquad \ldots \bar{a}_r \cdot \bar{a}_z = 1, \ \bar{a}_\phi \cdot \bar{a}_r = 0 \ r = 4$$

$$= (4)^3 \int_{z=0}^{1} \int_{\phi=0}^{2\pi} dz \cos^2 \phi \, d\phi = 64 \int_0^1 dz \int_{\phi=0}^{2\pi} \frac{1+\cos 2\phi}{2} d\phi$$

$$= 64 \times [z]_0^1 \times \frac{1}{2} \times \left\{ [\phi]_0^{2\pi} + \left[+ \frac{\sin 2\phi}{2} \right]_0^{2\pi} \right\}$$

$$= 64 \times 1 \times \frac{1}{2} \times 2\pi = 64\pi$$

$$\therefore \quad \oint_S \bar{F} \cdot d\bar{S} = 0 + 64\pi + 0 = 64\pi$$

Let us verify divergence theorem which states that,

$$\oint_S \bar{F} \cdot d\bar{S} = \oint_v (\nabla \cdot \bar{F}) \, dv$$

Now $\quad dv = r \, dr \, d\phi \, dz$

$$\nabla \cdot \bar{F} = \frac{1}{r} \frac{\partial}{\partial r} (r \, F_r) + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$= \frac{1}{r} \frac{\partial}{\partial r} (r \times r^2 \cos^2 \phi) + \frac{1}{r} \frac{\partial}{\partial \phi} (z \sin \phi) + 0$$

$$= \frac{\cos^2 \phi}{r} \times 3r^2 + \frac{z}{r} (+\cos \phi) = 3 \, r \cos^2 \phi + \frac{z \cos \phi}{r}$$

$$\therefore \quad \oint_v (\nabla \cdot \bar{F}) \, dv = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \int_{r=0}^{4} \left(3r \cos^2 \phi + \frac{z \cos \phi}{r} \right) r \, dr \, d\phi \, dz$$

$$= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \left[\frac{3 r^3}{3} \cos^2 \phi + z \cos \phi \, r \right]_0^4 d\phi \, dz$$

$$= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \left\{ 4^3 \left[\frac{1+\cos 2\phi}{2} \right] + 4z \cos \phi \right\} d\phi \, dz$$

$$= \int_{z=0}^{1} \left\{ 32 \left[\phi + \frac{\sin 2\phi}{2} \right]_0^{2\pi} + 4z [\sin \phi]_0^{2\pi} \right\} dz$$

$$= \int_{z=0}^{1} \{ 32 \times [2\pi + 0] + 4z[0] \} \, dz = \int_{z=0}^{1} 64 \pi \, dz$$

$$= 64 \, \pi [z]_0^1 = 64\pi$$

Thus $\int_S \bar{F} \cdot d\bar{S} = \oint_v (\nabla \cdot \bar{F}) \, dv$ and divergence theorem is verified.

1.17 Gradient of a Scalar

Consider that in space let W be the unique function of x, y and z co-ordinates in the cartesian system. This is the **scalar function** and denoted as W (x, y, z). Consider the vector operator in cartesian system denoted as ∇ called del. It is defined as,

$$\boxed{\nabla \text{ (del)} = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z}$$

Key Point: *The operation of the vector operator del (∇) on a scalar function is called gradient of a scalar.*

$$\text{Grad W} = \nabla W = \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right) W$$

$$\therefore \quad \boxed{\text{Grad W} = \frac{\partial W}{\partial x}\bar{a}_x + \frac{\partial W}{\partial y}\bar{a}_y + \frac{\partial W}{\partial z}\bar{a}_z}$$

Key Point: *Gradient of a scalar is a vector.*

The gradient of a scalar W in various co-ordinate systems are given by,

Sr. No	Co-ordinate system	Grad W = ∇ W
1.	Cartesian	$\nabla W = \frac{\partial W}{\partial x}\bar{a}_x + \frac{\partial W}{\partial y}\bar{a}_y + \frac{\partial W}{\partial z}\bar{a}_z$
2.	Cylindrical	$\nabla W = \frac{\partial W}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial W}{\partial \phi}\bar{a}_\phi + \frac{\partial W}{\partial z}\bar{a}_z$
3.	Spherical	$\nabla W = \frac{\partial W}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\bar{a}_\phi$

Table 1.4

1.17.1 Properties of Gradient of a Scalar

The various properties of a gradient of a scalar field W are,

1. The gradient ∇ W gives the maximum rate of change of W per unit distance.
2. The gradient ∇ W always indicates the direction of the maximum rate of change of W.
3. **The gradient ∇ W at any point is perpendicular to the constant W surface,** which passes through the point.
4. The **directional derivative** of W along the unit vector $\bar{a}$ is $\nabla W \cdot \bar{a}$ (dot product), which is projection of ∇ W in the direction of unit vector $\bar{a}$.

If U is the another scalar function then,

5. $\nabla(U+W) = \nabla U + \nabla W$
6. $\nabla(UW) = U\nabla W + W\nabla U$

7. $$\nabla\left(\frac{U}{W}\right) = \frac{W\nabla U - U\nabla W}{W^2}$$

Example 1.20 : *A particular scalar field α is given by,*

a) $\alpha = 20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)$ ... *In cartesian*

b) $\alpha = 25\, r \sin\phi$ ... *In cylindrical*

c) $\alpha = \frac{40\cos\theta}{r^2}$ ... *In spherical*

Find its gradient at $P(0,1,1)$ for cartesian, $P\left(\sqrt{2}, \frac{\pi}{2}, 5\right)$ for cylindrical and $P(3, 60°, 30°)$ for the spherical.

Solution : a) $\alpha = 20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)$ in cartesian

$$\nabla\alpha = \frac{\partial\alpha}{\partial x}\bar{a}_x + \frac{\partial\alpha}{\partial y}\bar{a}_y + \frac{\partial\alpha}{\partial z}\bar{a}_z$$

$$\frac{\partial\alpha}{\partial x} = \frac{\partial}{\partial x}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = -20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)$$

$$\frac{\partial\alpha}{\partial y} = \frac{\partial}{\partial y}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = 20\, e^{-x} \cos\left(\frac{\pi y}{6}\right) \times \frac{\pi}{6}$$

$$\frac{\partial\alpha}{\partial z} = \frac{\partial}{\partial z}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = 0$$

$\therefore$ $$\nabla\alpha = -20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\bar{a}_x + 20\, e^{-x}\frac{\pi}{6}\cos\left(\frac{\pi y}{6}\right)\bar{a}_y$$

$\therefore$ At P (0, 1, 1) the $\nabla\alpha = -10\,\bar{a}_x + 9.0689\,\bar{a}_y$

b) $\alpha = 25\, r \sin\phi$ in cylindrical.

$\therefore$ $$\nabla\alpha = \frac{\partial\alpha}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial\alpha}{\partial\phi}\bar{a}_\phi + \frac{\partial\alpha}{\partial z}\bar{a}_z$$

$$\frac{\partial\alpha}{\partial r} = 25\sin\phi \qquad \frac{\partial\alpha}{\partial\phi} = 25\, r\cos\phi, \qquad \frac{\partial\alpha}{\partial z} = 0$$

$\therefore$ $$\nabla\alpha = 25\sin\phi\,\bar{a}_r + 25\cos\phi\,\bar{a}_\phi$$

$\therefore$ At $P\left(\sqrt{2}, \frac{\pi}{2}, 5\right)$ the $\nabla\alpha = 25\,\bar{a}_r$

c) $\alpha = \dfrac{40\cos\theta}{r^2}$ in spherical.

$$\therefore \quad \nabla\alpha = \frac{\partial\alpha}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial\alpha}{\partial\theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial\alpha}{\partial\phi}\bar{a}_\phi$$

$$\frac{\partial\alpha}{\partial r} = 40\cos\theta\left[-2r^{-3}\right] = -80\frac{\cos\theta}{r^3}$$

$$\frac{\partial\alpha}{\partial\theta} = -\frac{40}{r^2}\sin\theta, \quad \frac{\partial\alpha}{\partial\phi} = 0$$

$$\therefore \quad \nabla\alpha = -\frac{80\cos\theta}{r^3}\bar{a}_r - \frac{40}{r^3}\sin\theta\,\bar{a}_\theta$$

$\therefore$ At P $(3, 60°, 30°)$ the $\nabla\alpha = -\mathbf{1.4814}\,\bar{a}_r\ \mathbf{-0.9362}\,\bar{a}_\theta$

1.18 Curl of a Vector

The circulation of a vector field around a closed path is given by curl of a vector. Mathematically it is defined as,

$$\boxed{\text{Curl of } \bar{F} = \lim_{\Delta S_N \to 0} \frac{\oint \bar{F} \cdot d\bar{l}}{\Delta S_N}} \quad \text{... (1)}$$

where ΔS_N = Area enclosed by the line integral in normal direction

Thus maximum circulation of $\bar{F}$ per unit area as area tends to zero whose direction is normal to the surface is called curl of $\bar{F}$.

Symbolically it is expressed as,

$$\boxed{\nabla \times \bar{F} = \text{curl of } \bar{F}} \quad \text{... (2)}$$

Key Point: *Curl indicates the rotational property of vector field. If curl of vector is zero, the vector field is irrotational.*

In various co-ordinate sysems, the curl of $\bar{F}$ is given by,

$$\nabla \times \bar{F} = \left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial F_x}{\partial_z} - \frac{\partial F_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right]\bar{a}_z$$

i.e.

$$\boxed{\nabla \times \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \quad \text{Cartesian}} \quad \text{...(3)}$$

$$\nabla \times \bar{F} = \left[\frac{1}{r}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial(rF_\phi)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial\phi}\right]\bar{a}_z$$

i.e.

$$\nabla \times \bar{F} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ F_r & F_\phi & F_z \end{vmatrix} \quad \text{Cylindrical} \quad \text{...(4)}$$

Key Point: In $\dfrac{\partial(r F_\phi)}{\partial r}$, r cannot be taken outside as differentiation is with respect to r.

$$\nabla\times\bar{F} = \frac{1}{r\sin\theta}\left[\frac{\partial F_\phi \sin\theta}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(r F_\phi)}{\partial r}\right]\bar{a}_\theta$$

$$+\frac{1}{r}\left[\frac{\partial(r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\bar{a}_\phi$$

i.e.

$$\boxed{\nabla\times\bar{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ F_r & rF_\theta & r\sin\theta\, F_\phi\end{vmatrix} \quad \text{Spherical}}$$

Key Point: *The physical significance and concept of curl is discussed in detail in section 7.10 of chapter 7.*

1.19 Stoke's Theorem

The Stoke's theorem relates the line integral to a surface integral. It states that,

The line integral of $\bar{F}$ around a closed path L is equal to the integral of curl of $\bar{F}$ over the open surface S enclosed by the closed path L.

Mathematically it is expressed as,

$$\boxed{\oint_L \bar{F}\cdot d\bar{L} = \int_S (\nabla\times\bar{F})\cdot d\bar{S}} \quad \ldots (1)$$

where dL = Perimeter of total surface S

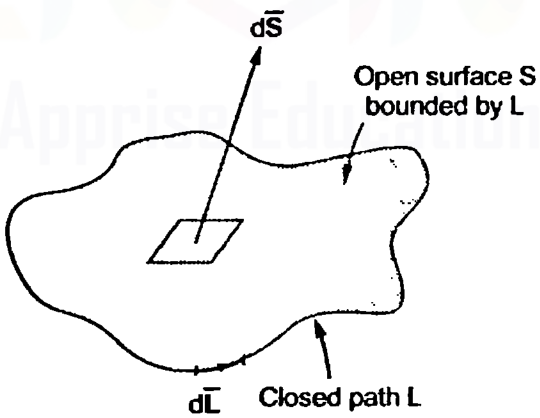


Fig. 1.51

Key Point: *Stoke's theorem is applicable only when $\bar{F}$ and $\nabla\times\bar{F}$ are continuous on the surface S. The path L and open surface S enclosed by path L for which Stoke's theorem is applicable are shown in the Fig. 1.51.*

Key Point: *The proof of Stoke's theorem is included in the section 7.11 of chapter 7.*

Example 1.21 : *Verify Stoke's theorem for a vector field*

$$\bar{F} = r^2\cos\phi\ \bar{a}_r + z\sin\phi\ \bar{a}_z$$

around the path L defined by $0 \le r \le 3$, $0 \le \phi \le 45°$ and $z = 0$ as shown in the Fig. 1.52.

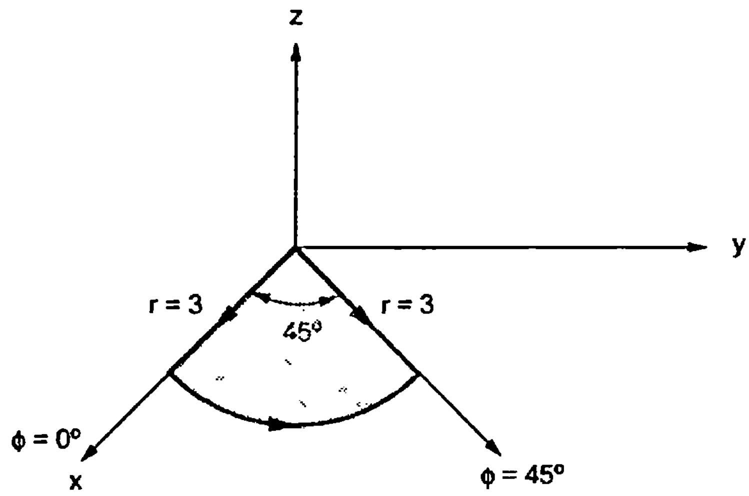


Fig. 1.52

Solution : From Stoke's theorem,

$$\oint_L \bar{F} \cdot d\bar{L} = \int_S (\nabla \times \bar{F}) \cdot d\bar{S}$$

The L.H.S. of Stoke's theorem is already evaluated in Ex. 1.18, which is **2.636**. (Refer Page 1-50).

To evaluate R.H.S., find $\nabla \times \bar{F}$

$$\nabla \times \bar{F} = \left[\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial (r F_\phi)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \phi}\right]\bar{a}_z$$

$$F_r = r^2 \cos\phi, \quad F_\phi = 0, \quad F_z = z \sin\phi$$

$$\therefore \quad \nabla \times \bar{F} = \left[\frac{1}{r} \times 0 - 0\right]\bar{a}_r + [0 - 0]\,\bar{a}_\phi + \left[\frac{1}{r}(0) - \frac{1}{r}(r)^2(-\sin\phi)\right]\bar{a}_z$$

$$= r \sin\phi\, \bar{a}_z$$

$d\bar{S} = r\, dr\, d\phi\, \bar{a}_z$ as surface is in x-y plane i.e. z = 0 plane for which normal direction is $\bar{a}_z$.

$$\therefore \int_S (\nabla \times \bar{F}) \cdot d\bar{S} = \int_S (r \sin\phi\, \bar{a}_z) \cdot (r\, dr\, d\phi)\, \bar{a}_z = \int_{\phi=0}^{45^\circ} \int_{r=0}^{3} r^2 \sin\phi\, dr\, d\phi = \left[\frac{r^3}{3}\right]_0^3 [-\cos\phi]_0^{45^\circ}$$

$$= [9]\,[-0.707 - (-1)] = 9 \times 0.2928 = \mathbf{2.636}$$

Thus Stoke's theorem is verified.

1.20 Laplacian of a Scalar

The divergence of a vector and gradient of a scalar are discussed earlier. The composite operator of these two is called Laplacian of a scalar.

If V is a scalar field, then the Laplacian of scalar V is denoted as $\nabla^2 V$ and mathematically defined as the divergence of the gradient of V.

Key Point: *The operator ∇^2 is called laplacian operator.*

In cartesian co-ordinate system,

$$\nabla^2 V = \nabla \cdot \nabla V = \left[\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right] \cdot \left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$\therefore \quad \boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}} \quad \text{...In cartesian system}$$

Key Point: *The Laplacian of a scalar is always a scalar.*

In cylindrical co-ordinate system it is given by,

$$\therefore \quad \boxed{\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}} \quad \text{...In cylindrical system}$$

In spherical co-ordinate system it is given by,

$$\therefore \quad \boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}} \quad \text{...In spherical system}$$

Harmonic Field : A scalar field is said to be harmonic in a given region, if its Laplacian vanishes in that region.

Mathematically for a scalar field V to be **harmonic,**

$$\boxed{\nabla^2 V = 0}$$

This equation is called Laplace's equation. Its solution and applications are discussed throughly in the chapter 6.

Example 1.22 : *Find the Laplacian of the scalar fields and comment on, which fields are harmonic.*

i) $W = x^2 y + xyz - yz^2$ *ii)* $U = rz\sin\phi + z^2\cos^2\phi + r^2$

iii) $V = 2r\cos\theta\cos\phi$

Solution : i) $W = x^2y + xyz - yz^2$

$$\therefore \quad \nabla^2 W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(x^2 + xz - z^2) + \frac{\partial}{\partial z}(xy - 2yz)$$

$$= 2y + 0 + 0 + 0 + 0 - 2y = 0$$

As $\nabla^2 W = 0$, the scalar field W is **harmonic.**

ii) $U = rz \sin\phi + z^2 \cos^2\phi + r^2$

$$\therefore \quad \nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}[r\,(z\sin\phi + 2r)] + \frac{1}{r^2}\frac{\partial}{\partial \phi}[rz\cos\phi - 2z^2 \sin\phi\cos\phi] + \frac{\partial}{\partial z}[r\sin\phi + 2\,z\cos^2\phi]$$

... $2\sin\phi\cos\phi = \sin 2\phi$

$$= \frac{1}{r}[\,z\sin\phi + 4r] + \frac{1}{r^2}[-rz\sin\phi - z^2\, 2\cos 2\phi] + [0 + 2\cos^2\phi]$$

$$= \frac{z}{r}\sin\phi + 4 - \frac{z}{r}\sin\phi - \frac{2z^2}{r^2}\cos 2\phi + 2\cos^2\phi$$

$$= 4 + 2\cos^2\phi - \frac{2z^2}{r^2}\cos 2\phi$$

As $\nabla^2 U \neq 0$, the scalar field U is **not harmonic.**

iii) $V = 2r\cos\theta\cos\phi$

$$\therefore \quad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\,(2\cos\theta\cos\phi] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta(-2r\cos\phi\sin\theta)] + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}(-2r\cos\theta\sin\phi)$$

$$= \frac{1}{r^2}[4r\cos\theta\cos\phi] + \frac{1}{r^2\sin\theta}[-2r\cos\phi \times 2\sin\theta\cos\theta] + \frac{1}{r^2\sin^2\theta}[-2r\cos\theta\cos\phi]$$

$$= \frac{4\cos\theta\cos\phi}{r} - \frac{4\cos\theta\cos\phi}{r} - \frac{2\cos\theta\cos\phi}{r\sin^2\theta} = \frac{-2}{r}\cot\theta\,\mathrm{cosec}\,\theta\cos\phi$$

As $\nabla^2 V \neq 0$, the scalar field V is **not harmonic.**

Examples with Solutions

Example 1.23 : *Given $\bar{A} = 5\bar{a}_x$ and $\bar{B} = 4\bar{a}_x + B_y\bar{a}_y$ then find B_y such that angle between $\bar{A}$ and $\bar{B}$ is 45°. If $\bar{B}$ also has a term $B_z\bar{a}_z$, what relationship must exist between B_y and B_z ?*

Solution : $\bar{A} = 5\,\bar{a}_x$ and $\bar{B} = 4\,\bar{a}_x + B_y\,\bar{a}_y$, $\theta_{AB} = 45°$

Now $\bar{A} \cdot \bar{B} = A_x B_x + A_y B_y + A_z B_z$

$= (5 \times 4) + (0) + (0) = 20$

But $\bar{A} \cdot \bar{B} = |\bar{A}|\,|\bar{B}| \cos\theta_{AB}$

$\therefore \quad 20 = \sqrt{(5)^2} \times \sqrt{(4)^2 + (B_y)^2} \times \cos 45°$

$\therefore \quad \sqrt{16 + B_y^2} = 5.6568$

$\therefore \quad B_y^2 = 16$

$\therefore \quad B_y = \pm 4$

Now $\bar{B} = 4\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$

Still $\bar{A} \cdot \bar{B} = 20$

$\therefore \quad 20 = \sqrt{(5)^2} \times \sqrt{(4)^2 + (B_y)^2 + (B_z)^2} \times \cos 45°$

$\therefore \quad \sqrt{16 + B_y^2 + B_z^2} = 5.6568$

$\therefore \quad B_y^2 + B_z^2 = 16$

This is the required relation between B_y and B_z.

Example 1.24 : *Find the unit vector directed towards the point (x_1, y_1, z_1) from an arbitrary point in the plane $y = -5$.*

Solution : The plane y = – 5 is parallel to xz plane as shown in the Fig. 1.53.

The coordinates of point P are (x, – 5, z) as $y = -5$ is constant. While Q is arbitrary point having co-ordinates (x_1, y_1, z_1). To find unit vector along the direction PQ.

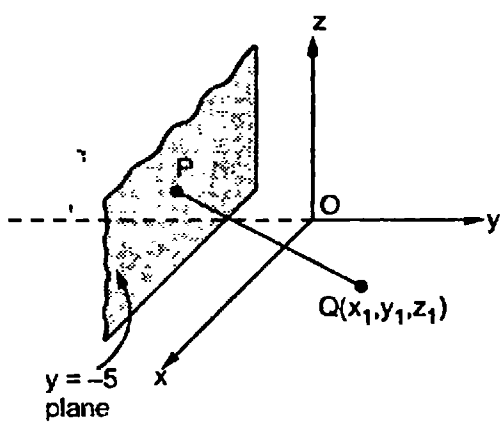


Fig. 1.53

$$\bar{a}_{PQ} = \frac{\overline{PQ}}{|\overline{PQ}|}$$

where $\overline{PQ} = \bar{Q} - \bar{P}$

$\overline{PQ} = (x_1 - x)\bar{a}_x + (y_1 - (-5))\bar{a}_y + (z_1 - z)\bar{a}_z$

$$\therefore \qquad |\overline{PQ}| = \sqrt{(x_1 - x)^2 + (y_1 + 5)^2 + (z_1 - z)^2}$$

$$\therefore \qquad \bar{a}_{PQ} = \frac{(x_1 - x)\,\bar{a}_x + (y_1 + 5)\,\bar{a}_y + (z_1 - z)\,\bar{a}_z}{\sqrt{(x_1 - x)^2 + (y_1 + 5)^2 + (z_1 - z)^2}}$$

Example 1.25 : *Use spherical co-ordinates to write the differential surface areas dS_1 and dS_2 as shown and integrate to obtain the surfaces areas A and B as shown in the Fig. 1.54.*

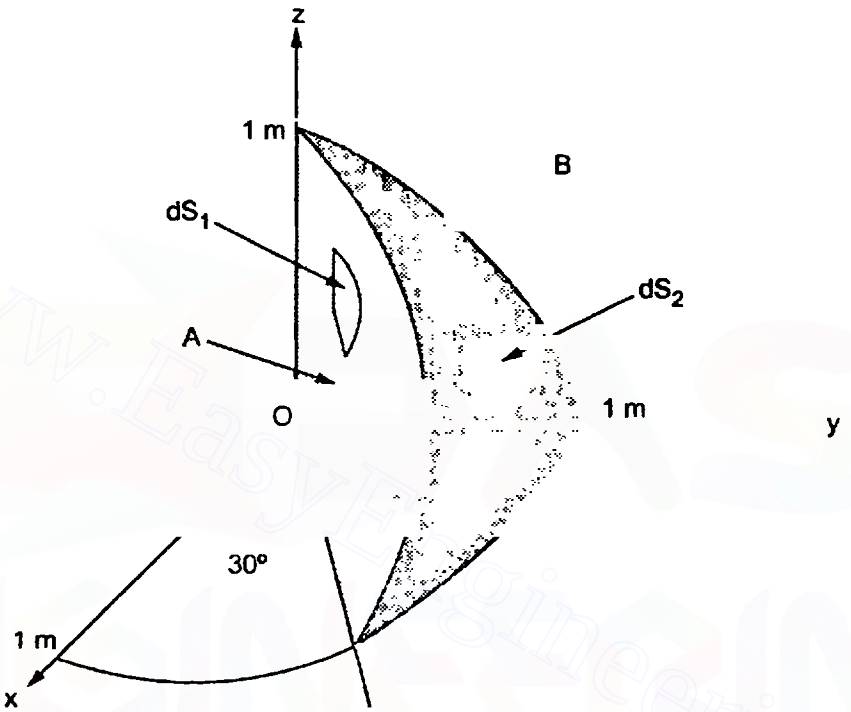


Fig. 1.54

Solution : Consider differential surface area dS_1. The unit vector perpendicular to it is in the direction of increasing ϕ i.e. $\bar{a}_\phi$. Hence $d\,\bar{S}_1$ is $d\bar{S}_\phi$.

$$\therefore \qquad d\bar{S}_1 = r\,dr\,d\theta\;\bar{a}_\phi$$

$$\therefore \qquad A = \iint r\,dr\,d\theta$$

Now r is changing from 0 to 1 while θ is changing from 0 to 90°. (Note that θ is measured from z axis.).

$$\therefore \qquad A = \int_0^{90^\circ}\int_0^1 r\,dr\,d\theta = \left[\frac{r^2}{2}\right]_0^1 [\theta]_0^{90^\circ}$$

But for areas angles must be taken in radians.

$$\therefore \qquad A = \frac{1}{2} \times [90^\circ] = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}\ m^2$$

The differential surface area dS_2 is on the curved surface of sphere, the direction normal to it is from origin radially going outward i.e. $\bar{a}_r$.

$$\therefore \quad d\bar{S}_2 = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

Now r is constant as 1m. The θ varies from 0 to 90° i.e.

0 to $\pi/2$ rad while ϕ is varying from 30° to 90° i.e. $\pi/6$ rad to $\pi/2$ rad.

$$\therefore \quad B = \int_{\pi/6}^{\pi/2}\int_{0}^{\pi/2} (1)^2 \sin\theta\, d\theta\, d\phi = \int_{\pi/6}^{\pi/2} [-\cos\theta]_0^{\pi/2}\, d\phi$$

$$= [0-(-1)]\,[\phi]_{\pi/6}^{\pi/2} = \frac{\pi}{2}-\frac{\pi}{6} = \frac{\pi}{3}\ m^2$$

Example 1.26 : *Given points $P(r=5, \phi=60°, z=2)$ and $Q(r=2, \phi=110°, z=-1)$ in cylindrical co-ordinate system. Find*
i) Unit vector in cartesian co-ordinates at P directed towards Q
ii) Unit vector in cylindrical co-ordinates at P directed towards Q.

Solution : Let us obtain the cartesian co-ordinates of P and Q.

It is known that $x = r\cos\phi$, $y = r\sin\phi$ and $z = z$

$\therefore$ P (2.5, 4.33, 2) and Q (– 0.684, 1.8793, – 1)

i) The unit vector from P to Q is,

$$\bar{a}_{PQ} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{\bar{Q}-\bar{P}}{|\overline{PQ}|} \quad \text{where } \bar{P} \text{ and } \bar{Q} \text{ are position vectors}$$

$$= \frac{(-0.684-2.5)\bar{a}_x + (1.8793-4.33)\bar{a}_y + (-1-2)\bar{a}_z}{|\overline{PQ}|}$$

$$= \frac{-3.184\bar{a}_x - 2.4507\bar{a}_y - 3\bar{a}_z}{\sqrt{(-3.184)^2+(-2.4507)^2+(-3)^2}}$$

$$\therefore \quad \bar{a}_{PQ} = \mathbf{-0.6349\,\bar{a}_x - 0.4887\,\bar{a}_y - 0.5983\,\bar{a}_z}$$

ii) The vector $\overline{PQ} = -3.184\,\bar{a}_x - 2.4507\,\bar{a}_y - 3\bar{a}_z$ ... As obtained earlier.

Let us transform this into cylindrical coordinates.

$$(PQ)_r = \overline{PQ}\cdot\bar{a}_r = -3.184\bar{a}_x\cdot\bar{a}_r - 2.4507\bar{a}_y\cdot\bar{a}_r - 3\bar{a}_z\cdot\bar{a}_r$$

$$= -3.184\cos\phi - 2.4507(-\sin\phi) + 0 \quad \text{... Refer Table 1.2}$$

At point P, $\phi = 60°$

$$\therefore \quad (PQ)_r = -3.184\times 0.5 - 2.4507(-0.866) = 0.5303$$

$$(PQ)_\phi = \overline{PQ} \cdot \bar{a}_\phi = -3.184\, \bar{a}_x \cdot \bar{a}_\phi - 2.4507\, \bar{a}_y \cdot \bar{a}_\phi - 3\bar{a}_z \cdot \bar{a}_\phi$$

$$= -3.184\,(-\sin\phi) - 2.4507 \cos\phi$$

$\therefore$ $$(PQ)_\phi = -3.184\,(-0.866) - 2.4507 \times 0.5 = 1.5319$$

and $$(PQ)_z = \overline{PQ} \cdot \bar{a}_z = -3 \qquad \ldots\ \bar{a}_x \cdot \bar{a}_z = \bar{a}_y \cdot \bar{a}_z = 0$$

$\therefore$ $$\overline{PQ} = 0.5303\, \bar{a}_r + 1.5319\, \bar{a}_\phi - 3\bar{a}_z$$

$\therefore$ $$\bar{a}_{PQ} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{0.5303\bar{a}_r + 1.5319\bar{a}_\phi - 3\bar{a}_z}{\sqrt{(0.5303)^2 + (1.5319)^2 + (-3)^2}}$$

$$= 0.155\, \bar{a}_r + 0.449\, \bar{a}_\phi - 0.88\, \bar{a}_z$$

Example 1.27 : *Find the area of the curved surface using the cylindrical co-ordinates which lies on the right circular cylinder of radius 2 m, height 8 m and* $40° \le \phi \le 90°$.

[UPTU : 2002-03]

Solution : The surface is shown in the Fig. 1.55.

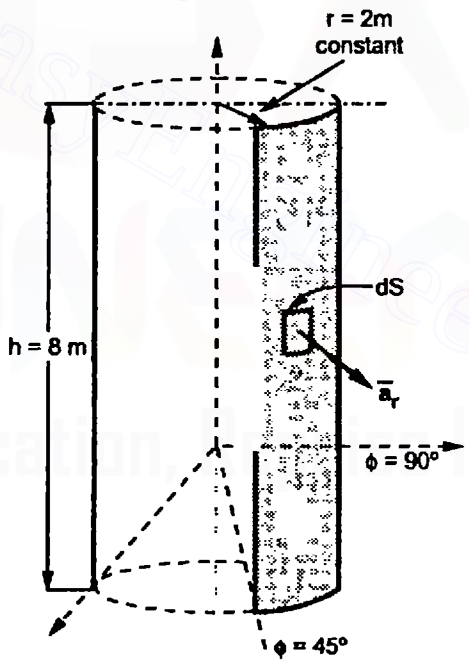


Fig. 1.55

The differential area normal to $\bar{a}_r$ is,

$$d\bar{S} = r\, d\phi\, dz\, \bar{a}_r$$

The surface is constant r surface and normal to it is unit vector $\bar{a}_r$.

$\therefore$ $$S = \int dS = \iint r\, d\phi\, dz$$

$$= \int_{z=0}^{8} \int_{\phi=45°}^{90°} r\, d\phi\, dz \qquad \ldots\ r = 2\text{ m}$$

$$= r\,[\phi]_{45^\circ}^{90^\circ}\,[z]_0^8$$

$$= 2\times[90^\circ-45^\circ]\times\frac{\pi}{180^\circ}\times[8-0] \qquad \text{...Use } \phi \text{ in radians}$$

$$= \frac{2\times 45^\circ\times\pi\times 8}{180^\circ} = \mathbf{12.5663\ m^2}$$

Example 1.28 : *Convert point P (1,3,5) from Cartesian to cylindrical and spherical co-ordinates.* **[UPTU : 2003-04]**

Solution : P(1, 3, 5) i.e. x = 1, y = 3, z = 5

In cylindrical system

$$r = \sqrt{x^2+y^2} = \sqrt{1+3^2} = 3.1622$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{1} = 71.56^\circ$$

$$z = z = 5$$

$\therefore$ **P(3.1622, 71.56°, 5)** in cylindrical

In spherical system :

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+3^2+5^2} = 5.916$$

$$\theta = \tan^{-1}\frac{z}{r} = \cos^{-1}\frac{5}{5.916} = 32.31^\circ$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{1} = 71.56^\circ$$

$\therefore$ **P(5.916, 32.31°, 71.56°)** in spherical.

Example 1.29 : *Given a vector function*

$$\bar{A} = (3x+c_1 z)\ \bar{a}_x + (c_2 x - 5z)\,\bar{a}_y + (4x - c_3 y + c_4 z)\,\bar{a}_z$$

Calculate c_1, c_2, c_3 and c_4 if A is irrotational and solenoidal. **[UPTU : 2003-04]**

Solution : For $\bar{A}$ to be irrotational, $\nabla\times\bar{A} = 0$

$$\nabla\times\bar{A} = \left[\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right]\bar{a}_z$$

$A_x = 3x + c_1 z, \quad A_y = c_2 x - 5z \quad A_z = 4x - c_3 y + c_4 z$

$\therefore \quad \nabla\times\bar{A} = [-c_3+5]\,\bar{a}_x + [c_1-4]\bar{a}_y + [c_2-0]\bar{a}_z = 0$

$\therefore \quad c_3 = 5, \quad c_1 = 4, \quad c_2 = 0$ ... For $\bar{A}$ to be irrotational

For $\bar{A}$ to be solenoidal, $\nabla \bullet \bar{A} = 0$

$$\therefore \quad \nabla \bullet \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

$$\therefore \quad 3 + 0 + c_4 = 0$$

$$\therefore \quad c_4 = -3$$

... For $\bar{A}$ to be irrotational

Example 1.30 : *Verify that vector field $\bar{A} = yz\bar{a}_x + zx\bar{a}_y + xy\bar{a}_z$ is irrotational and solenoidal.* **(UPTU : 2005-06, 5 Marks)**

Solution : For $\bar{A}$ to be irrotational, $\nabla \times \bar{A} = 0$

$$\nabla \times \bar{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\bar{a}_z$$

$A_x = yz$, $A_y = zx$ and $A_z = xy$ ...Given

$$\therefore \frac{\partial A_x}{\partial y} = z, \frac{\partial A_x}{\partial z} = y, \frac{\partial A_y}{\partial x} = z, \frac{\partial A_y}{\partial z} = x, \frac{\partial A_z}{\partial x} = y, \frac{\partial A_z}{\partial y} = x$$

$$\therefore \quad \nabla \times \bar{A} = [x - x]\bar{a}_x + [y - y]\bar{a}_y + [z - z]\bar{a}_z = \mathbf{0}$$

Thus $\bar{A}$ **is irrotational.**

For $\bar{A}$ to be solenoidal, $\nabla \bullet \bar{A} = 0$

$$\nabla \bullet \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

$$\frac{\partial A_x}{\partial x} = 0, \quad \frac{\partial A_y}{\partial y} = 0, \quad \frac{\partial A_z}{\partial z} = 0$$

$\therefore \quad \nabla \bullet \bar{A} = 0$ **hence $\bar{A}$ is solenoidal**

Example 1.31 : *If $\bar{A} = \alpha\,\bar{a}_x + 2\,\bar{a}_y + 10\,\bar{a}_z$ and*

$\bar{B} = 4\alpha\,\bar{a}_x + 8\,\bar{a}_y - 2\alpha\,\bar{a}_z$, find out the value of α for which the two vectors become perpendicular. **(UPTU : 2006-07, 5 Marks)**

Solution : $\bar{A} = \alpha\bar{a}_x + 2\,a_y + 10\,\bar{a}_z$, $\quad \bar{B} = 4\alpha\bar{a}_x + 8\,\bar{a}_y - 2\alpha\,\bar{a}_z$

For perpendicular vectors, $\bar{A} \bullet \bar{B} = 0$

$\therefore (\alpha)\ (4\alpha) + (2)\ (8) + 10\ (-2\ \alpha) = 0$

$\therefore \quad 4\alpha^2 - 20\ \alpha + 16 = 0$

$\therefore \quad \alpha = $ **4 or 1**

Example 1.32 : *Given points $A(x = 2, y = 3, z = -1)$ and $B(\rho = 4, \phi = -50°, z = 2)$ find the distance A to B.* **(UPTU : 2006-07, 5 Marks)**

Solution : A (x = 2, y = 3, z = – 1), B(ρ = 4, ϕ = – 50°, z = 2)

Converting point B to cartesian system,

$$x = \rho \cos \phi = 4 \cos(-50°) = 2.57115$$

$$y = \rho \sin \phi = 4 \sin(-50°) = -3.0641$$

$$z = z = 2$$

$$\therefore \quad d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

$$= \sqrt{(2.57115-2)^2 + (-3.0641-3)^2 + [2-(-1)]^2}$$

$$= \sqrt{0.326212 + 36.77331 + 9} = \mathbf{6.7896}$$

Example 1.33 : *Show that the vector fields*

$$\bar{A} = \bar{a}_r \frac{\sin 2\theta}{r^2} + 2\bar{a}_\theta \frac{(\sin\theta)}{r^2} \text{ and}$$

$\bar{B} = r \cos\theta\, \bar{a}_r + r\, \bar{a}_\theta$ *are every where parallel to each other.* **(UPTU : 2007-08, 5 Marks)**

Solution : For parallel vectors, $\bar{A} \times \bar{B} = 0$

Given $\bar{A}$ and $\bar{B}$ are in spherical co-ordinates.

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\phi \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix}$$

$$A_r = \frac{\sin 2\theta}{r^2}, \quad A_\theta = \frac{2\sin\theta}{r^2}, \quad A_\phi = 0$$

$$B = r\cos\theta, \quad B_\theta = r, \quad B_\phi = 0$$

$$\therefore \quad \bar{A} \times \bar{B} = \bar{a}_r[A_\theta B_\phi - B_\theta A_\phi] - \bar{a}_\theta[A_r B_\phi - B_r A_\phi] + \bar{a}_\phi[A_r B_\theta - B_r A_\theta]$$

$$= 0\,\bar{a}_r - 0\,\bar{a}_\theta + \left[\frac{\sin 2\theta}{r^2} \times r - \frac{r\cos\theta \times 2\sin\theta}{r^2}\right]\bar{a}_\phi$$

$$= \left[\frac{2\sin\theta\cos\theta}{r} - \frac{2\sin\theta\cos\theta}{r}\right]\bar{a}_\phi = 0$$

As $\bar{A} \times \bar{B} = 0$, the two vector fields are parallel to each other.

Example 1.34 : *Express the field* $\bar{E} = \frac{A}{r^2}\bar{a}_r$ *in (i) rectangular components, ii) cylindrical components.* **(UPTU : 2007-08, 5 Marks)**

Solution :

$$\bar{E} = \frac{A\bar{a}_r}{r^2} \qquad \text{...In spherical co-ordinates}$$

i) Spherical to rectangular

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A/r^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad E_x = \frac{A}{r^2}\sin\theta\cos\phi,\ E_y = \frac{A}{r^2}\sin\theta\sin\phi,\ E_z = \frac{A\cos\theta}{r^2}$$

$$\therefore \quad \bar{E} = \frac{A}{r^2}\sin\theta\cos\phi\ \bar{a}_x + \frac{A}{r^2}\sin\theta\sin\phi\ \bar{a}_y + \frac{A\cos\theta}{r^2}\bar{a}_z$$

But
$$r = \sqrt{x^2+y^2+z^2},\quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z},\quad \phi = \tan^{-1}\frac{y}{x}$$

$$\therefore \quad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}},\quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}},\quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore \quad \bar{E} = \frac{A}{x^2+y^2+z^2}\times\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\times\frac{x}{\sqrt{x^2+y^2}}\bar{a}_x$$
$$+ \frac{A}{x^2+y^2+z^2}\times\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\times\frac{y}{\sqrt{x^2+y^2}}\bar{a}_y$$
$$+ \frac{A}{x^2+y^2+z^2}\times\frac{z}{\sqrt{x^2+y^2+z^2}}\bar{a}_z$$

$$\therefore \quad \bar{E} = \frac{Ax}{(x^2+y^2+z^2)^{3/2}}\ \bar{a}_x + \frac{Ay}{(x^2+y^2+z^2)^{3/2}}\ \bar{a}_y + \frac{Az}{(x^2+y^2+z^2)^{3/2}}\ \bar{a}_z$$

ii) Spherical to cylindrical

$$\begin{bmatrix} E_\rho \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A/r^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad E_\rho = \frac{A\sin\theta}{r^2},\ E_\phi = 0,\ E_z = \frac{A\cos\theta}{r^2}$$

$$r = \sqrt{\rho^2+z^2}, \quad \theta = \tan^{-1}\frac{\rho}{z}$$

$$\therefore \quad \sin\theta = \frac{\rho}{\sqrt{\rho^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}$$

$$\therefore \quad \bar{E} = \frac{A\rho}{(\rho^2+z^2)^{3/2}}\,\bar{a}_\rho + \frac{Az}{(\rho^2+z^2)^{3/2}}\,\bar{a}_z$$

Example 1.35 : *Find the divergence and curl of the following function :*

$$\bar{A} = 2xy\,\bar{a}_x + x^2z\,\bar{a}_y + z^3\bar{a}_z$$

(UPTU : 2007-08, 5 Marks)

Solution : $\bar{A} = 2xy\,\bar{a}_x + x^2z\,\bar{a}_y + z^3\bar{a}_z$

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} = 2y + 0 + 3z^2 = 2y + 3z^2$$

$$\nabla\times\bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2z & z^3 \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial z^3}{\partial y}-\frac{\partial x^2z}{\partial z}\right]-\bar{a}_y\left[\frac{\partial z^3}{\partial x}-\frac{\partial 2xy}{\partial z}\right]+\bar{a}_z\left[\frac{\partial x^2z}{\partial x}-\frac{\partial 2xy}{\partial y}\right]$$

$$= \bar{a}_x[0-x^2]-\bar{a}_y[0]+\bar{a}_z[2xz-2x]$$

$$= -x^2\bar{a}_x + 2x\,(z-1)\bar{a}_z$$

Review Questions

1. *What is a scalar and scalar field ? Give two examples.*
2. *What is a vector and vector field ? Give two examples.*
3. *What is a unit vector ? What is its significance in the vector representation ? How to find unit vector along a particular vector ?*
4. *Explain cartesian co-ordinate system and differential elements in cartesian co-ordinate system.*
5. *Explain cylindrical co-ordinate system and differential elements in cylindrical co-ordinate system.*
6. *Explain spherical co-ordinate system and differential elements in spherical co-ordinate system.*
7. *What is a dot product ? Explain its significance and applications.*
8. *What is a cross product ? Explain its properties and applications.*
9. *Explain the relationship between cartesian and cylindrical as well as cartesian and spherical systems.*
10. *How to transform the vectors from one coordinate system to other ?*

11. *Given two points A (5, 4, 3) and B (2, 3, 4).*
    Find : i) $\overline{A}+\overline{B}$ *ii)* $\overline{A}\bullet\overline{B}$ *iii)* θ_{AB} *iv)* $\overline{A}\times\overline{B}$
    v) Unit vector normal to the plane containing $\overline{A}$ *and* $\overline{B}$.
    vi) Area of parallelogram of which $\overline{A}$ *and* $\overline{B}$ *are adjacent sides.*

    [Ans. : $7\,\bar{a}_x+7\,\bar{a}_y+7\,\bar{a}_z$, 34, 26.762°, $0.41\,\bar{a}_x-0.82\,\bar{a}_y+0.41\,\bar{a}_z$, 17.1464]

    [Hint. : For area $|\overline{A}|\,|\overline{B}|\sin\theta_{AB}=|\overline{A}\times\overline{B}|$]

12. *If two positions vectors given are,* $\overline{A}=-2\,\bar{a}_x-5\,\bar{a}_y-4\,\bar{a}_z$ *and* $\overline{B}=2\,\bar{a}_x+3\,\bar{a}_y+5\,\bar{a}_z$ *then find,*
    i) $\overline{AB}$ *ii)* $\bar{a}_A$ *iii)* $\bar{a}_B$ *iv)* $\bar{a}_{AB}$ *v) Unit vector in the direction from C to A where C is (3, 5, 8).*

    [Ans. : $4\,\bar{a}_x+8\,\bar{a}_y+9\,\bar{a}_z$, $-0.298\,\bar{a}_x-0.745\,\bar{a}_y-0.596\,\bar{a}_z$, $-0.324\,\bar{a}_x+0.486\,\bar{a}_y-0.811\,\bar{a}_z$, $0.315\,\bar{a}_x+0.63\,\bar{a}_y+0.71\,\bar{a}_z$, $-0.304\,\bar{a}_x-0.61\,\bar{a}_y-0.732\,\bar{a}_z$]

13. *Find the value of* B_z *such that the angle between the vectors* $\overline{A}=2\,\bar{a}_x+\bar{a}_y+4\,\bar{a}_z$ *and* $\overline{B}=-2\,\bar{a}_x-\bar{a}_y+B_z\,\bar{a}_z$ *is 45°.* **[Ans. : 7.9]**

14. *For the vectors,* $\overline{A}=2\,\bar{a}_x-2\,\bar{a}_y+\bar{a}_z$ *and* $\overline{B}=3\,\bar{a}_x+5\,\bar{a}_y-2\,\bar{a}_z$ *find* $\overline{A}\bullet\overline{B}$, $\overline{A}\times\overline{B}$ *and show that* $\overline{A}\times\overline{B}=-(\overline{B}\times\overline{A})$.

    [Ans. : -6, $-\bar{a}_x+7\,\bar{a}_y+16\bar{a}_z$]

15. *Show that* $\overline{A}=4\,\bar{a}_x-2\,\bar{a}_y-\bar{a}_z$ *and* $\overline{B}=\bar{a}_x+4\,\bar{a}_y-4\,\bar{a}_z$ *are mutually perpendicular vectors.*

    [Hint. : Show $\overline{A}\bullet\overline{B}=0$]

16. *Find the angle between the vectors,* $\overline{A}=2\,\bar{a}_x+4\,\bar{a}_y-\bar{a}_z$ *and* $\overline{B}=3\,\bar{a}_x+6\,\bar{a}_y-4\,\bar{a}_z$ *using dot product and cross product.* **[Ans. : 18.21°]**

17. *Consider two vectors* $\overline{P}=4\bar{a}_y+10\bar{a}_z$ *and* $\overline{Q}=2\bar{a}_x+3\bar{a}_y$. *Find the projection of* $\overline{P}$ *and* $\overline{Q}$.

    [Ans. : 3.328]

18. *Given the points A (x = 2, y = 3, z = − 1) and* $B(r=4,\Phi=-50^\circ.z=2)$, *find the distance of A and B from the origin. Also find distance A to B.* **[Ans. : 3.74, 4.47, 6.78]**

19. *Given the two points A (x = 2, y = 3, z = − 1) and* $B(r=4,\theta=25^\circ.\phi=120^\circ)$. *Find the spherical coordinates of A, cartesian coordinates of B and distance AB.*

    [Ans. : A (3.74, 105.5°, 56.31°), B (− 0.845, 1.46, 3.627, 5.64)]

20. *Transform the vector* $5\,\bar{a}_x$ *at Q (x = 3, y = 4, z = − 2) to the cylindrical co-ordinates.*

    [Ans. : $3\,\bar{a}_r-4\,\bar{a}_\phi$]

21. *What is Laplacian of a scalar field ? What is its significance.*

22. *Find the Laplacian of the following scalar fields :*
    i) $W=e^{-z}\sin 2x\cosh y$ **[Ans. : $-2\,e^{-z}\sin 2x\cosh y$]**
    ii) $V=10\,r\sin^2\theta\cos\phi$ **[Ans. : $\frac{10\cos\phi}{r}(1+2\cos 2\theta)$]**

University Questions

1. *What do you mean by Scalar and Vector Fields ? Show the difference between the two.* **[UPTU : 2002-03, 5 Marks]**
2. *Give the physical interpretation of gradiant and curl of a vector.* **[UPTU : 2003-04(A), 5 Marks]**
3. *Represent the dot product of ∇ with vector field in spheircal co-ordinate system.* **[UPTU : 2003-04(B), 5 Marks]**
4. *Give the physical interpretation of gradient, divergence and curl of a vector field.* **[UPTU : 2003-04(B), 5 Marks]**
5. *Discuss the Stokes' theorem and its application.* **[UPTU : 2003-04(B), 5 Marks]**
6. *Verify that vector field* $\bar{A} = yz\bar{a}_x + zx\bar{a}_y + xy\bar{a}_z$ *is irrotational and solenoidal.* **[UPTU : 2005-06, 5 Marks]**
7. *Write down gradient of any scalar and divergence and curl of any vector.* $\bar{A}$ *in different co-ordiante system.* **[UPTU : 2006-07, 5 Marks]**
8. *If* $\bar{A} = \alpha\,\bar{a}_x + 2\,\bar{a}_y + 10\,\bar{a}_z$ *and* $\bar{B} = 4\alpha\,\bar{a}_x + 8\,\bar{a}_y - 2\alpha\,\bar{a}_z$, *find out the value of* α *for which the two vectors become perpendicular.* **[UPTU : 2006-07, 5 Marks]**
9. *Given ponts* $A(x = 2,\ y = 3,\ z = -1)$ *and* $B(\rho = 4,\ \phi = -50°,\ z = 2)$ *find the distance A to B.* **[UPTU : 2006-07, 5 Marks]**
10. *Show that the vector fields*

    $\bar{A} = \bar{a}_r \dfrac{\sin 2\theta}{r^2} + 2\bar{a}_\theta \dfrac{(\sin\theta)}{r^2}$ *and*

    $\bar{B} = r\cos\theta\bar{a}_r + r\bar{a}_\theta$ *are every where parallel to each other.* **[UPTU : 2007-08, 5 Marks]**
11. *Express the field* $\bar{E} = \dfrac{A}{r^2}\bar{a}_r$ *in (i) rectangular components, ii) cylindrical components.* **[UPTU : 2007-08, 5 Marks]**
12. *Write down the word statement of divergence theorem and Stokes theorem. Find out the divergence and curl of the following function.*

    $\bar{A} = 2xy\,\bar{a}_x + x^2 z\bar{a}_y + z^3\bar{a}_z$ **[UPTU : 2007-08, 5 Marks]**
13. *Establish the following vector identities :*

    i) $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\bullet\vec{C})\vec{B} - (\vec{A}\bullet\vec{B})\vec{C}$

    ii) $\nabla\cdot(\nabla\times\vec{A}) = 0$ **[UPTU : 2008-09, 10 Marks]**
14. *Discuss the following terms as applied to vector fields :*

    i) Gradient

    ii) Divergence

    iii) Curl and its physical interpretation **[UPTU : 2008-09, 10 Marks]**

2 Electric Field Intensity

2.1 Introduction

Electrostatics is a very important step in the study of engineering electromagnetics. Electrostatics is a science related to the electric charges which are static i.e. are at rest. An electric charge has its effect in a region or a space around it. This region is called an **electric field** of that charge. Such an electric field produced due to stationary electric charge does not vary with time. It is time invariant and called **static electric field**. The study of such time invariant electric fields in a space or vacuum, produced by various types of static charge distributions is called **electrostatics**. A very common example of such a field is a field used in cathode ray tube for focusing and deflecting a beam. Electrostatics plays a very important role in our day to day life. Most of the computer peripheral devices like keyboards, touch pads, liquid crystal displays etc. work on the principle of electrostatics. A variety of machines such as X-ray machine and medical instruments used for electrocardiograms, scanning etc. use the principle of electrostatics. Many industrial processes like spray painting, electrodeposition etc. also use the principle of electrostatics. Electrostatics is also used in the agricultural activities like sorting seeds, spraying to plants etc. Many components such as resistors, capacitors etc. and the devices such as bipolar transistors, field effect transistors function based on electrostatics. Hence this chapter introduces the basic concepts of electrostatics.

2.2 Coulomb's Law

The study of electrostatics starts with the study of the results of the experiements performed by an engineer from the French Army Engineers, **Colonel Charles Coulomb.** The experiments are related to the force exerted between the two point charges, which are placed near each other. The force exerted is due to the electric fields produced by the point charges.

A **point charge** means that electric charge which is spreaded on a surface or space whose geometrical dimensions are very very small compared to the other dimensions, in which the effect of its electric field is to be studied. Thus a point charge has a location but not the dimensions. A charge can be a positive or negative. A charge is actually the deficiency or excess of electrons in the atoms of a particle. An electron possesses a

negative charge. So the deficiency of an electron produces positive charge while excess of an electron produces negative charge. The charge is measured in **Coulombs (C).** The smellest possible charge is that corresponding to the charge on one electron which is 1.602×10^{-19} C. Hence **one Coulomb** of charge is defined as the charge possessed by $(1/1.602\times10^{-19})$ i.e. 6×10^{18} number of electrons. There can be an isolated positive or negative charge which exerts force on other charge placed in its vicinity. It is well known that the like charges repel while unlike charges attract each other. The Coulomb's law formulated in 1785 is related to such a force exerted by one charge on the other.

2.2.1 Statement of Coulomb's Law

The Coulomb's law states that force between the two point charges Q_1 and Q_2,

1. Acts along the line joining the two point charges.
2. Is directly proportional to the product $(Q_1 Q_2)$ of the two charges.
3. Is inversely proportional to the square of the distance between them.

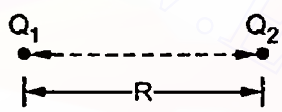


Fig. 2.1

Consider the two point charges Q_1 and Q_2 as shown in the Fig. 2.1, separated by the distance R. The charge Q_1 exerts a force on Q_2 while Q_2 also exerts a force on Q_1. The force acts along the line joining Q_1 and Q_2. The force exerted between them is repulsive if the charges are of same polarity while it is attractive if the charges are of different polarity.

Mathematically the force F between the charges can be expressed as,

$$\boxed{F \propto \frac{Q_1 Q_2}{R^2}} \qquad \text{... (1)}$$

where $\quad Q_1 Q_2 =$ Product of the two charges

$\quad R =$ Distance between the two charges

The Coulomb's law also states that this force depends on the **medium** in which the point charges are located. The effect of medium is introduced in the equation of force as a constant of proportionality denoted as k.

$$\therefore \qquad \boxed{F = k\frac{Q_1 Q_2}{R^2}} \qquad \text{... (2)}$$

where $\quad k =$ Constant of proportionality

2.2.1.1 Constant of Proportionality (k)

The constant of proportionality takes into account the effect of medium, in which charges are located. In the International System of Units (SI), the charges Q_1 and Q_2 are expressed in Coulombs (C), the distance R in metres (m) and the force F in newtons (N). Then to satisfy Coulomb's law, the constant of proportionality is defined as,

$$\boxed{k = \frac{1}{4\pi\varepsilon}} \qquad \text{... (3)}$$

where ε = Permittivity of the medium in which charges are located

The units of ε are farads/metre (F/m).

In general ε is expressed as,

$$\boxed{\varepsilon = \varepsilon_0 \varepsilon_r} \qquad \text{... (4)}$$

where ε_0 = Permittivity of the free space or vacuum

ε_r = Relative permittivity or dielectric constant of the medium with respect to free space

ε = Absolute permittivity

For the **free space or vacuum**, the relative permittivity $\varepsilon_r = 1$, hence

$$\varepsilon = \varepsilon_0$$

$$\therefore \quad F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2} \qquad \text{... (5)}$$

The value of permittivity of free space ε_0 is,

$$\boxed{\varepsilon_0 = \frac{1}{36\pi}\times 10^{-9} = 8.854\times 10^{-12} \text{ F/m}} \qquad \text{... (6)}$$

$$\therefore \quad k = \frac{1}{4\pi\varepsilon_0} = \frac{1}{4\pi\times 8.854\times 10^{-12}} = 8.98\times 10^{9} = 9\times 10^{9} \text{ m/F} \qquad \text{... (7)}$$

Hence the Coulomb's law can be expressed as,

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} \qquad \text{... (8)}$$

This is the force between the two point charges located in **free space or vacuum.**

Key Point: *As Q is measured in Coulomb, R in metre and F in newton, the units of* ε_0 *are,*

$$\text{Unit of } \varepsilon_0 = \frac{(C)\,(C)}{(N)\,(m^2)} = \frac{C^2}{N\text{-}m^2} = \frac{C^2}{N\text{-}m}\times\frac{1}{m}$$

But $\frac{C^2}{N\text{-}m}$ = Farad which is practical unit of capacitance

$$\therefore \quad \boxed{\text{Unit of } \varepsilon_0 = \text{F/m}}$$

2.2.2 Vector Form of Coulomb's Law

The force exerted between the two point charges has a fixed direction which is a straight line joining the two charges. Hence the force exerted between the two charges can be expressed in a vector form.

Consider the two point charges Q_1 and Q_2 located at the points having position vectors $\bar{r}_1$ and $\bar{r}_2$ as shown in the Fig. 2.2.

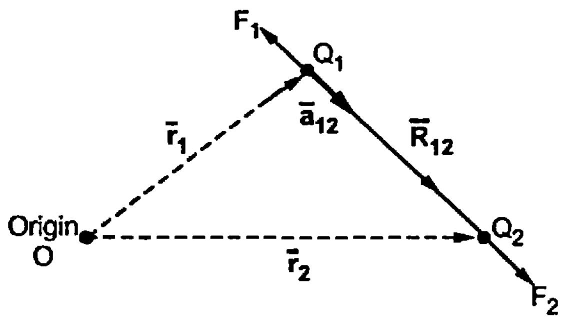


Fig. 2.2 Vector form of Coulomb's law

Then the force exerted by Q_1 on Q_2 acts along the direction $\bar{R}_{12}$ where $\bar{a}_{12}$ is unit vector along $\bar{R}_{12}$. Hence the force in the vector form can be expressed as,

$$\boxed{\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}} \quad \text{... (9)}$$

where $\bar{a}_{12}$ = Unit vector along $\bar{R}_{12} = \dfrac{\text{Vector}}{\text{Magnitude of vector}}$

$$\therefore \quad \bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|} \quad \text{... (10)}$$

where $|\bar{R}_{12}|$ = R = distance between the two charges

The following **observations** are important :

1. As shown in the Fig. 2.3, the force $\bar{F}_1$ is the force exerted on Q_1 due to Q_2. It can be expressed as,

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}\bar{a}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2} \times \frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|} \quad \text{... (11)}$$

But $\bar{r}_1 - \bar{r}_2 = -[\bar{r}_2 - \bar{r}_1]$

$\therefore \quad \bar{a}_{21} = -\bar{a}_{12}$

Hence substituting in (11),

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}(-\bar{a}_{21}) = -\bar{F}_2 \quad \text{... (12)}$$

Hence force exerted by the two charges on each other is equal but opposite in direction.

2. The like charges repel each other while the unlike charges attract each other. This is shown in the Fig. 2.3. These are experiement conclusions though not reflected in the mathematical expression.

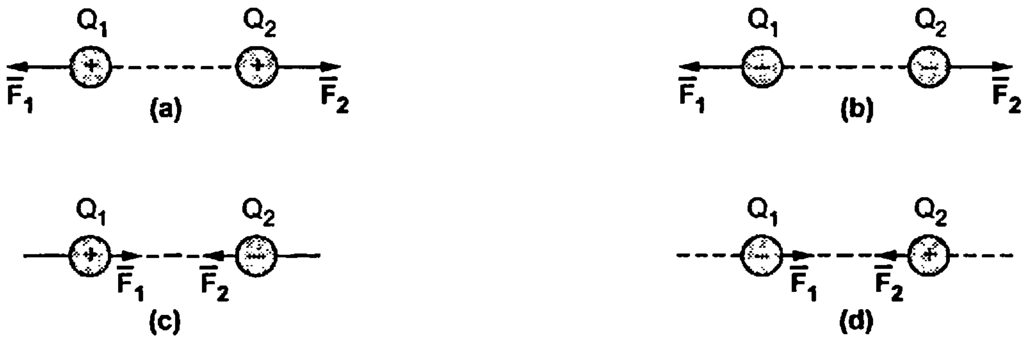


Fig. 2.3

3. It is necessary that the two charges are the point charges and stationary in nature.

4. The two point charges may be positive or negative. Hence their **signs** must be **considered** while using equation (9) to calculate the force exerted.

5. The Coulomb's law is linear which shows that if any one charge is increased 'n' times then the force exerted also increase by n times.

$$\therefore \quad \bar{F}_2 = -\bar{F}_1 \quad \text{then} \quad n\bar{F}_2 = -n\bar{F}_1$$

where $\quad n = \text{Scalar}$

2.2.3 Principle of Superposition

If there are more than two point charges, then each will exert force on the other, then the net force on any charge can be obtained by the **principle of superposition.**

Consider a point charge Q surrounded by three other point charges Q_1, Q_2 and Q_3, as shown in the Fig. 2.4.

The total force on Q in such a case is **vector sum** of all the forces exerted on Q due to each of the other point charges Q_1, Q_2 and Q_3.

Consider force exerted on Q due to Q_1. At this time, according to principle of superposition effects of Q_2 and Q_3 are to be suppressed.

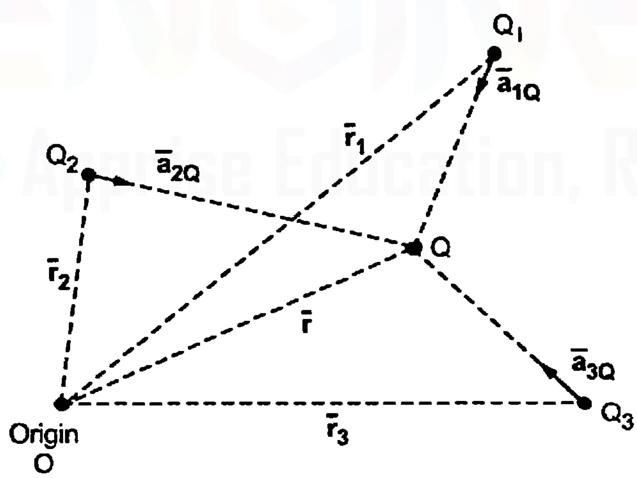


Fig. 2.4

$$\therefore \quad \bar{F}_{Q_1 Q} = \frac{Q_1 Q}{4\pi\varepsilon_0 R_{1Q}^2}\bar{a}_{1Q} \quad \text{... (13)}$$

where $\quad \bar{a}_{1Q} = \dfrac{\bar{r}-\bar{r}_1}{|\bar{r}-\bar{r}_1|}$

Similarly force exerted due to Q_2 on Q is,

$$\bar{F}_{Q_2Q} = \frac{Q_2Q}{4\pi\varepsilon_0 R_{2Q}^2}\bar{a}_{2Q} \quad \text{... (14)}$$

where $\bar{a}_{2Q} = \frac{\bar{r}-\bar{r}_2}{|\bar{r}-\bar{r}_2|}$

And force exerted due to Q_3 on Q is,

$$\bar{F}_{Q_3Q} = \frac{Q_3Q}{4\pi\varepsilon_0 R_{3Q}^2}\bar{a}_{3Q} \quad \text{... (15)}$$

where $\bar{a}_{3Q} = \frac{\bar{r}-\bar{r}_3}{|\bar{r}-\bar{r}_3|}$

Hence the total force on Q is,

$$\bar{F}_Q = \bar{F}_{Q_1Q} + \bar{F}_{Q_2Q} + \bar{F}_{Q_3Q} \quad \text{... (16)}$$

In general if there are n other charges then force exerted on Q due to all other n charges is,

$$\bar{F}_Q = \bar{F}_{Q_1Q} + \bar{F}_{Q_2Q} + \ldots\ldots + \bar{F}_{Q_nQ} \quad \text{... (17)}$$

$$\therefore \quad \boxed{\bar{F}_Q = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{Q_i}{R_{iQ}^2}\frac{\bar{r}-\bar{r}_i}{|\bar{r}-\bar{r}_i|}} \quad \text{... (18)}$$

2.2.4 Steps to Solve Problems on Coulomb's Law

Step 1 : Obtain the position vectors of the points where the charges are located.

Step 2 : Obtain the unit vector along the straight line joining the charges. The direction is towards the charge on which the force exerted is to be calculated.

Step 3 : Using Coulomb's law, express the force exerted in the vector form.

Step 4 : If there are more charges, repeat steps 1 to 3 for each charge exerting a force on the charge under consideration.

Step 5 : Using the principle of superposition, the vector sum of all the forces calculated earlier is the resultant force, exerted on the charge under consideration.

➡ **Example 2.1 :** *A charge $Q_1 = -20\ \mu C$ is located at P (– 6, 4, 6) and a charge $Q_2 = 50\ \mu C$ is located at R (5, 8, – 2) in a free space. Find the force exerted on Q_2 by Q_1 in vector form. The distances given are in metres.*

Solution : From the co-ordinates of P and R, the respective position vectors are –

$$\bar{P} = -6\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z$$

and $\quad \bar{R} = 5\bar{a}_x + 8\bar{a}_y - 2\bar{a}_z$

The force on Q_2 is given by,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

$$\bar{R}_{12} = \bar{R}_{PR} = \bar{R} - \bar{P} = [5-(-6)]\,\bar{a}_x + (8-4)\,\bar{a}_y + [-2-(6)\bar{a}_z]$$

$$= 11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z$$

$$\therefore \quad |R_{12}| = \sqrt{(11)^2 + (4)^2 + (-8)^2} = 14.1774$$

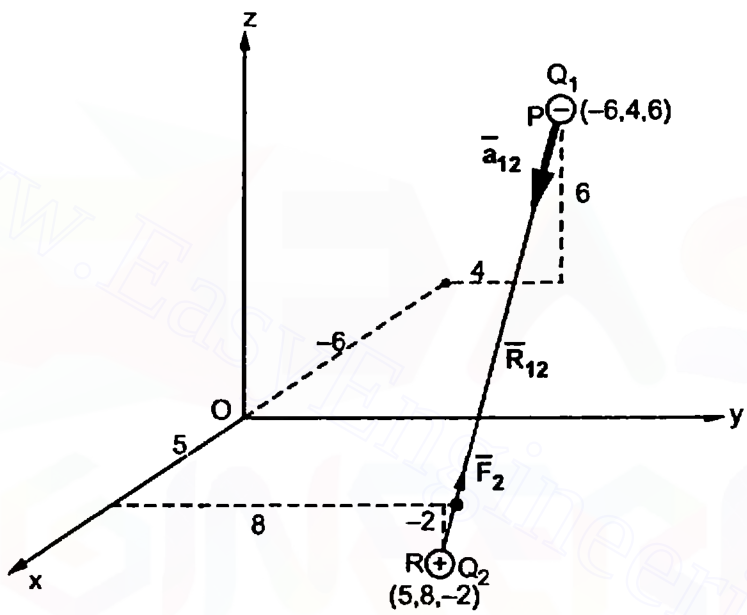


Fig. 2.5

$$\therefore \quad \bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z}{14.1774}$$

$$\therefore \quad \bar{a}_{12} = 0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z$$

$$\therefore \quad \bar{F}_2 = \frac{-20\times 10^{-6}\times 50\times 10^{-6}}{4\pi\times 8.854\times 10^{-12}\times (14.1774)^2}[\bar{a}_{12}]$$

$$= -0.0447\,[0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z] \quad \ldots (1)$$

$$= -0.0346\,\bar{a}_x - 0.01261\,\bar{a}_y + 0.02522\,\bar{a}_z \ \text{N} \quad \ldots (2)$$

This is the required force exerted on Q_2 by Q_1.

The magnitude of the force is,

$$|\bar{F}_2| = \sqrt{(0.0346)^2 + (0.01261)^2 + (-0.02522)^2} = \mathbf{44.634\ mN}$$

Key Point: *Note that as the two charges are of opposite polarity, the force $\bar{F}_2$ is attractive in nature. As shown in the Fig. 2.5, it acts in opposite direction to $\bar{a}_{12}$, which is indicated by negative sign in the equation (A).*

Example 2.2 : *Four point charges each of 10 μC are placed in free space at the points (1, 0, 0), (–1, 0, 0), (0, 1, 0) and (0, – 1, 0) m respectively. Determine the force on a point charge of 30 μC located at a point (0, 0, 1) m.*

Solution : Use the principle of superposition as there are four charges exerting a force on the fifth charge. The locations of charges are shown in the Fig. 2.6.

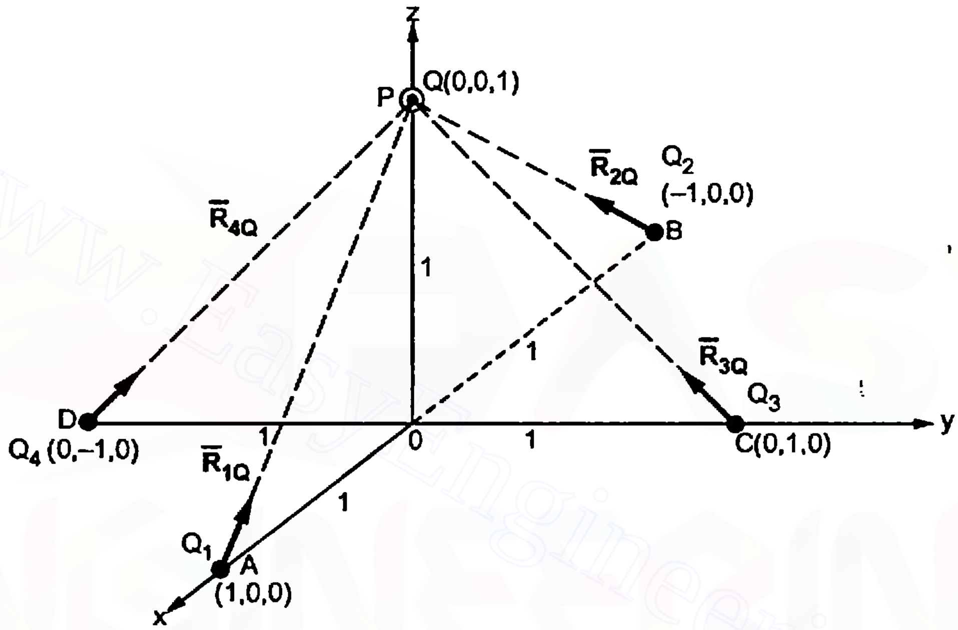


Fig. 2.6

The position vectors of four points at which the charges Q_1 to Q_4 are located can be obtained as,

$$\bar{A} = \bar{a}_x\,,\quad \bar{B} = -\bar{a}_x\,,\quad \bar{C} = \bar{a}_y \quad \text{and} \quad \bar{D} = -\bar{a}_y$$

while position vector of point P where charge of 30 μC is situated is,

$$\bar{P} = \bar{a}_z$$

Consider force on Q due to Q_1 alone,

$$\bar{F}_1 = \frac{QQ_1}{4\pi\varepsilon_0 R_{1Q}^2}\,\bar{a}_{1Q} = \frac{QQ_1}{4\pi\varepsilon_0 R_{1Q}^2}\cdot\frac{\bar{R}_{1Q}}{|\bar{R}_{1Q}|}$$

where $\bar{R}_{1Q} = \bar{P} - \bar{A} = \bar{a}_z - \bar{a}_x$ and $|\bar{R}_{1Q}| = \sqrt{1^2+1^2} = \sqrt{2}$

$$\therefore \quad \bar{F}_1 = \frac{30\times10^{-6}\times10\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(\sqrt{2})^2}\left[\frac{\bar{a}_z - \bar{a}_x}{\sqrt{2}}\right]$$

$$= 0.9533\,[\bar{a}_z - \bar{a}_x] \qquad \ldots (1)$$

It can be seen from the Fig. 2.6 that due to symmetry,

$$|\bar{R}_{1Q}| = |\bar{R}_{2Q}| = |\bar{R}_{3Q}| = |\bar{R}_{4Q}| = \sqrt{2}$$

Now $\bar{R}_{2Q} = \bar{P}-\bar{B} = \bar{a}_z + \bar{a}_x, \quad \bar{a}_{2Q} = \bar{a}_z + \bar{a}_x/\sqrt{2}$

$\bar{R}_{3Q} = \bar{P}-\bar{C} = \bar{a}_z - \bar{a}_y, \quad \bar{a}_{3Q} = \bar{a}_z - \bar{a}_y/\sqrt{2}$

$\bar{R}_{4Q} = \bar{P}-\bar{D} = \bar{a}_z + \bar{a}_y, \quad \bar{a}_{4Q} = \bar{a}_z + \bar{a}_y/\sqrt{2}$

$\therefore \quad \bar{F}_2 = \text{Force on Q due to } Q_2 = \dfrac{QQ_2}{4\pi\varepsilon_0 R_{2Q}^2}\bar{a}_{2Q}$

$\therefore \quad \bar{F}_3 = \text{Force on Q due to } Q_3 = \dfrac{QQ_3}{4\pi\varepsilon_0 R_{3Q}^2}\bar{a}_{3Q}$

$\therefore \quad \bar{F}_4 = \text{Force on Q due to } Q_4 = \dfrac{QQ_4}{4\pi\varepsilon_0 R_{4Q}^2}\bar{a}_{4Q}$

$$\frac{QQ_2}{4\pi\varepsilon_0 R_{2Q}^2} = \frac{QQ_3}{4\pi\varepsilon_0 R_{3Q}^2} = \frac{QQ_4}{4\pi\varepsilon_0 R_{4Q}^2} = \frac{30\times10^{-6}\times10\times10^{-6}}{4\pi\times 8.854\times10^{-12}\times(\sqrt{2})^2} = 1.3481$$

$$\therefore \quad \bar{F}_2 = 1.3481\left[\frac{\bar{a}_z + \bar{a}_x}{\sqrt{2}}\right] = 0.9533\,(\bar{a}_z + \bar{a}_x) \quad \text{... (2)}$$

$$\therefore \quad \bar{F}_3 = 1.3481\left[\frac{\bar{a}_z - \bar{a}_y}{\sqrt{2}}\right] = 0.9533\,(\bar{a}_z - \bar{a}_y) \quad \text{... (3)}$$

$$\therefore \quad \bar{F}_4 = 1.3481\left[\frac{\bar{a}_z + \bar{a}_y}{\sqrt{2}}\right] = 0.9533\,(\bar{a}_z + \bar{a}_y) \quad \text{... (4)}$$

Hence the total force F_t exerted on Q due to all four charges is vector sum of the individual forces exerted on Q, by the charges.

$$\therefore \quad \bar{F}_t = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= 0.9533\,[\bar{a}_z - \bar{a}_x + \bar{a}_z + \bar{a}_x + \bar{a}_z - \bar{a}_y + \bar{a}_z + \bar{a}_y] = 3.813\,\bar{a}_z \text{ N}$$

2.3 Electric Field Intensity

Consider a point charge Q_1 as shown in Fig. 2.7 (a).

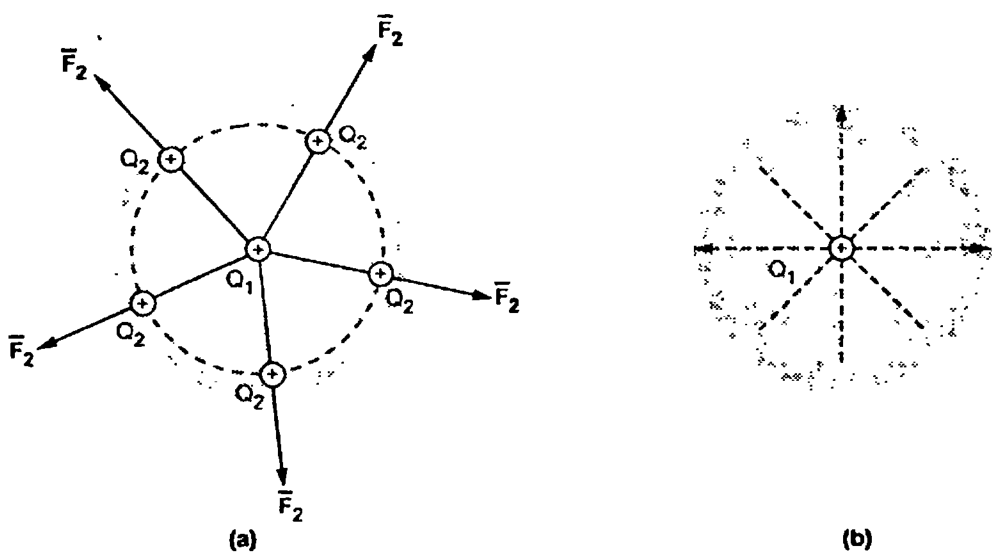


Fig. 2.7 Electric field

If any other similar charge Q_2 is brought near it, Q_2 experiences a force. Infact if Q_2 is moved around Q_1, still Q_2 experiences a force as shown in the Fig. 2.7 (a).

Thus there exists a region around a charge in which it exerts a force on any other charge. This region where a particular charge exerts a force on any other charge located in that region is called **electric field** of that charge. The electric field of Q_1 is shown in the Fig. 2.7 (b).

The force experienced by the charge Q_2 due to Q_1 is given by Coulomb's law as,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

Thus force per unit charge can be written as,

$$\frac{\bar{F}_2}{Q_2} = \frac{Q_1}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12} \quad \text{... (1)}$$

This force exerted per unit charge is called **electric field intensity** or **electric field strength**. It is a **vector quantity** and is directed along a segment from the charge Q_1 to the position of any other charge. It is denoted as $\bar{E}$.

$$\therefore \quad \boxed{\bar{E} = \frac{Q_1}{4\pi\varepsilon_0 R_{1p}^2}\bar{a}_{1p}} \quad \text{... (2)}$$

where $\quad p$ = Position of any other charge around Q_1

The equation (2) is the electric field intensity due to a single point charge Q_1 in a free space or vacuum.

Another definition of electric field intensity is the force experienced by a unit positive test charge i.e. Q_2 = 1C.

Consider a charge Q_1 as shown in the Fig. 2.8. The unit positive charge Q_2 = 1C is placed at a distance R from Q_1. Then the force acting on Q_2 due to Q_1 is along the unit vector $\bar{a}_R$. As the charge Q_2 is **unit charge,** the force exerted on Q_2 is nothing but electric field intensity $\bar{E}$ of Q_1 at the point where unit charge is placed.

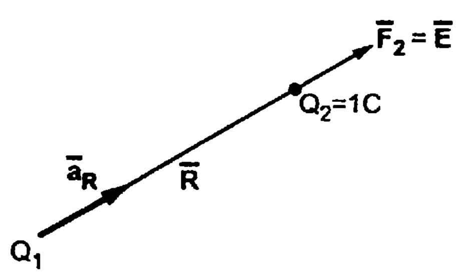


Fig. 2.8 Concept of electric field intensity

$$\therefore \quad \boxed{\bar{E} = \frac{Q_1}{4\pi\varepsilon_0 R^2}\bar{a}_R} \quad \text{... (3)}$$

If a charge Q_1 is located at the center of the spherical coordinate system then unit vector $\bar{a}_R$ in the equation (3) becomes the radial unit vector $\bar{a}_r$ coming radially outwards from Q_1. And the distance R is the radius of the sphere r.

$$\therefore \quad \bar{E} = \frac{Q_1}{4\pi\varepsilon_0 r^2}\,\bar{a}_r \text{ in spherical system} \quad \text{... (4)}$$

2.3.1 Units of $\bar{E}$

The definition of electric field intensity is,

$$\bar{E} = \frac{\text{Force}}{\text{Unit charge}} = \frac{\text{(N) Newtons}}{\text{(C) Coulomb}}$$

Hence units of $\bar{E}$ is N/C. But the electric potential has units J/C i.e. Nm/C and hence $\bar{E}$ is also measured in units **V/m (volts per metre)**. This unit is used practically to express $\bar{E}$.

2.3.2 Method of Obtaining $\bar{E}$ in Cartesian System

Consider a charge Q_1 located at point $A(x_1, y_1, z_1)$ as shown in the Fig. 2.9. It is required to obtain $\bar{E}$ at any point $B(x, y, z)$ in the cartesian system. Then $\bar{E}$ at point B can be obtained using following steps :

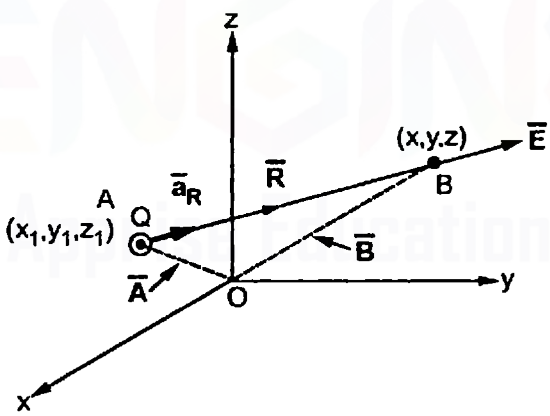


Fig. 2.9 $\bar{E}$ in cartesian system

Step 1 : Obtain the position vectors of points A and B.

$\therefore \quad \bar{r}_A = \bar{A}$ while $\bar{r}_B = \bar{B}$ from their co-ordinates

$$\therefore \quad \bar{A} = x_1\,\bar{a}_x + y_1\,\bar{a}_y + z_1\,\bar{a}_z \quad \text{and}$$

$$\bar{B} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z.$$

Step 2 : Find the distance vector $\bar{R}$ directed from A to B.

$$\therefore \quad \bar{R} = \bar{B} - \bar{A} = (x - x_1)\bar{a}_x + (y - y_1)\bar{a}_y + (z - z_1)\bar{a}_z$$

Step 3 : Find the unit vector $\bar{a}_R$ along the direction from A to B.

$$\therefore \quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{\bar{B} - \bar{A}}{|\bar{B} - \bar{A}|}$$

Step 4 : Obtain the $\bar{E}$ at the point B as,

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{Q}{4\pi\varepsilon_0 R^2}\frac{\bar{R}}{|\bar{R}|} \text{ V/m}$$

where $R^2 = |\bar{R}|^2$ = Distance between the points A and B

Step 5 : Magnitude of $\bar{E}$ is given by,

$$|\bar{E}| = \frac{Q}{4\pi\varepsilon_0 R^2} \text{ V/m}$$

Substituting $\bar{R}$ and $|\bar{R}|$ interms of the cartesian co-ordinates of A and B, the required electric field intensity $\bar{E}$ at the point B can be obtained.

2.3.3 Electric Field due to Discrete Charges

Similar to a force exerted on a charge due to n number of charges is the vector sum of the individual force exerted by each charge, the electric field at a point due to n number of charges is to be obtained using law of superposition.

Consider n charges $Q_1, Q_2 \ldots Q_n$ as shown in the Fig. 2.10. The combined electric field intensity is to be obtained at point P. The distances of point P from $Q_1, Q_2 \ldots Q_n$ are $R_1, R_2, \ldots R_n$ respectively. The unit vectors along these directions are $\bar{a}_{R1}, \bar{a}_{R2} \ldots\ldots \bar{a}_{Rn}$ respectively.

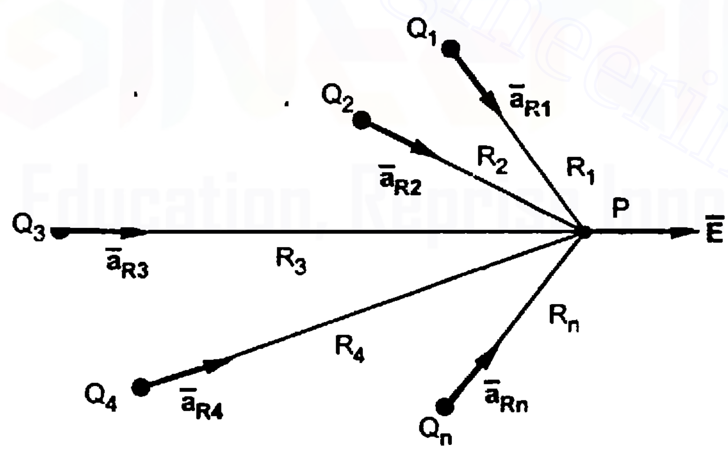


Fig. 2.10 $\bar{E}$ due to n number of charges

Then the total electric field intensity at point P is the vector sum of the individual field intensities produced by the various charges at the point P.

$$\therefore \quad \bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \ldots + \bar{E}_n$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2} + \ldots + \frac{Q_n}{4\pi\varepsilon_0 R_n^2}\bar{a}_{Rn}$$

$$\therefore \quad \boxed{\bar{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{Q_i}{R_i^2} \bar{a}_{Ri}} \quad \ldots (5)$$

Each unit vector can be obtained by using the method discussed earlier.

$$\bar{a}_{Ri} = \frac{\bar{r}_P - \bar{r}_i}{|\bar{r}_P - \bar{r}_i|}$$

where $\bar{r}_P$ = Position vector of point P

$\bar{r}_i$ = Position vector of point where charge Q_i is placed.

2.3.4 Important Observations

The important observations related to $\bar{E}$ are,

1. $\bar{E}$ around a charge Q_1 is directly proportional to the charge Q_1.
2. $\bar{E}$ around a charge Q_1 is inversely proportional to the distance between charge Q_1 and point at which $\bar{E}$ is to be calculated. More is the distance less is the electric field intensity and less will be the force experience by a unit charge placed at that point.
3. The field intensity $\bar{E}$ at any point and force $\bar{F}$ exerted on a charge placed at the same point are always in the same direction.
4. Placing unit charge is a method of detecting the presence of electric field around a charge. Without any unit test charge placed nearby, every charge has its electric field always existing, around it.
5. The test charge placed must be small enough so that the electric field intensity $\bar{E}$ to be measured should not get disturbed.

Example 2.3 : *Determine the electric field intensity at P(– 0.2, 0, – 2.3) m due to a point charge of +5 nC at Q(0.2, 0.1, – 2.5) m in air.*

Solution :

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 R^2} \bar{a}_R$$

$$\bar{a}_R = \frac{\bar{R}_{QP}}{|\bar{R}_{QP}|} = \frac{\bar{P}-\bar{Q}}{|\bar{P}-\bar{Q}|}$$

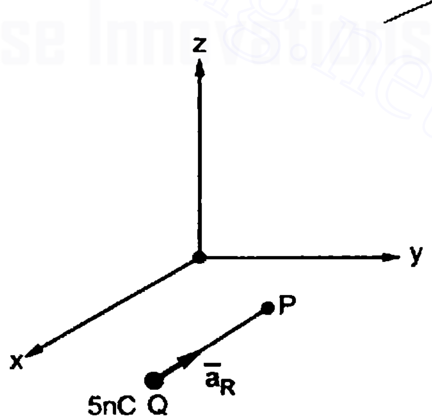


Fig. 2.11

$$\bar{P}-\bar{Q} = (-0.2-0.2)\bar{a}_x + (0-0.1)\bar{a}_y + [-2.3-(-2.5)]\bar{a}_z$$

$$= -0.4\bar{a}_x - 0.1\bar{a}_y + 0.2\bar{a}_z$$

$$\therefore \quad \bar{a}_R = \frac{-0.4\bar{a}_x - 0.1\bar{a}_y + 0.2\bar{a}_z}{\sqrt{(-0.4)^2 + (0.1)^2 + (0.2)^2}}$$

$$= \frac{-0.4\bar{a}_x - 0.1\bar{a}_y + 0.2\bar{a}_z}{0.45825}$$

$$= -0.8728\,\bar{a}_x - 0.2182\,\bar{a}_y + 0.4364\,\bar{a}_z$$

$$\therefore \quad R = \left|\bar{P} - \bar{Q}\right| = 0.45825$$

$$\therefore \quad \bar{E} = \frac{5\times10^{-9}}{4\pi\times 8.854\times10^{-12}\times(0.45825)^2}\,[\bar{a}_R] = 214\,\bar{a}_R$$

Substituting value of $\bar{a}_R$,

$$\bar{E} = -186.779\,\bar{a}_x - 46.694\,\bar{a}_y + 93.389\,\bar{a}_z \ \text{V/m}$$

This is electric field intensity at point P.

Example 2.4 : *Calculate the field intensity at a point on a sphere of radius 3 m, if a positive charge of 2 μC is placed at the origin of the sphere.*

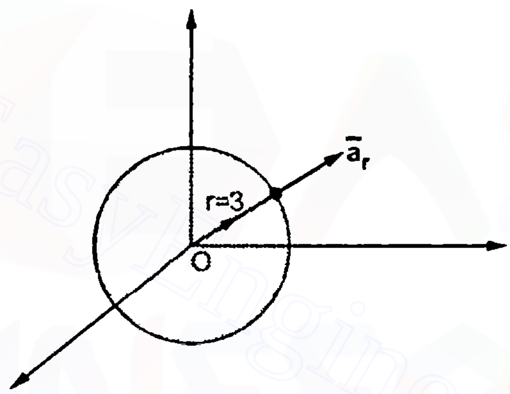


Fig. 2.12

Solution : Let us use spherical co-ordinate system.

The sphere of radius r = 3 m.

$$\therefore \quad R = r = 3 \text{ m}$$

And $\bar{E}$ acts radially outwards along the unit vector $\bar{a}_r$ in spherical co-ordinate system.

$$\therefore \quad \bar{a}_R = \bar{a}_r \quad \text{in this case.}$$

$$\therefore \quad \bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\bar{a}_r$$

$$\therefore \quad \bar{E} = \frac{2\times10^{-6}}{4\pi\times 8.854\times10^{-12}\times(3)^2}\,\bar{a}_r$$

$$= \mathbf{1.9972\,\bar{a}_r \ kV/m}$$

Note that in this case $\bar{a}_r$ is specifically unit vector in spherical co-ordinate system, which is special case of general $\bar{a}_R$.

Example 2.5 : *A charge of 1 C is at (2, 0, 0). What charge must be placed at (– 2, 0, 0) which will make y component of total $\bar{E}$ zero at the point (1, 2, 2) ?*

Solution : The various points and charges are shown in the Fig. 2.13.

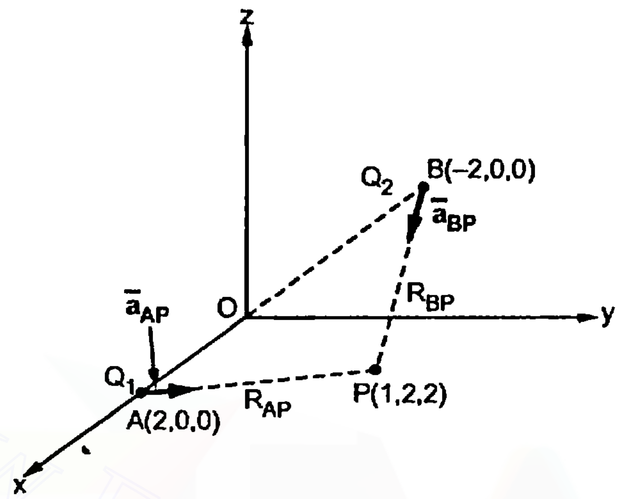


Fig. 2.13

The position vectors of points A, B and P are,

$$\bar{A} = 2\bar{a}_x \ , \ \bar{B} = -2\bar{a}_x$$

$$\bar{P} = \bar{a}_x + 2\bar{a}_y + 2\bar{a}_z$$

$\bar{E}_A$ is field at P due to Q_1, and will act along $\bar{a}_{AP}$. $\bar{E}_B$ is field at P due to Q_2 and will act along $\bar{a}_{BP}$.

$$\therefore \quad \bar{E}_A = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2}\bar{a}_{AP} = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2} \times \frac{\bar{P}-\bar{A}}{|\bar{P}-\bar{A}|}$$

$$\therefore \quad \bar{E}_B = \frac{Q_2}{4\pi\varepsilon_0 R_{BP}^2}\bar{a}_{BP} = \frac{Q_2}{4\pi\varepsilon_0 R_{BP}^2} \times \frac{\bar{P}-\bar{B}}{|\bar{P}-\bar{B}|}$$

$$\therefore \ \bar{E} \text{ at } P = \bar{E}_A + \bar{E}_B = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{R_{AP}^2}\frac{\bar{P}-\bar{A}}{|\bar{P}-\bar{A}|} + \frac{Q_2}{R_{BP}^2}\frac{\bar{P}-\bar{B}}{|\bar{P}-\bar{B}|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1[-\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z]}{(\sqrt{9})^2\sqrt{(1)^2+(2)^2+(2)^2}} + \frac{Q_2[3\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z]}{(\sqrt{17})^2\sqrt{(3)^2+(2)^2+(2)^2}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z}{27} + \frac{Q_2[3\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z]}{70.0927}\right]$$

The y component of $\bar{E}$ must be zero.

$$\therefore \quad \frac{2}{27} + \frac{2Q_2}{70.0927} = 0$$

$$\therefore \qquad Q_2 = -\frac{2}{27} \times \frac{70.0927}{2} = -2.596 \text{ C}$$

This is the required charge Q_2 to be placed at $(-2, 0, 0)$ which will make y component of $\overline{E}$ zero at point P.

2.4 Types of Charge Distributions

Uptill now the forces and electric fields due to only point charges are considered. In addition to the **point charges,** there is possibility of continuous charge distributions along a line, on a surface or in a volume. Thus there are four types of charge distributions which are,

1. Point charge 2. Line charge 3. Surface charge 4. Volume charge

2.4.1 Point Charge

It is seen that if the dimensions of a surface carrying charge are very very small compared to region surrounding it then the surface can be treated to be a point. The corresponding charge is called **point charge**. The point charge has a position but not the dimensions. This is shown in the Fig. 2.14 (a). The point charge can be positive or negative.

2.4.2 Line Charge

It is possible that the charge may be spreaded all along a line, which may be finite or infinite. Such a charge uniformly distributed along a line is called a **line charge.** This is shown in the Fig. 2.14 (b).

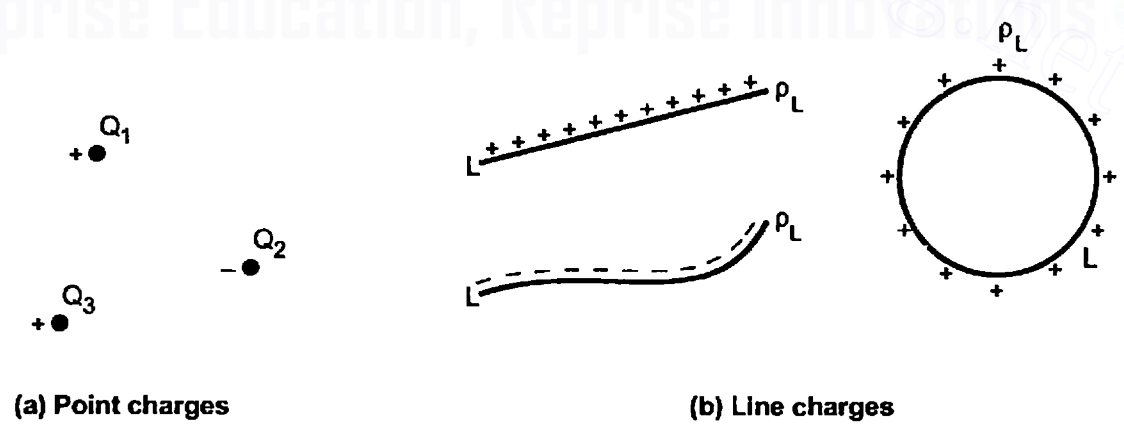


Fig. 2.14 Charge distributions

The charge density of the line charge is denoted as ρ_L and defined as charge per unit length.

$$\therefore \quad \boxed{\rho_L = \frac{\text{Total charge in coulomb}}{\text{Total length in metres}} \ (\text{C/m})}$$

Thus ρ_L is measured in C/m. The ρ_L is constant all along the length L of the line carrying the charge.

2.4.2.1 Method of Finding Q from ρ_L

In many cases, ρ_L is given to be the function of coordinates of the line i.e. $\rho_L = 3x$ or $\rho_L = 4y^2$ etc. In such a case it is necessary to find the total charge Q by considering differential length dl of the line. Then by integrating the charge dQ on dl, *for the entire length, total charge Q is to be obtained. Such an integral is called* **line integral.**

Mathematically, $dQ = \rho_L \, dl$ = charge on differential length dl

$$\therefore \quad Q = \int_L dQ = \int_L \rho_L \, dl \quad \text{... (1)}$$

If the line of length L is a closed path as shown in the Fig. 2.14 (b) then integral is called closed contour integral and denoted as,

$$\boxed{Q = \oint_L \rho_L \, dl} \quad \text{... (2)}$$

A sharp beam in a cathode ray tube or a charged circular loop of conductor are the examples of line charge. The charge distributed may be positive or negative along a line.

Example 2.6 : *A charge is distributed on x axis of cartesian system having a line charge density of $3x^2$ μC/m. Find the total charge over the length of 10 m.*

Solution : Given $\rho_L = 3x^2$ μC/m and L = 10 m along x axis.

The differential length be dl = dx in x direction and corresponding charge is $dQ = \rho_L \, dl = \rho_L \, dx$.

$$\therefore \quad Q = \int_L \rho_L \, dl = \int_0^{10} 3x^2 \, dx = \left[\frac{3x^3}{3}\right]_0^{10}$$

$$= 1000 \ \mu C = \mathbf{1 \ mC}$$

2.4.3 Surface Charge

If the charge is distributed uniformly over a two dimensional surface then it is called a **surface charge or a sheet of charge.** The surface charge is shown in the Fig. 2.15.

The two dimensional surface has area in square metres. Then the surface charge density is denoted as ρ_S and defined as the charge per unit surface area.

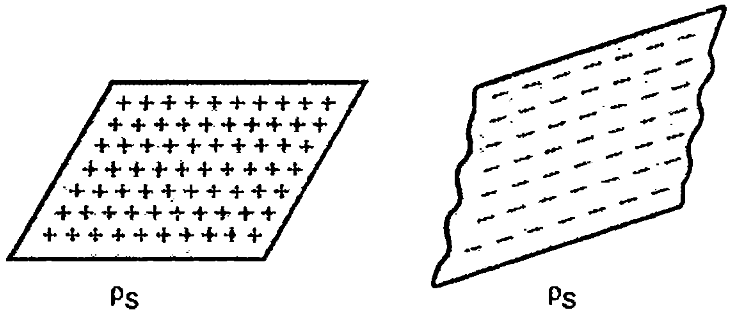


Fig. 2.15 Surface charge distributions

$$\therefore \quad \boxed{\rho_S = \frac{\text{Total charge in coulomb}}{\text{Total area in square metres}} \left(C/m^2\right)}$$

Thus ρ_S is expressed in C/m^2. The ρ_S is constant over the surface carrying the charge.

2.4.3.1 Method of Finding Q from ρ_S

In case of surface charge distribution, it is necessary to find the total charge Q by considering elementary surface area dS. The charge dQ on this differential area is given by ρ_S dS. Then integrating this dQ over the given surface, the total charge Q is to be obtained. Such an integral is called a **surface integral** and mathematically given by,

$$\boxed{Q = \int_S dQ = \int_S \rho_S \, dS} \quad \text{... (3)}$$

The plate of a charged parallel plate capacitor is an example of surface charge distribution. If the dimensions of the sheet of charge are very large compared to the distance at which the effects of charge are to be considered then the distribution is called infinite sheet of charge.

2.4.4 Volume Charge

If the charge distributed uniformly in a volume then it is called **volume charge**. The volume charge is shown in the Fig. 2.16.

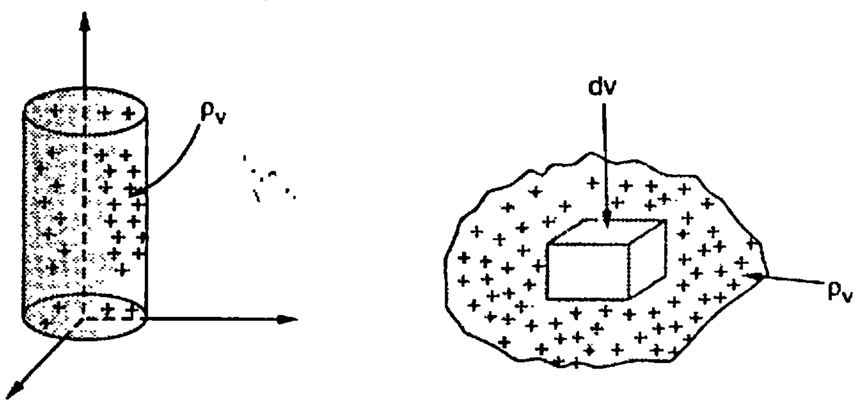


Fig. 2.16 Volume charge distribution

The **volume charge density** is denoted as ρ_v and defined as the charge per unit volume.

$$\boxed{\rho_v = \frac{\text{Total charge in coulomb}}{\text{Total volume in cubic metres}} \left(\frac{C}{m^3}\right)}$$

Thus ρ_v is expressed in C/m^3.

2.4.4.1 Method of Finding Q from ρ_v

In case of volume charge distribution, consider the differential volume dv as shown in the Fig. 2.16. Then the charge dQ possessed by the differential volume is $\rho_v dv$. Then the total charge within the finite given volume is to be obtained by integrating the dQ throughout that volume. Such an integral is called **volume integral.** Mathematically it is given by,

$$\boxed{Q = \int_{vol} \rho_v \, dv} \quad \text{... (4)}$$

The charged cloud is an example of volume charge.

Key Point: *In all the integrals line, surface and volume a single integral sign is used but practically for surface integral it becomes double integration while to integrate throughout the volume it becomes triple integration. Similarly ρ_S and ρ_v can be functions of the co-ordinates of the system used e.g. $\rho_S = 4xy\ C/m^2$, $\rho_v = 20z\, e^{-0.2y}\ C/m^3$ etc.*

Example 2.7 : *A volume charge density is expressed as $\rho_v = 10z^2 x \sin \pi y$. Find the total charge inside the volume $(-1 \le x \le 2)$, $(0 \le y \le 1)$, $(3 \le z \le 3.6)$*

[UPTU, 2003-04, 5 Marks]

Solution : $\rho_v = 10z^2 x \sin \pi y \quad C/m^3$

Consider differential volume in cartesian system as,

$$dv = dx\, dy\, dz$$

$$\therefore \quad dQ = \rho_v\, dv = 10z^2 x \sin \pi y \;\; dx\, dy\, dz$$

$$\therefore \quad Q = \int_{vol} \rho_v\, dv = \int_{z=3}^{3.6} \int_{y=0}^{1} \int_{x=-1}^{2} 10z^2 x \sin \pi y \; dx\, dy\, dz$$

$$= 10\left[\frac{z^3}{3}\right]_3^{3.6} \left[\frac{x^2}{2}\right]_{-1}^{2} \left[\frac{-\cos \pi y}{\pi}\right]_0^1$$

$$= 10\left[\frac{3.6^3}{3} - \frac{3^3}{3}\right]\left[\frac{4}{2} - \frac{1}{2}\right]\left[-\frac{\cos \pi}{\pi} + \frac{\cos 0}{\pi}\right]$$

$$= \mathbf{62.57\ C}$$

2.5 Electric Field Intensity due to Various Charge Distributions

It is known that the electric field intensity due to a point charge Q is given by,

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

Let us consider various charge distributions.

2.5.1 $\bar{E}$ due to Line Charge

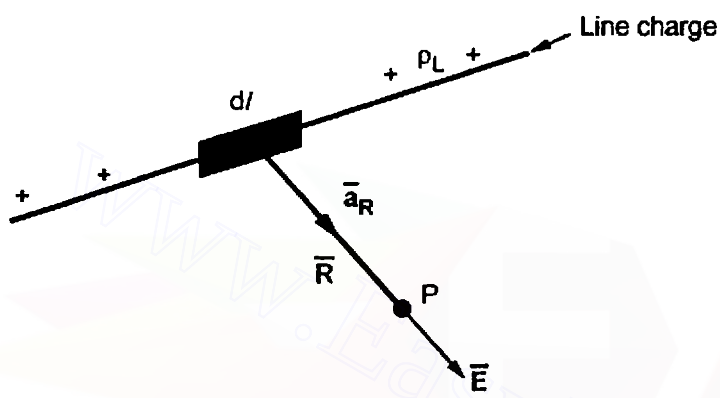


Fig. 2.17

Consider a line charge distribution having a charge density ρ_L as shown in the Fig. 2.17.

The charge dQ on the differential length dl is,

$$dQ = \rho_L \, dl$$

Hence the **differential electric field** d$\bar{E}$ at point P due to dQ is given by,

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2}\bar{a}_R \quad \text{... (1)}$$

Hence the total $\bar{E}$ at a point P due to line charge can be obtained by integrating d$\bar{E}$ over the length of the charge.

$$\therefore \quad \boxed{\bar{E} = \int_L \frac{\rho_l \, dl}{4\pi\varepsilon_0 R^2}\bar{a}_R} \quad \text{... (2)}$$

The $\bar{a}_R$ and dl is to be obtained depending upon the co-ordinate system used.

2.5.2 $\bar{E}$ due to Surface Charge

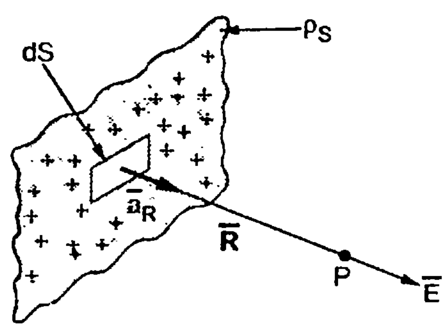


Fig. 2.18

Consider a surface charge distribution having a charge density ρ_S as shown in the Fig. 2.18.

The charge dQ on the differential surface area dS is,

$$dQ = \rho_S \, dS$$

Hence the differential electric field $d\bar{E}$ at a point P due to dQ is given by,

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{\rho_S\, dS}{4\pi\varepsilon_0 R^2}\bar{a}_R \quad \text{... (3)}$$

Hence the total $\bar{E}$ at a point P is to be obtained by integrating $d\bar{E}$ over the surface area on which charge is distributed. Note that this will be a double integration.

$$\therefore \quad \boxed{\bar{E} = \int_S \frac{\rho_S\, dS}{4\pi\varepsilon_0 R^2}\bar{a}_R} \quad \text{... (4)}$$

The $\bar{a}_R$ and dS to be obtained according to the position of the sheet of charge and the coordinate system used.

2.5.3 $\bar{E}$ due to Volume Charge

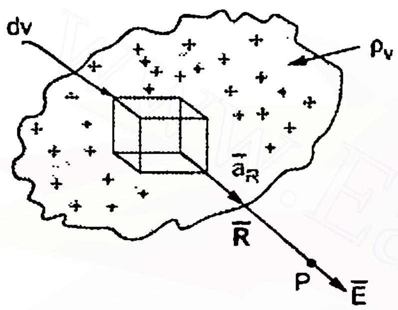


Fig. 2.19

Consider a volume charge distribution having a charge density ρ_v as shown in the Fig. 2.19.

The charge dQ on the differential volume dv is,

$$dQ = \rho_v\, dv$$

Hence the differential electric field $d\bar{E}$ at a point P due to dQ is given by,

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{\rho_v\, dv}{4\pi\varepsilon_0 R^2}\bar{a}_R \quad \text{... (5)}$$

Hence the total $\bar{E}$ at a point P is to be obtained by integrating $d\bar{E}$ over the volume in which charge is accumulated. Note that this integration will be a triple integration.

$$\therefore \quad \boxed{\bar{E} = \int_{Vol} \frac{\rho_v\, dv}{4\pi\varepsilon_0 R^2}\bar{a}_R} \quad \text{... (6)}$$

The $\bar{a}_R$ and dv must be obtained according to the co-ordinate system used.

Thus if there are all possible types of charge distributions, then the total $\bar{E}$ at a point is the vector sum of individual electric field intensities produced by each of the charges at a point under consideration.

$$\therefore \quad \bar{E}_{total} = \bar{E}_P + \bar{E}_l + \bar{E}_S + \bar{E}_v \quad \text{... (7)}$$

where $\bar{E}_P, \bar{E}_l, \bar{E}_S$ and $\bar{E}_v$ are the field intensities due to point, line, surface and volume charge distributions respectively.

Let us discuss and learn the method of obtaining electric field intensities under widely varying charge distributions.

2.6 Electric Field due to Infinite Line Charge

Consider an infinitely long straight line carrying uniform line charge having density ρ_L C/m. Let this line lies along z-axis from $-\infty$ to ∞ and hence called **infinite line charge** Let point P is on y-axis at which electric field intensity is to be determined. The distance of point P from the origin is 'r' as shown in the Fig. 2.20.

Consider a small differential length d*l* carrying a charge dQ, along the line as shown in the Fig. 2.20. It is along z axis hence d*l* = dz.

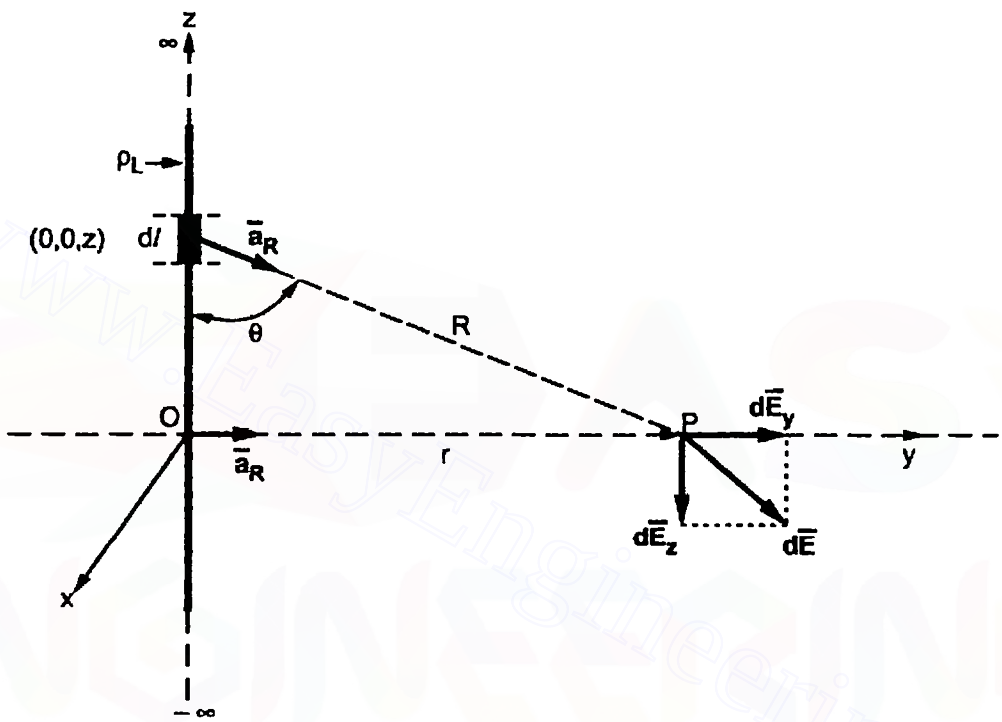


Fig. 2.20 Field due to infinite line charge

$$\therefore dQ = \rho_L\, dl = \rho_L\, dz \qquad \text{... (1)}$$

The co-ordinates of dQ are (0, 0, z) while the co-ordinates of point P are (0, r, 0). Hence the distance vector $\bar{R}$ can be written as,

$$\bar{R} = \bar{r}_P - \bar{r}_{dl} = [r\bar{a}_y - z\bar{a}_z]$$

$$\therefore |\bar{R}| = \sqrt{r^2 + z^2}$$

$$\therefore \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{r\bar{a}_y - z\bar{a}_z}{\sqrt{r^2+z^2}} \qquad \text{... (2)}$$

$$\therefore d\,\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$= \frac{\rho_L\, dz}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\left[\frac{r\bar{a}_y - z\bar{a}_z}{\sqrt{r^2+z^2}}\right] \qquad \text{... (3)}$$

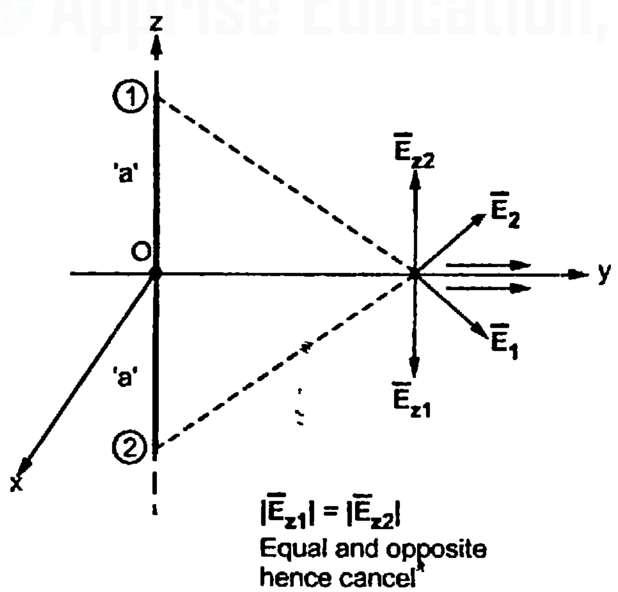


Fig. 2.21

Note : For every charge on positive z-axis there is equal charge present on negative z-axis. Hence the z component of electric field intensities produced by such charges at point P will cancel each other. Hence effectively there will not be any z component of $\bar{E}$ at P. This is shown in the Fig. 2.21.

Hence the equation of $d\bar{E}$ can be written by eliminating $\bar{a}_z$ component,

$$\therefore \quad d\bar{E} = \frac{\rho_L\, dz}{4\pi\varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \frac{r\,\bar{a}_y}{\sqrt{r^2+z^2}} \qquad \text{... (4)}$$

Now by integrating $d\bar{E}$ over the z-axis from $-\infty$ to ∞ we can obtain total $\bar{E}$ at point P.

$$\therefore \quad \bar{E} = \int_{-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon_0 \left(r^2+z^2\right)^{3/2}}\, r\, dz\, \bar{a}_y$$

Note : For such an integration, use the substitution

$$z = r\tan\theta \quad \text{i.e.} \quad r = \frac{z}{\tan\theta}$$

$$\therefore \quad dz = r\sec^2\theta\, d\theta$$

Here r is not the variable of integration.

$$\left.\begin{aligned} &\text{For } z = -\infty, \quad \theta = \tan^{-1}(-\infty) = -\pi/2 = -90^\circ \\ &\text{For } z = +\infty, \quad \theta = \tan^{-1}(\infty) = \pi/2 = +90^\circ \end{aligned}\right\} \text{Changing the limits}$$

$$\therefore \quad \bar{E} = \int_{\theta=-\pi/2}^{\pi/2} \frac{\rho_L}{4\pi\varepsilon_0 [r^2 + r^2\tan^2\theta]^{3/2}}\, r \times r\sec^2\theta\, d\theta\, \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\pi/2}^{\pi/2} \frac{r^2\sec^2\theta\, d\theta}{r^3[1+\tan^2\theta]^{3/2}}\, \bar{a}_y$$

But $\quad 1+\tan^2\theta = \sec^2\theta$

$$\therefore \quad \bar{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta\, d\theta}{r\sec^3\theta}\, \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta\, \bar{a}_y \qquad \text{... } \sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} [\sin\theta]_{-\pi/2}^{\pi/2}\, \bar{a}_y = \frac{\rho_L}{4\pi\varepsilon_0 r} \left[\sin\frac{\pi}{2} - \sin\left(\frac{-\pi}{2}\right)\right] \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} [1-(-1)]\, \bar{a}_y = \frac{\rho_L}{4\pi\varepsilon_0 r} \times 2\, \bar{a}_y$$

$$\therefore \quad \boxed{\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\, \bar{a}_y \quad \text{V/m}} \qquad \text{... (5)}$$

Key Point: *If without considering symmetry of charges and without cancelling z component from $d\bar{E}$, if integration is carried out, it gives the same answer. The integration results the z component of $\bar{E}$ to be mathematically zero.*

The result of equation (5) which is specifically in cartesian system can be generalized. The $\bar{a}_y$ is unit vector along the distance r which is perpendicular distance of point P from the line charge. Thus in general $\bar{a}_y = \bar{a}_r$.

Hence the result of $\bar{E}$ can be expressed as,

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \ \text{V/m}} \qquad \text{... (6)}$$

where r = Perpendicular distance of point P from the line charge

$\bar{a}_r$ = Unit vector in the direction of the perpendicular distance of point P from the line charge

Very important notes : 1. The field intensity $\bar{E}$ at any point has no component in the direction parallel to the line along which the charge is located and the charge is infinite. For example if line charge is parallel to z axis, $\bar{E}$ can not have $\bar{a}_z$ component, if line charge is parallel to y axis, $\bar{E}$ can not have a_y component. This makes the integration calculations easy.

2. The above equation consists r and $\bar{a}_r$ which **do not** have meanings of cylindrical co-ordinate system. The distance r is to be obtained by distance formula while $\bar{a}_r$ is unit vector in the direction of $\bar{r}$.

Key Point: *This result can be used as a standard result for solving other problems.*

Example 2.8 : *A uniform line charge, infinite in extent with ρ_L = 20 nC/m lies along the z axis. Find the $\bar{E}$ at (6,8,3) m.*

Solution : The line charge is shown in the Fig. 2.22.

Any point on the line is (0,0,z).

Key Point: *As line charge is along z axis, $\bar{E}$ can not have any component along z direction. So do not consider z co-ordinate while calculating $\bar{r}$.*

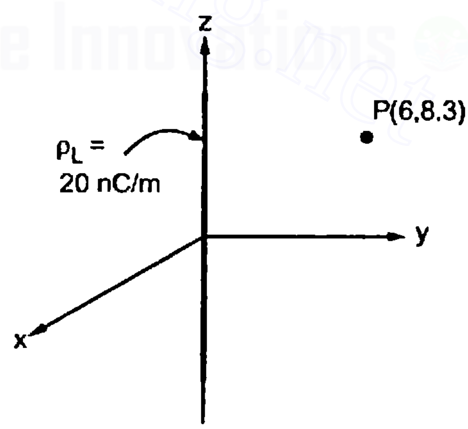


Fig. 2.22

$$\therefore \quad \bar{r} = (6-0)\bar{a}_x + (8-0)\bar{a}_y$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{6\,\bar{a}_x + 8\,\bar{a}_y}{\sqrt{6^2+8^2}} = \frac{6\,\bar{a}_x + 8\,\bar{a}_y}{10}$$

$$= 0.6\bar{a}_x + 0.8\,\bar{a}_y$$

Thus, $$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

$$= \frac{20\times10^{-9}}{2\pi\times 8.854\times10^{-12}\times10}[0.6\bar{a}_x + 0.8\,\bar{a}_y] = 10.7853\bar{a}_x + 14.38\,\bar{a}_y \ \text{V/m}$$

2.7 Electric Field due to Charged Circular Ring

Consider a charged circular ring of radius r placed in xy plane with centre at origin, carrying a charge uniformly along its circumference. The charge density is ρ_L C/m .

The point P is at a perpendicular distance 'z' from the ring as shown in the Fig. 2.23.

Consider a small differential length dl on this ring. The charge on it is dQ.

$$\therefore \quad dQ = \rho_L \, dl$$

$$\therefore \quad d\overline{E} = \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} \, \overline{a}_R \qquad \text{... (1)}$$

where R = Distance of point P from dl.

Consider the cylindrical co-ordinate system. For dl we are moving in ϕ direction where dl = r dϕ.

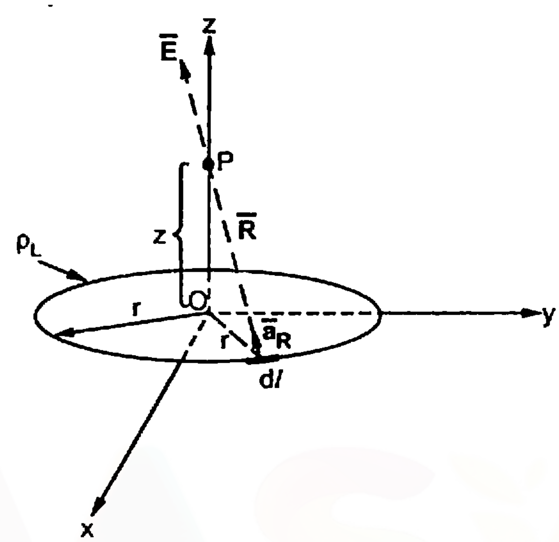


Fig. 2.23

$$\therefore \quad dl = r\, d\phi \qquad \text{... (2)}$$

$$\text{Now} \quad R^2 = r^2 + z^2 \qquad \text{... from Fig. 2.23}$$

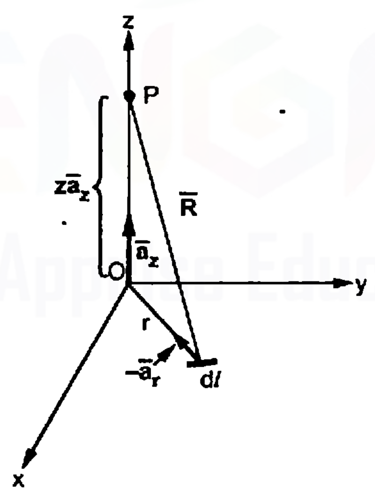


Fig. 2.23(a)

While $\overline{R}$ can be obtained from its two components, in cylindrical system as shown in the Fig. 2.23(a). The two components are,

1) Distance r in the direction of $-\overline{a}_r$, radially inwards i.e. $-r\overline{a}_r$.

2) Distance z in the direction of $\overline{a}_z$ i.e. $z\,\overline{a}_z$

$$\therefore \quad \overline{R} = -r\overline{a}_r + z\overline{a}_z \qquad \text{... (3)}$$

Key Point: *This method can be used conveniently to obtain $\overline{R}$ by identifying its components in the direction of unit vectors in the co-ordinate system considered.*

$$\therefore \quad |\overline{R}| = \sqrt{(-r)^2 + (z)^2} = \sqrt{r^2 + z^2} \qquad \text{... (4)}$$

$$\therefore \quad \overline{a}_R = \frac{\overline{R}}{|\overline{R}|} = \frac{-r\overline{a}_r + z\overline{a}_z}{\sqrt{r^2+z^2}} \qquad \text{... (5)}$$

$$\therefore \quad d\overline{E} = \frac{\rho_L \, dl}{4\pi\varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \times \frac{-r\overline{a}_r + z\overline{a}_z}{\sqrt{r^2+z^2}}$$

$$\therefore \quad d\bar{E} = \frac{\rho_L (r\, d\phi)}{4\pi\varepsilon_0 \left(r^2+z^2\right)^{3/2}} [-r\bar{a}_r + z\bar{a}_z] \quad \text{... (6)}$$

Note : The radial components of $\bar{E}$ at point P will be symmetrically placed in the plane parallel to xy plane and are going to cancel each other. This is shown in the Fig. 2.23 (b). Hence neglecting $\bar{a}_r$ component from $d\bar{E}$ we get,

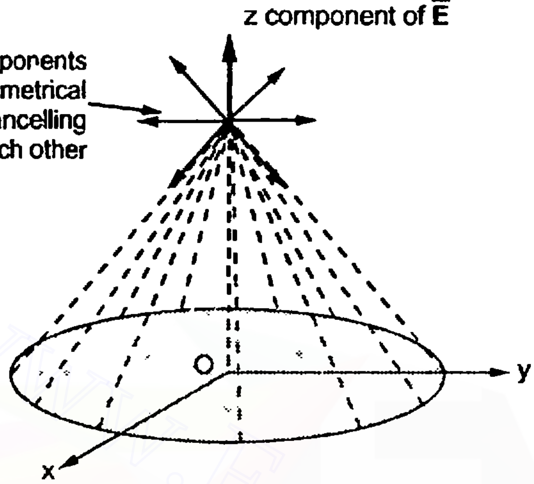


Fig. 2.23 (b)

$$d\bar{E} = \frac{\rho_L (r\, d\phi)}{4\pi\varepsilon_0 \left(r^2+z^2\right)^{3/2}} z\bar{a}_z \quad \text{... (7)}$$

$$\therefore \quad \bar{E} = \int_{\phi=0}^{2\pi} \frac{\rho_L\, r\, d\phi}{4\pi\varepsilon_0 \left(r^2+z^2\right)^{3/2}} z\bar{a}_z$$

$$= \frac{\rho_L\, r}{4\pi\varepsilon_0 \left(r^2+z^2\right)^{3/2}} z\,\bar{a}_z\, [\phi]_0^{2\pi}$$

... Integration w.r.t. ϕ

$$\therefore \quad \boxed{\bar{E} = \frac{\rho_L\, r z}{2\varepsilon_0 \left(r^2+z^2\right)^{3/2}} \bar{a}_z} \quad \text{... (8)}$$

where r = Radius of the ring

z = Perpendicular distance of point P from the ring along the axis of the ring

This is the electric field at a point P (0, 0, z) due to the circular ring of radius r placed in xy plane.

Example 2.9 : *Prove that the electric field intensity at a point P located at a distance r from an infinite line charge with uniform charge density of ρ_L C/m is, $\bar{E} = \dfrac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$ in cylindrical co-ordinate system.*

Solution : Consider that the line charge is located along z axis as shown in the Fig. 2.24 (a).

Consider the differential length dl carrying the charge dQ.

Now $dl = dz$ ... Along z axis

$\therefore\ dQ = \rho_L\, dl = \rho_l\, dz$

$$\therefore\ d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

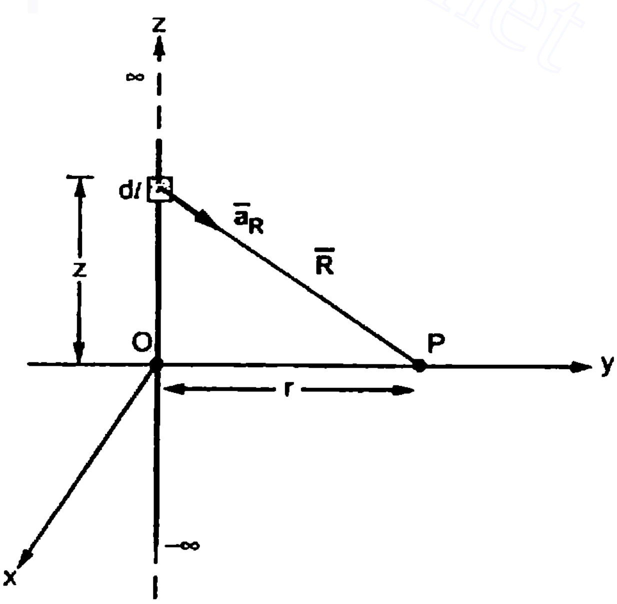


Fig. 2.24 (a)

In cylindrical co-ordinate system the distance vector $\bar{R}$ has two components as shown in the Fig. 2.24 (b).

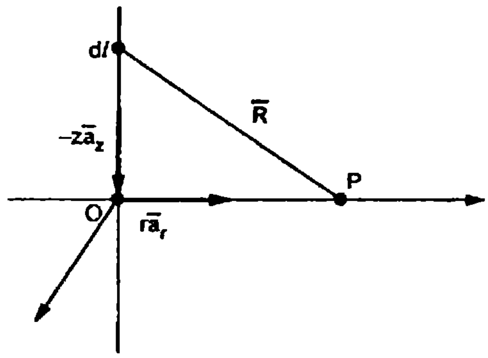


Fig. 2.24 (b)

1) The component along negative z direction i.e. $-z\,\bar{a}_z$

2) The component along $\bar{a}_r$ which is $r\,\bar{a}_r$. (Radial component).

$$\therefore \quad \bar{R} = r\,\bar{a}_r - z\,\bar{a}_z$$

$$\therefore |\bar{R}| = \sqrt{(r)^2+(-z)^2} = \sqrt{r^2+z^2}$$

$$\therefore \quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$\therefore \qquad d\bar{E} = \frac{\rho_L\,dz}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\left[\frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}\right]$$

Hence $\bar{E}$ can be obtained by integrating $d\bar{E}$ along z axis from $-\infty$ to ∞. It can be noted that due to symmetry z component will cancell in $\bar{E}$ but let us prove it mathematically.

$$\therefore \quad \bar{E} = \int_{z=-\infty}^{z=\infty} \frac{\rho_L\,dz}{4\pi\varepsilon_0\left(r^2+z^2\right)^{3/2}}[r\bar{a}_r - z\bar{a}_z]$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\left[\int_{-\infty}^{\infty}\frac{r\,dz}{\left(r^2+z^2\right)^{3/2}}\bar{a}_r - \int_{-\infty}^{\infty}\frac{z\,dz}{\left(r^2+z^2\right)^{3/2}}\bar{a}_z\right] \quad \text{... Separating variables}$$

Put $\quad z = r\tan\theta \quad$ i.e. $\quad r = \dfrac{z}{\tan\theta}$

$$\therefore \qquad dz = r\sec^2\theta\,d\theta$$

$$\left.\begin{array}{l} \text{For } z=-\infty, \quad \theta = \tan^{-1}(-\infty) = -\dfrac{\pi}{2} = -90^\circ \\ \text{For } z=+\infty, \quad \theta = \tan^{-1}(\infty) = +\dfrac{\pi}{2} = +90^\circ \end{array}\right\} \quad \text{... Change of limits}$$

$$\therefore \bar{E} = \frac{\rho_L}{4\pi\varepsilon_0}\left\{\int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{r\times r\sec^2\theta\,d\theta\,\bar{a}_r}{\left(r^2+r^2\tan^2\theta\right)^{3/2}} - \int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{r\tan\theta\,r\sec^2\theta\,d\theta\,\bar{a}_z}{\left(r^2+r^2\tan^2\theta\right)^{3/2}}\right\}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\left\{\int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{r^2\sec^2\theta\,d\theta\,\bar{a}_r}{r^3\sec^3\theta} - \int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{r\tan\theta\,r\sec^2\theta\,d\theta\,\bar{a}_z}{r^3\sec^3\theta}\right\} \quad \text{... } 1+\tan^2\theta = \sec^2\theta$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\left\{\int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{1}{r\sec\theta}\,d\theta\,\bar{a}_r - \int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{1}{r}\,\frac{\sin\theta}{\cos\theta}\,\frac{1}{\sec\theta}\,d\theta\,\bar{a}_z\right\}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\left\{\int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{1}{r}\cos\theta\, d\theta\, \bar{a}_r - \int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{1}{r}\sin\theta\, d\theta\, \bar{a}_z\right\} \qquad \ldots \frac{1}{\sec\theta} = \cos\theta$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\frac{1}{r}\left\{[\sin\theta]_{-\pi/2}^{\pi/2}\,\bar{a}_r - [-\cos\theta]_{-\pi/2}^{\pi/2}\,\bar{a}_z\right\}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\frac{1}{r}\left\{\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right]\bar{a}_r - \left[-\cos\frac{\pi}{2} - \left(-\cos -\frac{\pi}{2}\right)\right]\bar{a}_z\right\}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\frac{1}{r}\{[1-(-1)]\,\bar{a}_r - [0]\} \qquad \ldots \text{ as } \cos\frac{\pi}{2} = \cos\frac{-\pi}{2} = 0$$

$$\therefore \quad \boxed{\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0 r}(2)\,\bar{a}_r = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\bar{a}_r} \qquad \ldots \textbf{Proved.}$$

Note : Mathematically also z component is getting cancelled. Hence looking at the symmetry and cancelling the terms, makes the mathematical exercise much more easier.

Example 2.10 : *A uniform line charge ρ_L = 25 nC/m lies on the line x = –3 m and y = 4 m in free space. Find the electric field intensity at a point (2, 3, 15) m.*

Solution : The line is shown in the Fig. 2.25. The line with x = – 3 constant and y = 4 constant is a line parallel to z axis as z can take any value. The $\bar{E}$ at P (2, 3, 15) is to be calculated.

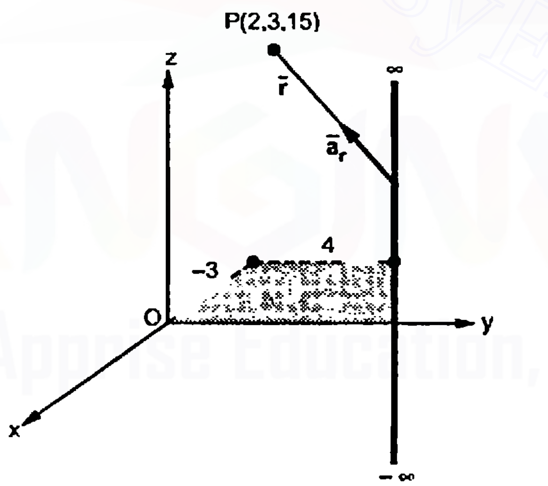


Fig. 2.25

The charge is infinite line charge hence $\bar{E}$ can be obtained by standard result,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

To find $\bar{r}$, consider two points, one on the line which is (–3, 4, z) while P (2, 3, 15). But as line is parallel to z axis, $\bar{E}$ can not have component in $\bar{a}_z$ direction hence z need not be considered while calculating $\bar{r}$.

$$\therefore \quad \bar{r} = [2-(-3)]\bar{a}_x + [3-4]\bar{a}_y = 5\bar{a}_x - \bar{a}_y \qquad \ldots \text{ z not considered}$$

$$\therefore \quad |\bar{r}| = \sqrt{(5)^2 + (-1)^2} = \sqrt{26}$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{5\bar{a}_x - \bar{a}_y}{\sqrt{26}}$$

$$\therefore \quad \bar{E} = \frac{\rho_L}{2\pi\varepsilon_0}\cdot\frac{1}{\sqrt{26}}\left[\frac{5\bar{a}_x - \bar{a}_y}{\sqrt{26}}\right] = \frac{25\times10^{-9}\,[5\bar{a}_x - \bar{a}_y]}{2\pi\times 8.854\times10^{-12}\times 26}$$

$$= \mathbf{86.42\ \bar{a}_x - 17.284\ \bar{a}_y \quad V/m}$$

2.8 Electric Field due to Infinite Sheet of Charge

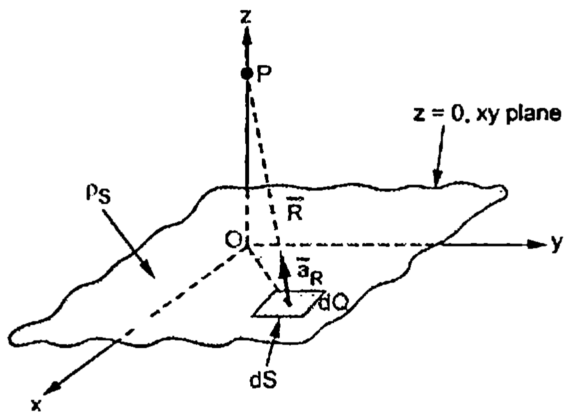


Fig. 2.26

Consider an infinite sheet of charge having uniform charge density ρ_S C/m^2, placed in xy plane as shown in the Fig. 2.26. Let us use cylindrical coordinates.

The point P at which $\bar{E}$ to be calculated is on z axis.

Consider the differential surface area dS carrying a charge dQ. The normal direction to dS is z direction hence dS normal to z direction is r dr dϕ.

Now $$dQ = \rho_S\, dS = \rho_S\; r\, dr\, d\phi \qquad \text{... (1)}$$

Hence, $$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R = \frac{\rho_S\, r\, dr\, d\phi}{4\pi\varepsilon_0 R^2}\,\bar{a}_R \qquad \text{... (2)}$$

The distance vector $\bar{R}$ has two components as shown in the Fig. 2.27.

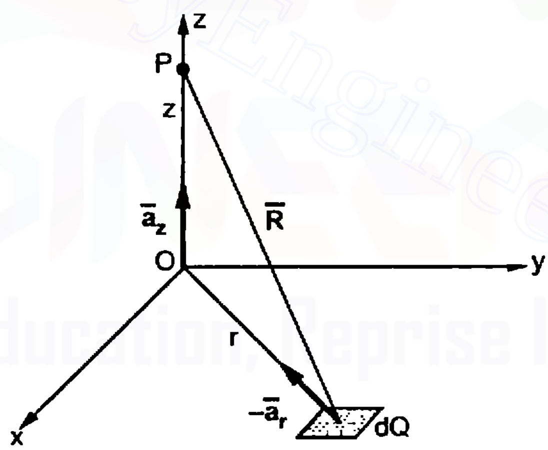


Fig. 2.27

1. The radial component r along $-\bar{a}_r$ i.e. $-r\,\bar{a}_r$.
2. The component z along $\bar{a}_z$ i.e. $z\,\bar{a}_z$.

With these two components $\bar{R}$ can be obtained from the differential area towards point P as,

$$\therefore \quad \bar{R} = -r\bar{a}_r + z\bar{a}_z \qquad \text{... (3)}$$

$$\therefore \quad |\bar{R}| = \sqrt{(-r)^2 + (z)^2} = \sqrt{r^2 + z^2} \qquad \text{... (4)}$$

$$\therefore \quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2 + z^2}} \qquad \text{... (5)}$$

$$\therefore \quad d\bar{E} = \frac{\rho_S \, r \, dr \, d\phi}{4\pi\varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \left[\frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}\right]$$

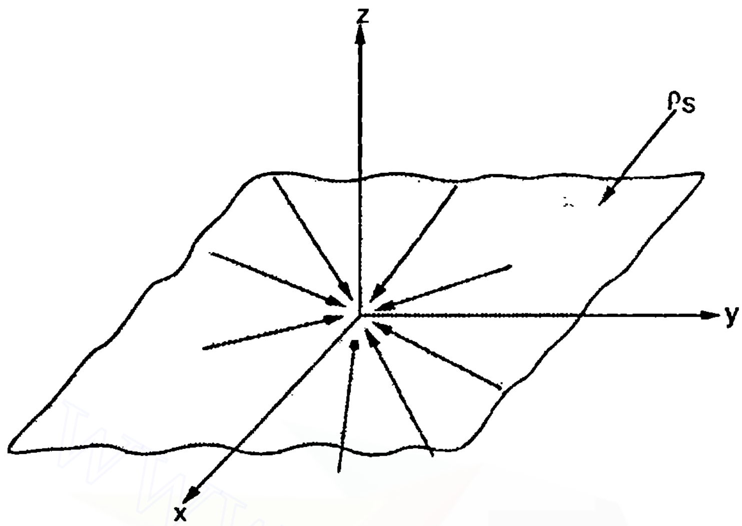


Fig. 2.28

For infinite sheet in xy plane, r varies from 0 to ∞ while ϕ varies from 0 to 2π.

Note : As there is symmetry about z axis from all radial direction, all $\bar{a}_r$ components of $\bar{E}$ are going to cancell each other and net $\bar{E}$ will not have any radial component.

Hence while integrating $d\bar{E}$ there is no need to consider $\bar{a}_r$ component. Though if considered, after integration procedure, it will get mathematically cancelled.

$$\therefore \quad \bar{E} = \int_{\phi=0}^{2\pi}\int_{r=0}^{\infty} d\bar{E} = \int_0^{2\pi}\int_0^{\infty} \frac{\rho_S \, r \, dr \, d\phi}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} (z\bar{a}_z)$$

Put $r^2+z^2 = u^2$ hence $2r\,dr = 2u\,du$

For $r = 0$, $u = z$ and $r = \infty$, $u = \infty$ ... Changing limits

$$\therefore \quad \bar{E} = \int_0^{2\pi}\int_{u=z}^{\infty} \frac{\rho_S}{4\pi\varepsilon_0} \frac{u\,du}{(u^2)^{3/2}} \, d\phi \, z\,\bar{a}_z$$

$$= \int_0^{2\pi}\int_{u=z}^{\infty} \frac{\rho_S}{4\pi\varepsilon_0} \frac{du}{u^2} \, d\phi \, (z\,\bar{a}_z)$$

$$= \int_0^{2\pi} \frac{\rho_S}{4\pi\varepsilon_0} d\phi \, z\,\bar{a}_z \left[-\frac{1}{u}\right]_z^{\infty} \qquad \text{... as } \int \frac{1}{u^2} = \int u^{-2} = \frac{u^{-1}}{-1} = -\frac{1}{u}$$

$$= \frac{\rho_S}{4\pi\varepsilon_0} [\phi]_0^{2\pi} (z\bar{a}_z) \left[-\frac{1}{\infty} - \left(-\frac{1}{z}\right)\right] = \frac{\rho_S}{4\pi\varepsilon_0} (2\pi)\,\bar{a}_z$$

$$\therefore \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0}\,\bar{a}_z \;\; \text{V/m} \qquad \text{... For points above xy plane}$$

Now $\bar{a}_z$ is direction normal to differential surface area dS considered. Hence in general if $\bar{a}_n$ is direction normal to the surface containing charge, the above result can be generalized as,

$\therefore$ $$\boxed{\bar{E} = \frac{\rho_S}{2\varepsilon_0}\bar{a}_n \text{ V/m}}$$ ... (6)

where $\bar{a}_n$ = Direction normal to the surface charge

Thus for the points below xy plane, $\bar{a}_n = -\bar{a}_z$ hence,

$\therefore$ $$\bar{E} = -\frac{\rho_S}{2\varepsilon_0}\bar{a}_z \text{ V/m}$$ ... For points below xy plane.

Note : The equation (6) is standard result and can be used directly to solve the problems.

Key Point: *Thus electric field due to infinite sheet of charge is everywhere normal to the surface and its magnitude is independent of the distance of a point from the plane containing the sheet of charge.*

Important Observations :

1. $\bar{E}$ due to infinite sheet of charge at a point is not dependent on the distance of that point from the plane containing the charge.
2. The direction of $\bar{E}$ is perpendicular to the infinite charge plane.
3. The magnitude of $\bar{E}$ is constant every where and given by $|\bar{E}| = \rho_S / 2\varepsilon_0$.

Example 2.11 : *Charge lies in y = – 5m plane in the form of an infinite square sheet with a uniform charge density of $\rho_S = 20$ nC/m². Determine $\bar{E}$ at all the points.*

Solution : The plane y = – 5 m constant is parallel to xz plane as shown in the Fig. 2.29.

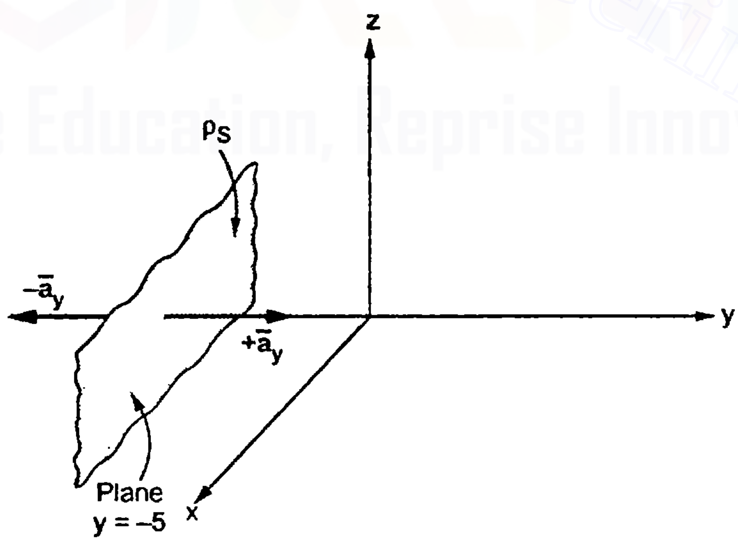


Fig. 2.29

For y > – 5, the $\bar{E}$ component will be along $+\bar{a}_y$ as normal direction to the plane y = – 5 m is $\bar{a}_y$.

$\therefore$ $$\bar{a}_n = \bar{a}_y$$

$$\therefore \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0}\bar{a}_n = \frac{\rho_S}{2\varepsilon_0}\bar{a}_y$$

$$= \frac{20\times10^{-9}}{2\times8.854\times10^{-12}}\bar{a}_y = 1129.43\ \bar{a}_y \quad \text{V/m}$$

For $y < -5$, the $\bar{E}$ component will be along $-\bar{a}_y$ direction, with same magnitude.

$$\therefore \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0}(-\bar{a}_y) = -1129.43\ \bar{a}_y \quad \text{V/m}$$

At any point to the left or right of the plane, $|\bar{E}|$ is constant and acts normal to the plane.

Example 2.12 : *Find $\bar{E}$ at P (1, 5, 2) m in free space if a point charge of 6 μC is located at (0,0,1), the uniform line charge density ρ_L = 180 nC/m along x axis and uniform sheet of charge with $\rho_S = 25$ nC/m² over the plane z = −1.*

Solution : **Case 1 :** Point charge Q_1 = 6 μC at A (0, 0, 1) and P (1, 5, 2)

$$\therefore \quad \bar{E}_1 = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2}\bar{a}_{AP} = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2}\left[\frac{\bar{R}_{AP}}{|\bar{R}_{AP}|}\right]$$

$$\bar{R}_{AP} = (1-0)\bar{a}_x + (5-0)\bar{a}_y + (2-1)\bar{a}_z = \bar{a}_x + 5\bar{a}_y + \bar{a}_z$$

$$\therefore \quad |\bar{R}_{AP}| = \sqrt{(1)^2+(5)^2+(1)^2} = \sqrt{27}$$

$$\therefore \quad \bar{E}_1 = \frac{6\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(\sqrt{27})^2}\left[\frac{\bar{a}_x+5\bar{a}_y+\bar{a}_z}{\sqrt{27}}\right]$$

$$\therefore \quad \bar{E}_1 = 384.375\ \bar{a}_x + 1921.879\ \bar{a}_y + 384.375\ \bar{a}_z \quad \text{V/m}$$

Case 2 : Line charge ρ_L along x axis.

It is infinite hence using standard result,

$$\bar{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r = \frac{\rho_L}{2\pi\varepsilon_0 r}\frac{\bar{r}}{|\bar{r}|}$$

Consider any point on line charge i.e. (x, 0, 0) while P (1, 5, 2). But as line is along x axis, no component of $\bar{E}$ will be along $\bar{a}_x$ direction. Hence while calculating $\bar{r}$ and $\bar{a}_r$, do not consider x co-ordinates of the points.

$$\therefore \quad \bar{r} = (5-0)\bar{a}_y + (2-0)\bar{a}_z = 5\bar{a}_y + 2\bar{a}_z$$

$$\therefore \quad |\bar{r}| = \sqrt{(5)^2+(2)^2} = \sqrt{29}$$

$$\therefore \quad \bar{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0\times\sqrt{29}}\left[\frac{5\bar{a}_y+2\bar{a}_z}{\sqrt{29}}\right] = \frac{180\times10^{-9}[5\bar{a}_y+2\bar{a}_z]}{2\pi\times8.854\times10^{-12}\times29}$$

$$= 557.859\ \bar{a}_y + 223.144\ \bar{a}_z \quad \text{V/m}$$

Case 3 : Surface charge ρ_S over the plane z = – 1. The plane is parallel to xy plane and normal direction to the plane is $\bar{a}_n = \bar{a}_z$, as point P is above the plane. At all the points above z = – 1 plane the $\bar{E}$ is constant along $\bar{a}_z$ direction.

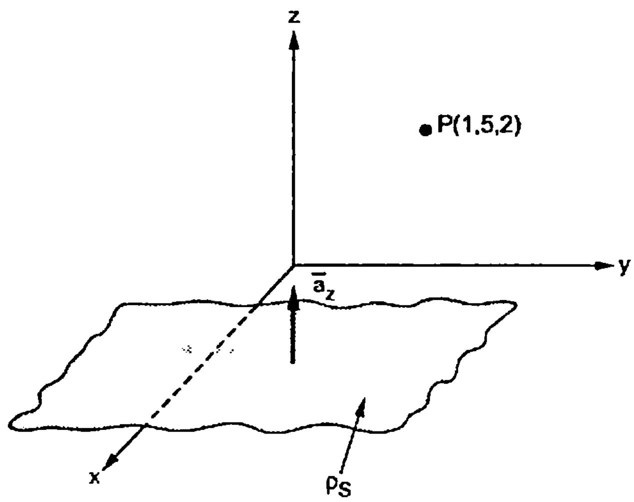


Fig. 2.30

$$\therefore \quad \bar{E}_3 = \frac{\rho_S}{2\varepsilon_0}\bar{a}_n$$

$$= \frac{25\times10^{-9}}{2\times8.854\times10^{-12}}\bar{a}_z$$

$$= 1411.7913 \ \bar{a}_z \text{V/m}$$

Hence the net $\bar{E}$ at point P is,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 = 384.375\,\bar{a}_x + 1921.879\,\bar{a}_y + 384.375\,\bar{a}_z + 557.859\,\bar{a}_y + 223.144\,\bar{a}_z + 1411.7913\,\bar{a}_z$$

$$= \mathbf{384.375\,\bar{a}_x + 2479.738\,\bar{a}_y + 2019.3103\,\bar{a}_z \ \ V/m}$$

Example 2.13 : *The charge lies on the circular disc $r \le 4$ m, $z = 0$, with density $\rho_S = [10^{-4}/r]$ C/m². Determine $\bar{E}$ at r = 0, z = 3 m.*

Solution : The sheet of charge is shown in the Fig. 2.31.

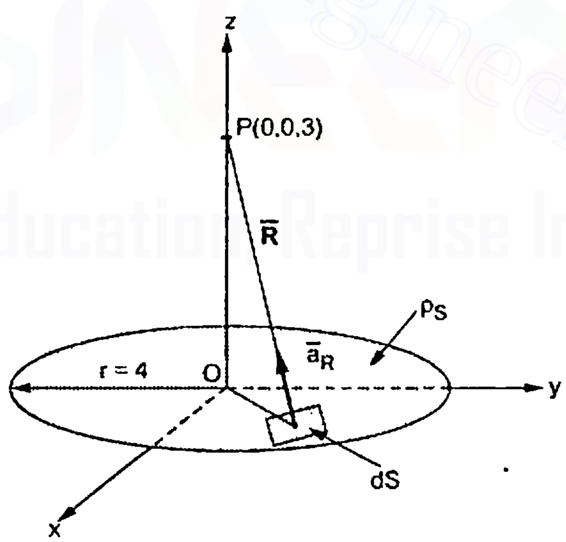


Fig. 2.31

Consider the differential area dS carrying the charge dQ. The normal direction to dS is $\bar{a}_z$ hence $dS_z = r\,dr\,d\phi$.

$$\therefore \quad dQ = \rho_S\,dS = \rho_S\,r\,dr\,d\phi$$

$$= \frac{10^{-4}}{r}\cdot r\,dr\,d\phi$$

$$\therefore \quad dQ = 10^{-4}\ dr\,d\phi$$

$$\therefore \qquad d\bar{E} = \frac{10^{-4}\, dr\, d\phi}{4\pi\varepsilon_0 R^2}\, \bar{a}_R$$

Consider $\bar{R}$ as shown in the Fig. 2.32, which has two components in cylindrical system,

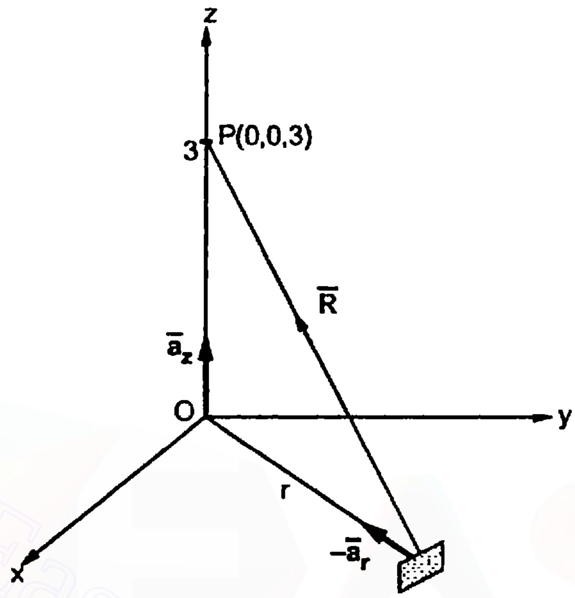


Fig. 2.32

1. The component along $-\bar{a}_r$ having radius r i.e. $-r\,\bar{a}_r$.
2. The component z = 3 along $\bar{a}_z$ i.e. $3\,\bar{a}_z$.

$$\therefore \qquad \bar{R} = -r\,\bar{a}_r + 3\bar{a}_z$$

$$|\bar{R}| = \sqrt{(-r)^2 + (3)^2} = \sqrt{r^2+9}$$

$$\therefore \qquad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a}_r + 3\bar{a}_z}{\sqrt{r^2+9}}$$

$$\therefore \qquad d\bar{E} = \frac{10^{-4}\, dr\, d\phi}{4\pi\varepsilon_0 \left(\sqrt{r^2+9}\right)^2} \left[\frac{-r\bar{a}_r + 3\bar{a}_z}{\sqrt{r^2+9}}\right]$$

It can be seen that due to symmetry about z axis, all radial components will cancell each other. Hence there will not be any component of $\bar{E}$ along $\bar{a}_r$. So in integration $\bar{a}_r$ need not be considered.

$$\therefore \qquad \bar{E} = \int_{\phi=0}^{2\pi} \int_{r=0}^{4} \frac{10^{-4}\, dr\, d\phi}{4\pi\varepsilon_0 \left(r^2+9\right)^{3/2}}\, (3\bar{a}_z)$$

As there is no r dr in the numerator, use

$$\left.\begin{aligned} r &= 3\tan\theta, \quad dr = 3\sec^2\theta\, d\theta \\ \text{For } r &= 0, \quad \theta_1 = 0 \\ \text{For } r &= 4, \quad \theta_2 = \tan^{-1} 4/3 \end{aligned}\right\} \quad \text{... Change of limits}$$

$$\therefore \quad \bar{E} = \int_{\phi=0}^{2\pi} \int_{\theta_1=0}^{\theta_2} \frac{10^{-4}\, 3\sec^2\theta\, d\theta\, d\phi}{4\pi\varepsilon_0 \left[9\tan^2\theta + 9\right]^{3/2}} (3\bar{a}_z)$$

$$\therefore \quad \bar{E} = \int_{\phi=0}^{2\pi} \int_{\theta_1=0^\circ}^{\theta_2} \frac{299.5914\times 10^3 \sec^2\theta\, d\theta\, d\phi}{\left[1+\tan^2\theta\right]^{3/2}} \bar{a}_z$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta_1=0^\circ}^{\theta_2} \frac{299.5914\times 10^3}{\sec\theta}\, d\theta\, d\phi\, \bar{a}_z$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta_1=0^\circ}^{\theta_2} 299.5914\times 10^3\, d\theta\, d\phi [\cos\theta] \bar{a}_z$$

$$= 299.5914 \times 10^3\, [\phi]_0^{2\pi}\, [\sin\theta]_{\theta_1=0^\circ}^{\theta_2}\, \bar{a}_z \qquad \text{... Separating variables}$$

$$= 1.8823 \times 10^6 \sin\theta_2\, \bar{a}_z \qquad \text{... } \sin 0^\circ = 0$$

Now $\theta_2 = \tan^{-1}\frac{4}{3}$ i.e. $\tan\theta_2 = \frac{4}{3}$

$$\therefore \quad \sin\theta_2 = \frac{4}{5} = 0.8$$

$$\therefore \quad \bar{E} = 1.8823\times 10^6 \times 0.8\, \bar{a}_z$$

$$= 1.5059 \times 10^6\, \bar{a}_z \text{ V/m}$$

$$= 1.5059\, \bar{a}_z \text{ MV/m}$$

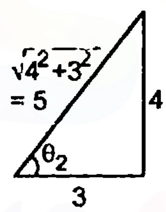


Fig. 2.33

Examples with Solutions

Example. 2.14 : *Q_1 and Q_2 are the point charges located at $(0,-4,3)$ and $(0,1,1)$. If Q_1 is 2nC, find Q_2 such that the force on a test charge at $(0,-3,4)$ has no z component.*

Solution : The charges are shown in the Fig. 2.34.

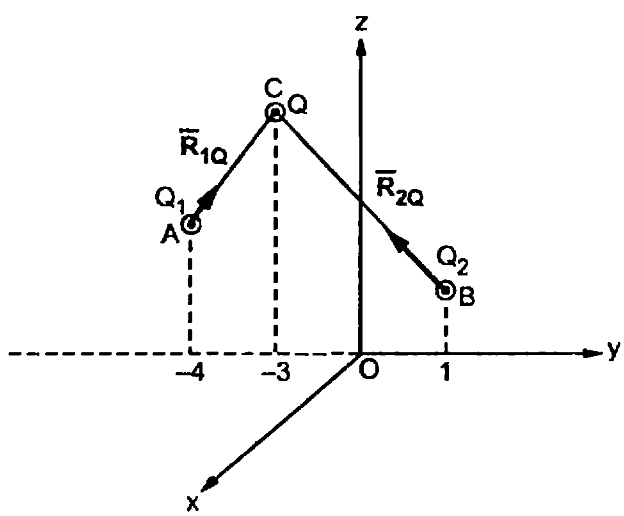


Fig. 2.34

The position vectors of the points A, B and C are,

$$\bar{A} = -4\bar{a}_y + 3\bar{a}_z$$

$$\bar{B} = \bar{a}_y + \bar{a}_z$$

$$\bar{C} = -3\bar{a}_y + 4\bar{a}_z$$

$$\therefore \quad \bar{R}_{1Q} = \bar{C} - \bar{A} = \bar{a}_y + \bar{a}_z$$

and $$\bar{R}_{2Q} = \bar{C} - \bar{B} = -4\bar{a}_y + 3\bar{a}_z$$

$$\therefore \quad |\bar{R}_{1Q}| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

and $$|\bar{R}_{2Q}| = \sqrt{(-4)^2 + (3)^2} = 5$$

$$\therefore \quad \bar{F}_1 = \text{Force on Q due to } Q_1 = \frac{Q\,Q_1}{4\pi\varepsilon_0 R_{1Q}^2}\,\bar{a}_{1Q}$$

and $$\bar{F}_2 = \text{Force on Q due to } Q_2 = \frac{QQ_2}{4\pi\varepsilon_0 R_{2Q}^2}\,\bar{a}_{2Q}$$

$$\therefore \quad \bar{F}_t = \bar{F}_1 + \bar{F}_2 = \frac{Q}{4\pi\varepsilon_0}\left[\frac{Q_1}{R_{1Q}^2}\,\bar{a}_{1Q} + \frac{Q_2}{R_{2Q}^2}\,\bar{a}_{2Q}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{2\times10^{-9}}{(\sqrt{2})^2}\left(\frac{\bar{a}_y + \bar{a}_z}{\sqrt{2}}\right) + \frac{Q_2}{(5)^2}\left(\frac{-4\bar{a}_y + 3\bar{a}_z}{5}\right)\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[7.071\times10^{-10}\,(\bar{a}_y + \bar{a}_z) + \frac{Q_2}{125}(-4\bar{a}_y + 3\bar{a}_z)\right]$$

$\therefore$ Total z component of $\bar{F}_t$ is,

$$= \frac{Q}{4\pi\varepsilon_0}\left[7.071\times10^{-10} + \frac{3\,Q_2}{125}\right]\bar{a}_z$$

To have this component zero,

$7.071\times10^{-10} + \dfrac{3\,Q_2}{125} = 0$ as Q is test charge and can not be zero.

$$\therefore \quad Q_2 = -\frac{7.071\times10^{-10}\times125}{3} = -\mathbf{29.462\ nC}$$

Example 2.15 : *In a Millikan oil drop experiment, the weight of a 1.6×10^{-14} kg drop is exactly balanced by the electric force in vertically directed 200 kV/m field. Calculate the charge on the drop in units of the electronic charge $(e = 1.6\times10^{-19}\ \text{C})$.*

Solution : Given E = 200 kV/m, m = 1.6×10^{-14} kg

$$\therefore \quad |E| = \frac{|F|}{Q}$$

$$\therefore \quad 200\times10^{3} = \frac{|F|}{Q}$$

$$\therefore \quad |F| = 200\times10^{3} Q \text{ N} \qquad \dots (1)$$

This is balanced by the weight mg

$$\therefore \quad |F| = mg = 1.6\times10^{-14}\times9.81$$

$$= 1.5696\times10^{-13} \text{ N} \qquad \dots (2)$$

Equating (1) and (2),

$$200\times10^{3}\, Q = 1.5696\times10^{-13}$$

$$\therefore \quad Q = 7.848\times10^{-19} \text{ C} \qquad \dots \text{Charge on drop}$$

Now $e = 1.6\times10^{-19}$ C hence Q in terms of e is,

$$Q = \frac{7.848\times10^{-19}}{1.6\times10^{-19}}$$

$$= \mathbf{4.905e\ C}$$

Example 2.16 : *The charge is distributed along the z axis from z = − 5 m to −∞ and z = +5 m to +∞ with a charge density of 20 nC/m. Find* $\bar{E}$ *at (2, 0, 0) m. Also express the answer in cylindrical coordinates.*

Solution : The charge is shown as in the Fig. 2.35.

Key Point: *If ρ_L is not distributed all along the length then standard result can not be used. The basic procedure is to be used.*

As charge is not infinite, let us use basic procedure of considering differential charge.

Consider the differential element d*l* in the z direction hence,

$$dl = dz$$

$$\therefore \quad dQ = \rho_L\, dl = \rho_L\, dz$$

$$\therefore \quad d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R = \frac{\rho_L\, dz}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

Any point on z axis is (0, 0, z) while point P at which $\bar{E}$ to be calculated is (2, 0, 0).

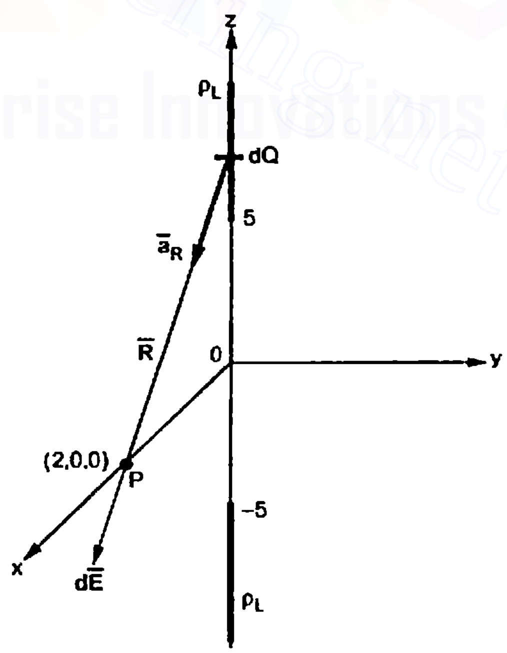


Fig. 2.35

$$\bar{R} = (2-0)\bar{a}_x + (0-z)\bar{a}_z = 2\bar{a}_x - z\bar{a}_z$$

$$|\bar{R}| = \sqrt{(2)^2+(-z)^2} = \sqrt{4+z^2}$$

$$\therefore \quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{2\bar{a}_x - z\bar{a}_z}{\sqrt{4+z^2}}$$

$$\therefore \quad d\bar{E} = \frac{\rho_l\, dz}{4\pi\varepsilon_0\left(\sqrt{4+z^2}\right)^2}\left[\frac{2\bar{a}_x - z\bar{a}_z}{\sqrt{4+z^2}}\right]$$

$$= \frac{\rho_L\, dz}{4\pi\varepsilon_0\left(4+z^2\right)^{3/2}}\left(2\bar{a}_x - z\bar{a}_z\right)$$

Now there is no charge between – 5 to 5 hence to find $\bar{E}$, $d\bar{E}$ to be integrated in two zones $-\infty$ to – 5 and 5 to ∞ in z direction.

$$\therefore \quad \bar{E} = \int_{\infty}^{-5} d\bar{E} + \int_{5}^{\infty} d\bar{E}$$

Looking at the symmetry it can be observed that z component of $\bar{E}$ produced by charge between 5 to ∞ will cancel the z component of $\bar{E}$ produced by charge between -5 to $-\infty$. Hence for integration $\bar{a}_z$ component from $d\bar{E}$ can be neglected.

$$\therefore \quad \bar{E} = \int_{-\infty}^{-5} \frac{\rho_L\, dz(2\bar{a}_x)}{4\pi\varepsilon_0\left(4+z^2\right)^{3/2}} + \int_{5}^{\infty} \frac{\rho_L\, dz(2\bar{a}_x)}{4\pi\varepsilon_0\left(4+z^2\right)^{3/2}}$$

Put $\quad z = 2\tan\theta$ and hence $dz = 2\sec^2\theta\, d\theta$

For $z = -\infty$, $\theta = -\pi/2$, For $z = -5$, $\theta = \tan^{-1} -\frac{5}{2} = -68.19°$

For $z = +\infty$, $\theta = +\pi/2$, For $z = +5$, $\theta = \tan^{-1}\frac{5}{2} = 68.19°$

$$\therefore \quad \bar{E} = \frac{2\rho_L\,\bar{a}_x}{4\pi\varepsilon_0}\left\{\int_{\theta=-90°}^{\theta=-68.19°} \frac{2\sec^2\theta\, d\theta}{\left(4+4\tan^2\theta\right)^{3/2}} + \int_{\theta=68.19°}^{\theta=90°} \frac{2\sec^2\theta\, d\theta}{\left(4+4\tan^2\theta\right)^{3/2}}\right\}$$

$$= \frac{2\rho_L\,\bar{a}_x}{4\pi\varepsilon_0}\left\{\int_{\theta=-90°}^{\theta=-68.19°} \frac{2\sec^2\theta\, d\theta}{4^{3/2}\sec^3\theta} + \int_{\theta=68.19°}^{\theta=90°} \frac{2\sec^2\theta\, d\theta}{4^{3/2}\sec^3\theta}\right\}$$

$$= \frac{2\rho_L\,\bar{a}_x\,(2)}{4\pi\varepsilon_0\left(4^{3/2}\right)}\left\{\int_{\theta=-90°}^{\theta=-68.19°} \frac{1}{\sec\theta}\, d\theta + \int_{\theta=68.19°}^{\theta=90°} \frac{1}{\sec\theta}\, d\theta\right\}$$

$$= \frac{\rho_L\,\bar{a}_x}{8\pi\varepsilon_0}\left\{[\sin\theta]_{-90°}^{-68.19°} + [\sin\theta]_{68.19°}^{90°}\right\}$$

$$= \frac{20\times10^{-9}\,\bar{a}_x}{8\pi\times 8.854\times10^{-19}}\left\{\sin(-68.19°) - \sin(-90°) + \sin(90°) - \sin(68.19°)\right\}$$

$= 12.87\ \bar{a}_x \approx 13\ \bar{a}_x$ V/m

To find cylindrical co-ordinates find the dot product of $\bar{E}$ with $\bar{a}_r, \bar{a}_\phi$ and $\bar{a}_z$, at point P, referring table of dot products of unit vectors.

$$\therefore \quad E_r = \bar{E} \cdot \bar{a}_r = 13\bar{a}_x \cdot \bar{a}_r = 13 \cos\phi$$

$$\therefore \quad E_\phi = \bar{E} \cdot \bar{a}_\phi = 13\bar{a}_x \cdot \bar{a}_\phi = -13 \sin\phi$$

$$\therefore \quad E_z = \bar{E} \cdot \bar{a}_z = 13\bar{a}_x \cdot \bar{a}_z = 0$$

At point P, x = 2, y = 0, z = 0

$$\therefore \quad r = \sqrt{x^2+y^2} = 2 \text{ and } \phi = \tan^{-1}\frac{y}{x} = \tan^{-1} 0 = 0°$$

$$\therefore \quad \cos\phi = 1 \text{ and } \sin\phi = 0$$

$$\therefore \quad E_r = 13, \quad E_\phi = 0, \quad E_z = 0$$

Hence the cylindrical co-ordinate systems $\bar{E}$ is,

$$\bar{E} = E_r\,\bar{a}_r + E_\phi\,\bar{a}_\phi + E_z\,\bar{a}_z$$

$$\therefore \quad \bar{E} = \mathbf{13}\ \bar{a}_r \ \mathbf{V/m}$$

Example 2.17 : *A circular ring of charge with radius 5 m lies in z = 0 plane with centre at origin. If the ρ_L = 10 nC/m, find the point charge Q placed at the origin which will produce same $\bar{E}$ at the point (0, 0, 5) m.*

Solution : The ring is shown in the Fig. 2.36 (a), in z = 0 i.e. xy plane.

The point P (0, 0, 5) m. Consider the differential length d*l* of the ring. It is in the ϕ direction hence **d*l* = r dϕ.**

The charge on d*l* is $dQ = \rho_L\, dl$

$$\therefore \quad dQ = \rho_L\ r\, d\phi$$

$$\therefore \quad d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$= \frac{\rho_L\ r\,d\phi}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

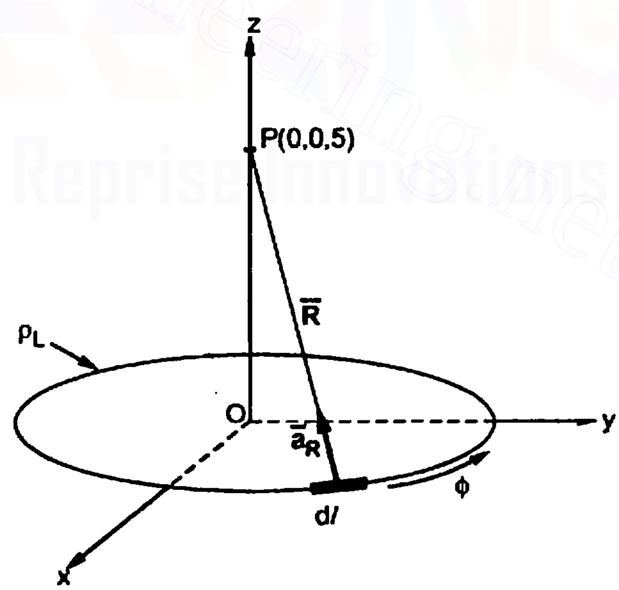


Fig. 2.36 (a)

Now $\bar{a}_R = \dfrac{\bar{R}}{|\bar{R}|}$ and $\bar{R}$ can be

resolved into two components as shown in the Fig. 2.36 (b).

The two components in cylindrical co-ordinate system are,

1. Along $-\bar{a}_r$ direction i.e. $-r\bar{a}_r$.

2. And z component in $\bar{a}_z$ direction i.e. $z\bar{a}_z$.

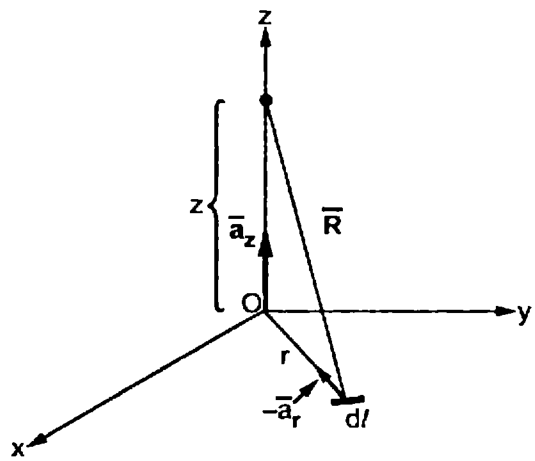


Fig. 2.36 (b)

$$\therefore \quad \bar{R} = -r\,\bar{a}_r + z\bar{a}_z$$

$$\text{hence} \quad |\bar{R}| = \sqrt{(r)^2 + z^2}$$

$$\therefore \quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$\therefore d\bar{E} = \frac{\rho_L\, r\, d\phi}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\left[\frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}\right]$$

Note : The $\bar{E}$ at P will have two components, in radial direction and z direction but radial components are symmetrical about z axis, from all the points of the ring and hence will cancel each other. So there is **no need to consider $\bar{a}_r$ component in integration.** Though if considered, mathematically will get cancelled.

$$\therefore \quad \bar{E} = \int_{\phi=0}^{\phi=2\pi} \frac{\rho_L\, r\, d\phi}{4\pi\varepsilon_0\left(r^2+z^2\right)^{3/2}} z\bar{a}_z \qquad \text{... Limit for } \phi = 0 \text{ to } 2\pi$$

$$= \frac{\rho_L\, r\, z}{4\pi\varepsilon_0\left(r^2+z^2\right)^{3/2}}\left[\int_0^{2\pi} d\phi\right]\bar{a}_z \qquad \text{... r = 5 m, z = 5m}$$

$$= \frac{\rho_L\, r\, z}{4\pi\varepsilon_0\left(r^2+z^2\right)^{3/2}}(2\pi)\,\bar{a}_z = \frac{10\times10^{-9}\times5\times5\times2\pi}{4\pi\times8.854\times10^{-12}\times[25+25]^{3/2}}\bar{a}_z$$

$$\therefore \quad \bar{E} = 39.9314\ \bar{a}_z\ \text{V/m} \qquad \text{... (1)}$$

Let Q be the point charge at the origin. From Q to point P, the distance vector $\bar{R} = 5\,\bar{a}_z$.

$$\therefore\ \bar{E} \text{ due to Q at P} = \frac{Q}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$\text{where} \quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{5\bar{a}_z}{5} = \bar{a}_z$$

$\therefore$ $\bar{E}$ due to Q at P $= \dfrac{Q}{4\pi\varepsilon_0 (5)^2} \bar{a}_z$ ... (2)

Equating (1) and (2),

$$\frac{Q}{4\pi\varepsilon_0 \times 25} = 39.9314$$

$$\therefore \quad Q = 111.071 \text{ nC}$$

Example 2.18 : *A line charge density ρ_L is uniformly distributed over a length of 2a with centre as origin along x axis. Find $\bar{E}$ at a point P which is on the z axis at a distance d.*

Solution : The line charge is shown in the Fig. 2.37 (a). As the charge distribution is not uniform, let us use the basic method of differential length. Consider differential length d*l* along the line charge. As it is along x axis, d*l* = dx.

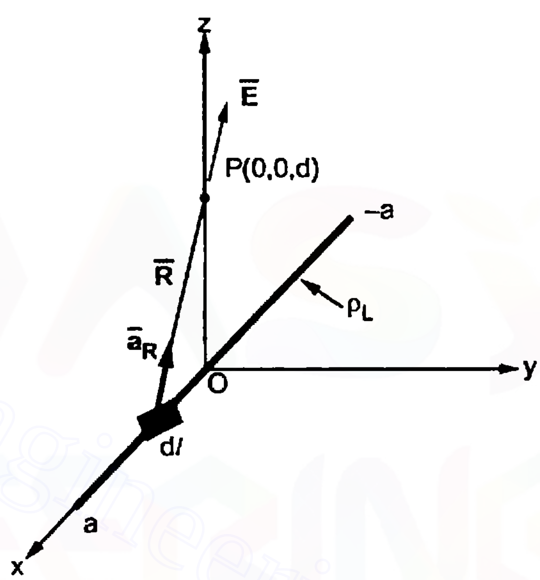


Fig. 2.37 (a)

$$\therefore \quad dQ = \rho_L \, dl = \rho_L \, dx$$

$$\text{Now} \quad d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2} \bar{a}_R = \frac{\rho_L \, dx}{4\pi\varepsilon_0 R^2} \bar{a}_R$$

To find $\bar{R}$, consider any point on the line charge which is say (x, 0, 0). And point P (0, 0, d).

$$\therefore \quad \bar{R} = (0-x)\bar{a}_x + (d-0)\bar{a}_z = -x\bar{a}_x + d\bar{a}_z$$

$$\therefore \quad |\bar{R}| = \sqrt{x^2 + d^2}$$

$$\therefore \quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-x\bar{a}_x + d\bar{a}_z}{\sqrt{x^2+d^2}}$$

$$\therefore \quad d\bar{E} = \frac{\rho_L \, dx}{4\pi\varepsilon_0 \left(\sqrt{x^2+d^2}\right)^2} \left[\frac{-x\bar{a}_x + d\bar{a}_z}{\sqrt{x^2+d^2}}\right]$$

But as charge is along x axis, $\bar{E}$ at P can not have any component in the direction of $\bar{a}_x$. Hence $\bar{a}_x$ component need not be considered in integration.

$$\therefore \qquad \bar{E} = \int_{x=-a}^{x=+a} \frac{\rho_L \, dx \, d\bar{a}_z}{4\pi\varepsilon_0 \left(x^2+d^2\right)^{3/2}}$$

The limits of x = − a to + a, as charge length is 2a.

Put $x = d \tan\theta$ hence $dx = d \sec^2\theta \; d\theta$

For $x = -a, \quad \theta_1 = \tan^{-1}\left(-\frac{a}{d}\right)$

For $x = +a, \quad \theta_2 = \tan^{-1}\left(\frac{a}{d}\right)$

$$\therefore \qquad \bar{E} = \int_{\theta_1}^{\theta_1} \frac{\rho_L \, d\sec^2\theta \, d\theta (d\bar{a}_z)}{4\pi\varepsilon_0 \left(d^2\tan^2\theta + d^2\right)^{3/2}}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{\theta_1}^{\theta_2} \frac{d^2 \sec^2\theta \, d\theta \, \bar{a}_z}{d^3 \left(1+\tan^2\theta\right)^{3/2}} \qquad \ldots.\ 1+\tan^2\theta = \sec^2\theta$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{\theta_1}^{\theta_2} \frac{1}{d} \frac{1}{\sec\theta} \, d\theta \, \bar{a}_z$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 d} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta \, \bar{a}_z$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 d} [\sin\theta]_{\theta_1}^{\theta_2} \, \bar{a}_z$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 d} [\sin\theta_2 - \sin\theta_1] \, \bar{a}_z$$

Now $\tan\theta_1 = -\frac{a}{d}$ and $\tan\theta_2 = \frac{a}{d}$ are shown in the Fig. 2.37 (b).

$$\therefore \qquad \sin\theta_2 = \frac{a}{\sqrt{a^2+d^2}}$$

$$\text{and} \qquad \sin\theta_1 = \frac{-a}{\sqrt{a^2+d^2}}$$

$$\therefore \ \bar{E} = \frac{\rho_L}{4\pi\varepsilon_0 d} \left\{ \frac{a}{\sqrt{a^2+d^2}} - \left[\frac{-a}{\sqrt{a^2+d^2}}\right] \right\} \bar{a}_z$$

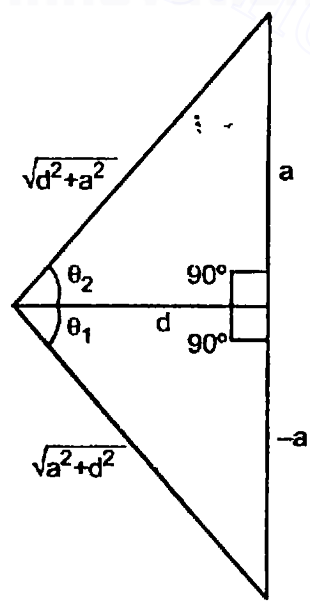


Fig. 2.37 (b)

$$= \frac{\rho_l \, 2a}{4\pi\varepsilon_0 d\sqrt{a^2+d^2}} \bar{a}_z$$

$$\therefore \quad \boxed{\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 d}\left[\frac{a}{\sqrt{a^2+d^2}}\right] a_z \quad V/m}$$

Example 2.19 : *A circular flat ring of inner radius 1 m, and outer radius 2m has $\rho_S = [100/r]\,\mu C/m^2$. Determine $\bar{E}$ on the axis of the ring 10 m away from the center.*

Solution : The ring is shown in the Fig. 2.38 (a) and kept in the xy plane with center as the origin. The z axis is the axis of the ring. Hence point P at which $\bar{E}$ to be calculated is (0, 0, 10).

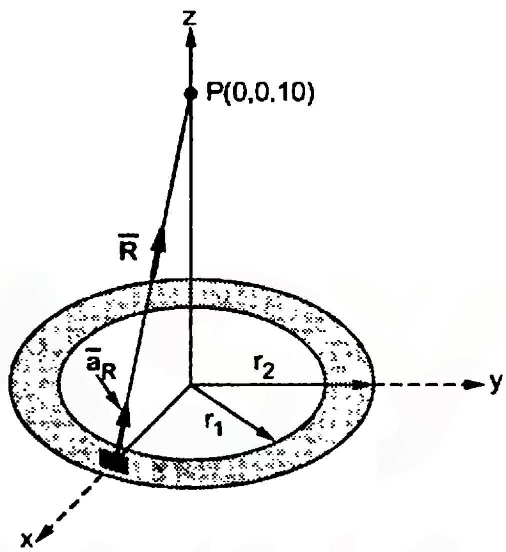


Fig. 2.38 (a)

Consider the differential surface area dS normal to z direction, i.e. normal to xy plane in which ring is placed.

$$\therefore \quad dS = r\,dr\,d\phi$$

Using cylindrical co-ordinate system.

$$\therefore \; dQ = \rho_S\,dS = \frac{100}{r}[r\,dr\,d\phi] = 100\,dr\,d\phi$$

$$\therefore \; d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{100\,dr\,d\phi}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

The $\bar{R}$ has two components as shown in the Fig. 2.38 (b).

1. Along $-\bar{a}_r$ direction having radius r i.e. $-r\bar{a}_r$.
2. Along $\bar{a}_z$ direction having component 10 i.e. $10\,\bar{a}_z$.

$$\therefore \quad \bar{R} = -r\,\bar{a}_r + 10\,\bar{a}_z$$

$$\therefore \quad |\bar{R}| = \sqrt{(-r)^2+(10)^2} = \sqrt{r^2+100}$$

$$\therefore \quad d\bar{E} = \frac{100\,dr\,d\phi}{4\pi\varepsilon_0 (r^2+100)^2}\left[\frac{-r\bar{a}_r + 10\bar{a}_z}{\sqrt{r^2+100}}\right]$$

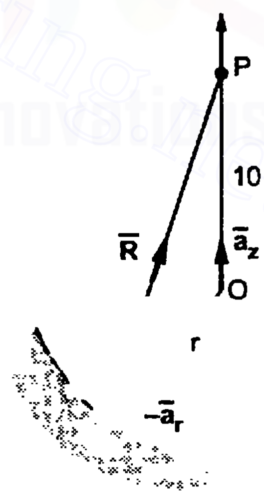


Fig. 2.38 (b)

The radial components of $\bar{E}$ which are in xy plane from all directions are going to cancell. Hence $\bar{a}_r$ component of $\bar{E}$ will be zero. Hence in integration $\bar{a}_r$ need not be considered.

$$\therefore \bar{E} = \int_{\phi=0}^{2\pi}\int_{r=r_1}^{r=r_2} \frac{100\,dr\,d\phi\,(10\bar{a}_z)}{4\pi\varepsilon_0 (r^2+100)^{3/2}} \qquad \text{... Limits of r are } r_1 \text{ to } r_2$$

Put $r = 10 \tan \theta$, hence $dr = 10 \sec^2 \theta \, d\theta$

For $r_1 = 1m$, $\tan \theta_1 = 1/10$, $\theta_1 = 5.7105°$

For $r_2 = 2m$, $\tan \theta_2 = 0.2$, $\theta_2 = 11.309°$

$$\therefore \quad \bar{E} = \int_{\phi=0}^{2\pi} \int_{\theta_1}^{\theta_2} \frac{100(10\sec^2\theta \, d\theta)\, d\phi (10\bar{a}_z)}{4\pi\varepsilon_0 (10)^3 (1+\tan^2\theta)^{3/2}}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta_1}^{\theta_2} \frac{10}{4\pi\varepsilon_0} \cdot \cos\theta \, d\theta \, d\phi \, \bar{a}_z \quad \text{... Using trigonometry results}$$

$$= [\phi]_0^{2\pi} [\sin\theta]_{\theta_1}^{\theta_2} \times \frac{10}{4\pi\varepsilon_0} \bar{a}_z$$

$$= \frac{10}{4\pi\varepsilon_0} \times 2\pi \times [\sin 11.309° - \sin 5.7105°] \, \bar{a}_z$$

$$= 5.455 \times 10^{10} \, \bar{a}_z \; \mu V/m$$

Note that ρ_s given is in $\mu C/m^2$ hence $\bar{E}$ in $\mu V/m$.

$$\therefore \quad \bar{\mathbf{E}} = \mathbf{54.55 \times 10^3 \; \bar{a}_z \;\; V/m}$$

Example 2.20 : *It is required to hold four equal point charges each in equilibrium at the corners of a square. Find the point charge which will do this, if placed at the centroid of the square.*

Solution : Let the sides of square be of length 'a' and each point charge is of magnitude Q coulombs. The square is shown in the Fig. 2.39.

The four corners of square are A, B, C and D while E is the centroid of the square. Let point charge 'q' is placed at E in order to hold the four charges in equilibrium. Let us calculate the force exerted on the charge at A placed at origin, due to all the charges.

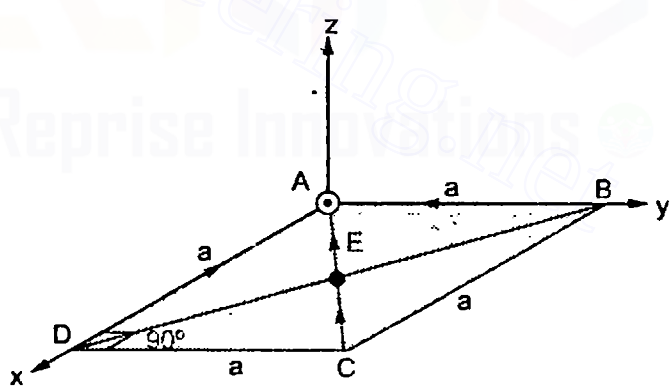


Fig. 2.39

A (0, 0, 0) B (0, a, 0) C (a, a, 0) D (a, 0, 0)

$\therefore \bar{B} = a\,\bar{a}_y$, $\bar{C} = a\,\bar{a}_x + a\,\bar{a}_y$, $D = a\,\bar{a}_x$

while point E is at $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$

$$\therefore \quad \bar{E} = 0.5\,a\,\bar{a}_x + 0.5\,a\,\bar{a}_y$$

Net force $\bar{F}_t$ on charge at point A due to all the charges is,

$$\bar{F}_t = \bar{F}_B + \bar{F}_C + \bar{F}_D + \bar{F}_E$$

$$= \frac{QQ}{4\pi\varepsilon_0 R_{BA}^2}\bar{a}_{BA} + \frac{QQ}{4\pi\varepsilon_0 R_{CA}^2}\bar{a}_{CA} + \frac{QQ}{4\pi\varepsilon_0 R_{DA}^2}\bar{a}_{DA} + \frac{Qq}{4\pi\varepsilon_0 R_{EA}^2}\bar{a}_{EA}$$

$$= \frac{Q^2}{4\pi\varepsilon_0}\left[\frac{1}{R_{BA}^2}\frac{\bar{A}-\bar{B}}{|\bar{A}-\bar{B}|} + \frac{1}{R_{CA}^2}\frac{\bar{A}-\bar{C}}{|\bar{A}-\bar{C}|} + \frac{1}{R_{DA}^2}\frac{\bar{A}-\bar{D}}{|\bar{A}-\bar{D}|}\right] + \frac{Qq}{4\pi\varepsilon_0 R_{EA}^2}\frac{\bar{A}-\bar{E}}{|\bar{A}-\bar{E}|}$$

$\bar{A}-\bar{B} = -a\,\bar{a}_y$ hence $R_{BA} = |\bar{A}-\bar{B}| = \sqrt{(-a)^2} = a$

$\bar{A}-\bar{C} = -a\bar{a}_x - a\bar{a}_y$ hence $R_{CA} = |\bar{A}-\bar{C}| = \sqrt{a^2+a^2} = \sqrt{2}\,a$

$\bar{A}-\bar{D} = -a\bar{a}_x$ hence $R_{DA} = |\bar{A}-\bar{D}| = \sqrt{(-a)^2} = a$

$\bar{A}-\bar{E} = -0.5\,a\,\bar{a}_x - 0.5\,a\,\bar{a}_y$ hence $R_{EA} = |\bar{A}-\bar{E}| = \sqrt{(-0.5\,a)^2 + (-0.5\,a)^2}$

$$= \sqrt{0.5}\,a$$

$$\therefore \quad \bar{F}_t = \frac{Q^2}{4\pi\varepsilon_0}\left[\frac{1}{a^2}\times\frac{-a\,\bar{a}_y}{a} + \frac{1}{2a^2}\times\frac{-a\,\bar{a}_x - a\,\bar{a}_y}{\sqrt{2}\,a} + \frac{1}{a^2}\,\frac{-a\,\bar{a}_x}{a}\right] + \frac{Qq}{4\pi\varepsilon_0}\times\frac{1}{0.5\,a^2}\,\frac{-0.5\,a\,\bar{a}_x - 0.5\,a\,\bar{a}_y}{\sqrt{0.5}\,a}$$

$$\therefore \quad \bar{F}_t = \frac{Q^2}{4\pi\varepsilon_0}\left[\frac{\bar{a}_x}{a^2}\left(-1-\frac{1}{2\sqrt{2}}\right) + \frac{\bar{a}_y}{a^2}\left(-1-\frac{1}{2\sqrt{2}}\right)\right] + \frac{Qq}{4\pi\varepsilon_0}\cdot\frac{1}{\sqrt{0.5}\,a^2}\left[-\bar{a}_x - \bar{a}_y\right]$$

$$\therefore \quad \bar{F}_t = \frac{Q^2}{4\pi\varepsilon_0 a^2}\left[-1.3535\,\bar{a}_x - 1.3535\,\bar{a}_y\right] + \frac{Qq}{4\pi\varepsilon_0\sqrt{0.5}\,a^2}\left[-\bar{a}_x - \bar{a}_y\right]$$

$$\therefore \quad \bar{F}_t = \frac{Q}{4\pi\varepsilon_0 a^2}\left[\left(-1.3535\,Q - \frac{q}{\sqrt{0.5}}\right)\bar{a}_x + \left(-1.3535\,Q - \frac{q}{\sqrt{0.5}}\right)\bar{a}_y\right]$$

To hold all the charges in equilibrium, the net force exerted on any of the charges due to other charges must be zero. i.e. $\bar{F}_t = 0$.

But $\dfrac{Q}{4\pi\varepsilon_0 a^2}$ can not be zero.

$$\therefore \quad -1.3535\,Q - \frac{q}{\sqrt{0.5}} = 0$$

$$\therefore \quad q = -1.3535 \times \sqrt{0.5}\,Q$$

$$= -0.9571\,Q \text{ C}$$

This is the charge required at the centroid to hold all the charges in equilibrium.

Thus if Q = 1 μC then q = – 0.9571 μC and so on.

Example 2.21 : *Two small identical conducting spheres have charges of 2 nC and –1 nC respectively. When they are separated by 4 cm apart, find the magnitude of the force between them. If they are brought into contact and then again separated by 4 cm, find the force between them.*

Solution : Case 1 : Before the charges are brought into contact

$$|\bar{F}| = \left|\frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\right| \quad \text{where } R_{12} = 4\text{ cm} = 4\times10^{-2}\text{ m}$$

$$= \left|\frac{2\times10^{-9}\times(-1\times10^{-9})}{4\pi\varepsilon_0\times(4\times10^{-12})^2}\right| = \mathbf{11.234\ \mu N}$$

Case 2 : The charges are brought into contact and then separated.

When charges are brought into contact, the charge distribution takes place due to transfer of charge. The transfer of charge continues till both the charges attain same value due to equal division of the two charges.

$$\therefore \text{ Charge on each sphere} = \frac{Q_1+Q_2}{2} = \frac{(2\times10^{-9})+(-1\times10^{-9})}{2}$$

$$= \frac{(2-1)\times10^{-9}}{2} = 0.5\text{ nC}$$

$$\therefore \quad |\bar{F}| = \left|\frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\right| = \left|\frac{0.5\times10^{-9}\times0.5\times10^{-9}}{4\pi\times\varepsilon_0\times(4\times10^{-2})^2}\right|$$

$$= \mathbf{1.404\ \mu N}$$

Note : that initially before charges are brought together the force between them was attractive as charges are of opposite polarity. But when they brought in contact and then seperated, the force is repulsive in nature.

Example 2.22 : *Two infinite sheets of uniform charge densities* $\rho_S = \frac{10^{-9}}{6\pi}$ *C / m² are located at z = –5 m and y = –5 m. Determine the uniform line charge density* ρ_L *necessary to produce same value of* $\bar{E}$ *at (4, 2, 2) m if the line charge is at y = 0, z = 0.*

Solution : The two sheets are shown in the Fig. 2.40.

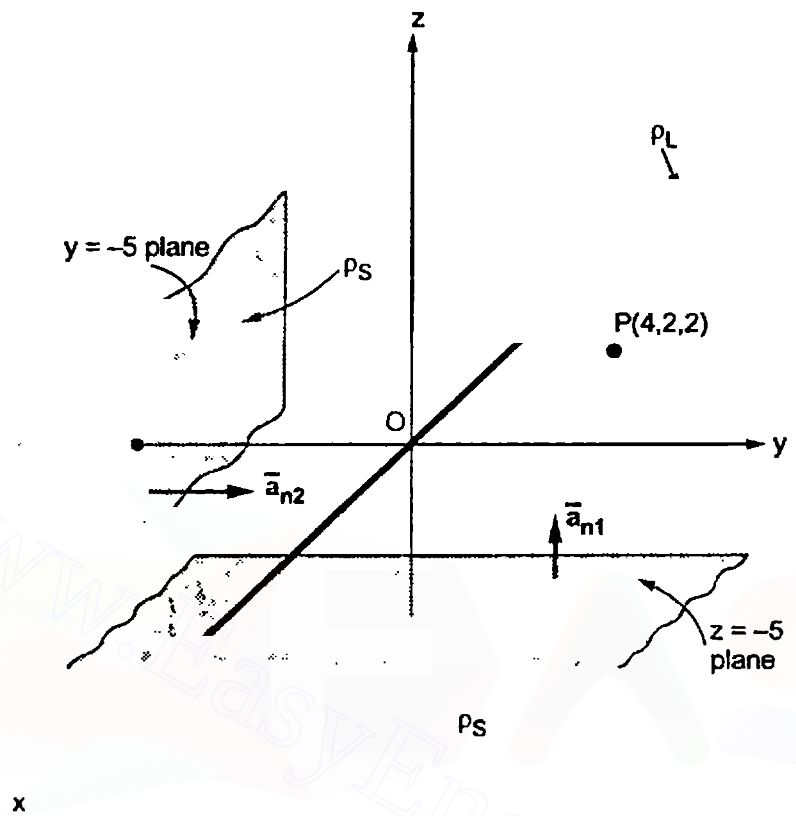


Fig. 2.40

The line charge ρ_L is to be located along y = 0, z = 0 line i.e. x axis.

For z = – 5 plane, the normal direction is $\bar{a}_{n1} = \bar{a}_z$ as the plane is parallel to xy plane. For y = – 5 plane, the normal direction is $\bar{a}_{n2} = \bar{a}_y$ as the plane is parallel to xz plane.

$$\therefore \qquad \bar{E}_1 = \frac{\rho_S}{2\varepsilon_0}\bar{a}_{n1} = \frac{\rho_S}{2\varepsilon_0}\bar{a}_z$$

and

$$\bar{E}_2 = \frac{\rho_S}{2\varepsilon_0}\bar{a}_{n2} = \frac{\rho_S}{2\varepsilon_0}\bar{a}_y$$

$$\therefore \qquad \bar{E} \text{ at P} = \bar{E}_1 + \bar{E}_2 = \frac{\rho_S}{2\varepsilon_0}\left[\bar{a}_y + \bar{a}_z\right] \text{ V/m} \qquad \ldots (1)$$

Consider line charge along x axis. As it is infinite,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r = \frac{\rho_L}{2\pi\varepsilon_0 r}\left[\frac{\bar{r}}{|\bar{r}|}\right]$$

For $\bar{r}$, consider a point on the line charge (x, 0, 0) while P (4, 2, 2). But as line charge is along x axis, $\bar{E}$ will not have component in $\bar{a}_x$ direction so the x coordinate should not be considered while calculating $\bar{r}$.

$$\therefore \quad \bar{r} = (2-0)\bar{a}_y + (2-0)\bar{a}_z = 2\bar{a}_y + 2\bar{a}_z$$

$$\therefore \quad |\bar{r}| = \sqrt{(2)^2 + (2)^2} = \sqrt{8}$$

$$\therefore \quad \bar{E} = \frac{\rho_L}{2\pi\varepsilon_0(\sqrt{8})}\left[\frac{2\bar{a}_y + 2\bar{a}_z}{\sqrt{8}}\right] = \frac{\rho_L}{8\pi\varepsilon_0}[\bar{a}_y + \bar{a}_z] \text{ V/m} \quad \text{... (2)}$$

To have same $\bar{E}$ at P (4, 2, 2) equate (1) and (2)

$$\therefore \quad \frac{\rho_S}{2\varepsilon_0} = \frac{\rho_L}{8\pi\varepsilon_0}$$

$$\therefore \quad \rho_L = 4\pi \times \frac{10^{-9}}{6\pi} = 0.666 \text{ nC/m}$$

This is the required line charge density.

Example 2.23 : *An infinite sheet with surface charge Q = 12 ε_0 Cm^{-2} is lying in the plane x – 2y + 3z = 4. Find an expression for the field-intensity on the side of the plane containing the origin.* **(UPTU : 2005-06, 5 Marks)**

Solution : The plane is shown in the Fig. 2.41. The plane can be defined uniquely from three points which can be obtained from the equation of plane x – 2y + 3z = 4.

For x = 0, y = 0, z = $\frac{4}{3}$

$$\therefore \quad P\left(0, 0, \frac{4}{3}\right)$$

For x = 0, z = 0, y = – 2

∴ Q (0, – 2, 0)

For y = 0, z = 0, x = 4

∴ (4, 0, 0)

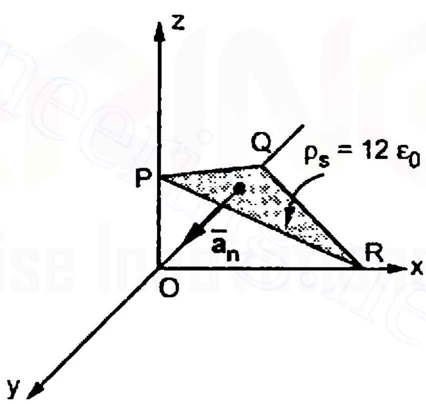


Fig. 2.41

The three poionts P, Q and R define a plane.

The plane is infinite sheet of charge.

$$\therefore \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0}\bar{a}_n = \frac{12\varepsilon_0}{2\varepsilon_0}\bar{a}_n = 6\,\bar{a}_n \text{ V/m}$$

$\bar{a}_n$ = Unit vector normal to the plane.

Note : If plane is defined as Ax + By + Cz = D then the unit vector normal to the plane is,

$$\bar{a}_n = \pm\left[\frac{A\,\bar{a}_x + B\bar{a}_y + C\bar{a}_z}{\sqrt{A^2+B^2+C^2}}\right]$$

Positive sign for front side of the plane.

Negative sign for back side of the plane.

In this case, A = 1, B = – 2, C = 3, D = 4

$$\therefore \qquad \bar{a}_n = \pm\left[\frac{\bar{a}_x - 2\bar{a}_y + 3\bar{a}_z}{\sqrt{1^2+2^2+3^2}}\right] = \pm\,[0.2672\,\bar{a}_x - 0.5345\,\bar{a}_y + 0.8017\,\bar{a}_z]$$

The origin is on the back side of the plane so use negative sign.

$$\therefore \qquad \bar{E} = 6\,[-0.2672\,\bar{a}_x - 0.5345\,\bar{a}_y + 0.8017\,\bar{a}_z]$$

$$\therefore \qquad \bar{E} = -1.6035\,\bar{a}_x - 3.207\,\bar{a}_y - 4.8102\,\bar{a}_z \text{ V/m}$$

Example 2.24 : *Three point charges $q_1 = 10^{-6}$ C, $q_2 = -10^{-6}$ C and $q_3 = 0.5 \times 10^{-6}$ C are located in air at the corners of an equilateral triangle of 50 cm side. Determine the magnitude and direction of the force on q_3.* **(UPTU : 2006-07, 5 Marks)**

Solution : The arrangement is shown in the Fig. 2.42. The triangle is placed in x-y plane with three corners at O(0, 0, 0), Q (0.5, 0, 0) and P.

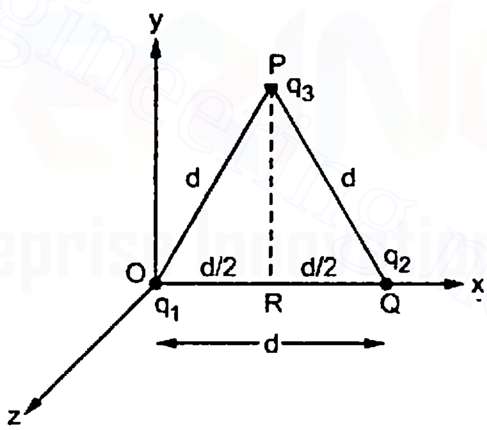


Fig. 2.42

The distance PR can be obtained as,

$$PR = \sqrt{d^2 - \left(\frac{d}{2}\right)^2}$$

$$= \frac{\sqrt{3}\,d}{2}$$

$\therefore$ Co-ordinates of P $\left(\frac{d}{2}, \frac{\sqrt{3}d}{2}, 0\right)$ i.e. (0.25, 0.433, 0).

Force on q_3 due to q_1 is,

$$\bar{F}_{31} = \frac{q_3 q_1}{4\pi\,\varepsilon_0 R_{OP}^2}\,\bar{a}_{OP} \quad \text{where } R_{OP} = 0.25\,\bar{a}_x + 0.433\,\bar{a}_y + 0\,\bar{a}_z$$

$$\therefore \quad \bar{F}_{31} = \frac{0.5\times10^{-6}\times10^{-6}}{4\pi\times8.854\times10^{-12}\times[(0.25)^2+(0.433)^2]}\ \frac{\bar{R}_{OP}}{|\bar{R}_{OP}|}$$

$$= 0.01797 \times \frac{[0.25\,\bar{a}_x + 0.433\,\bar{a}_y + 0\,\bar{a}_z]}{0.5}$$

$$= 8.985\times10^{-3}\,\bar{a}_x + 0.01556\,\bar{a}_y + 0\,\bar{a}_z \text{ N}$$

Force on q_3 due to q_2 is,

$$\bar{F}_{32} = \frac{q_3 q_2}{4\pi\,\varepsilon_0 R_{QP}^2}\,\bar{a}_{QP} \quad \text{where } \bar{R}_{QP} = (0.25-0.5)\,\bar{a}_x + (0.433-0)\,\bar{a}_y + 0\,\bar{a}_z$$

$$\therefore \quad \bar{R}_{QP} = -0.25\,\bar{a}_x + 0.433\,\bar{a}_y + 0\,\bar{a}_z, \quad |\bar{R}_{QP}| = 0.5$$

$$\therefore \quad \bar{F}_{32} = \frac{05\times10^{-6}\times(-10^{-6})}{4\times\pi\times8.854\times10^{-12}\times05^2}\times\frac{[-0.25\,\bar{a}_x + 0.433\bar{a}_y + 0\bar{a}_z]}{0.5}$$

$$= +8.985\times10^{-3}\,\bar{a}_x - 0.01556\,\bar{a}_y + 0\,\bar{a}_z \text{ N}$$

$$\therefore \quad \bar{F}_3 = \bar{F}_{31} + \bar{F}_{32} = 0.01797\,\bar{a}_x \text{ N}, \quad |\bar{F}_3| = \mathbf{0.01797\ N}$$

Example 2.25 : *Two uniform line charges of density $\rho_l = 4$ nC/m lie in the $x = 0$ plane at $y = \pm 4$ m.*

Find $\bar{E}$ at (4, 0, 10) m. **[UPTU : 2006-07, 5 Marks]**

Solution : The line charges are shown in the Fig. 2.43. The line charges with x = 0 and $y = \pm 4$ are parallel to the z-axis, as z can take any value.

Key Point: *As line charges are parallel to z axis, $\bar{E}$ at P can not have any component in z direction hence while calculating $\bar{r}$ and $\bar{a}_r$, the z co-ordinate need not be considered.*

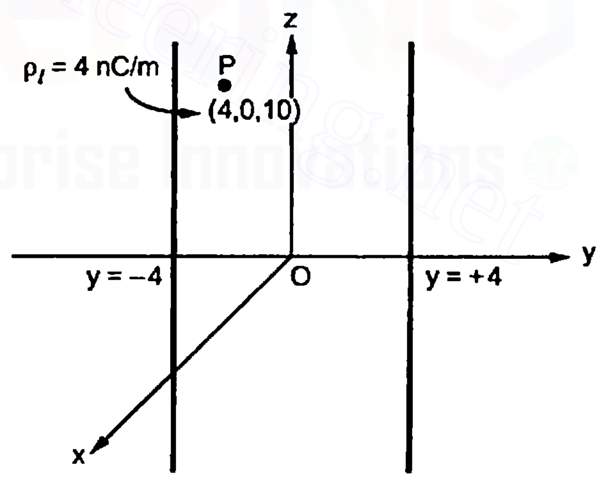


Fig. 2.43

$\bar{E}_1$ due to ρ_l at y = + 4 m : (0, 4, z) and P(4, 0, 10)

$$\therefore \ \bar{r} = (4-0)\,\bar{a}_x + (0-4)\,\bar{a}_y$$

$$= 4\,\bar{a}_x - 4\,\bar{a}_y, \quad |\bar{r}| = \sqrt{32}$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{4\bar{a}_x - 4\bar{a}_y}{\sqrt{32}}$$

$$\therefore \quad \bar{E}_1 = \frac{\rho_l}{2\pi\varepsilon_0\times r}\,\bar{a}_r = \frac{4\times10^{-9}}{2\pi\times8.854\times10^{-12}\times\sqrt{32}}\times\left[\frac{4\bar{a}_x - 4\bar{a}_y}{\sqrt{32}}\right]$$

$$= 8.9877\,\bar{a}_x - 8.9877\,\bar{a}_y \quad \text{V/m}$$

$\bar{E}_2$ due to ρ_l at y = – 4 m : (0, – 4, z) and P(4, 0, 10)

$$\therefore \quad \bar{r} = (4-0)\,\bar{a}_x + [0-(-4)]\,\bar{a}_y = 4\,\bar{a}_x + 4\,\bar{a}_y,\ |\bar{r}| = \sqrt{32}$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{4\bar{a}_x + 4\bar{a}_y}{\sqrt{32}}$$

$$\therefore \quad \bar{E}_2 = \frac{\rho_l}{2\pi\varepsilon_0 \times r}\,\bar{a}_r = \frac{4\times10^{-9}}{2\pi\times 8.854\times10^{-12}\times\sqrt{32}} \times \left[\frac{4\bar{a}_x + 4\bar{a}_y}{\sqrt{32}}\right]$$

$$= 8.9877\,\bar{a}_x + 8.9877\,\bar{a}_y \text{ V/m}$$

$$\therefore \quad \bar{E}_P = \bar{E}_1 + \bar{E}_2 = \mathbf{17.9755}\,\bar{a}_x \textbf{ V/m}$$

Example 2.26 : *A sphere of radius 2 cm is having volume charge density of ρ_v given by $\rho_v = \cos^2\theta$. Find the total charge Q contained in the sphere.* **[UPTU : 2007-08, 5 Marks]**

Solution : $\rho_v = \cos^2\theta$

$$Q = \int_{vol} \rho_v \, dv \text{ where } dv = r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{2} [r^2 \sin\theta\, dr\, d\theta\, d\phi]\cos^2\theta$$

$$= \left[\frac{r^3}{3}\right]_0^2 [\phi]_0^{2\pi} \left[\int_{\theta=0}^{\pi} \sin\theta\cos^2\theta\, d\theta\right] = \frac{8}{3}\times 2\pi\times I$$

Consider $\quad I = \int_{\theta=0}^{\pi} \sin\theta\cos^2\theta\, d\theta$

Put $\cos\theta = t$ i.e. $-\sin\theta d\theta = dt$

$$\therefore \quad I = \int_{\theta=0}^{\pi} -dt\times t^2 = -\left[\frac{t^3}{3}\right]_{\theta=0}^{\pi}$$

$$= -\left[\frac{\cos^3\theta}{3}\right]_0^{\pi} = -\left[\frac{\cos^3\pi - \cos^3 0}{3}\right]$$

$$= -\left[\frac{(-1)^3-(1)^3}{3}\right] = -\left[-\frac{1}{3}-\frac{1}{3}\right] = +\frac{2}{3}$$

$$\therefore \quad Q = \frac{8}{3}\times 2\pi\times\frac{2}{3} = \frac{32\pi}{9} = \mathbf{11.1701\ C}$$

Review Questions

1. *State Coulomb's law of force between any two point charges and state the units of force.*
2. *Define electric field intensity. Obtain an expression for the electric field intensity at a point which is at a distance of 'R' from a point charge Q.*
3. *State the units of electric field intensity $\bar{E}$ and explain the method of obtaining $\bar{E}$ at a point in cartesian system, due to a point charge Q.*
4. *Obtain an expression for total electric field intensity at a point due to infinite number of point charges.*
5. *Obtain an expression for total force experienced by a point charge due to infinite number of point charges around it.*
6. *Which are the various types of charge distributions ? Explain. State the units of line charge density, surface charge density and volume charge density.*
7. *A charge is distributed on y axis of cartesian system having a line charge density of* $5y^{3.5}\,\mu C/m$. *Find the total charge over the length of 15 m.* **[Ans. : 0.2178 C]**
8. *Find the total charge inside a volume having volume charge density as* $15z^3 e^{-0.3x}\sin\pi y\left(mC/m^3\right)$. *The volume is defined between* $-1 \le x \le 1$, $0 \le y \le 1$ *and* $2 \le z \le 5$. **[Ans. : 2.9515 C]**
9. *Explain the procedure of obtaining $\bar{E}$ due to the line charge, surface charge and volume charge.*
10. *Obtain an expression for an electric field due to infinite line charge having density ρ_L C/m, placed along z-axis, at a point P on y axis at a distance of d from the z axis.* **[Ans. : $\left(\rho_L / 2\pi\varepsilon_0 d\right)\bar{a}_y\, V/m$]**
11. *Obtain an exression for an electric field due to charged circular ring of radius 'h' placed in xy plane, at a point P (0, 0, z), having uniform line charge density of ρ_L C/m.* **[Ans. : $\dfrac{\rho_L hz}{2\varepsilon_0\left(h^2+z^2\right)^{3/2}}\bar{a}_z\, V/m$]**
12. *A charge of + 10 C is located at the point x = 0 and y = 1 and charge of − 5C is at the point x = 0 and y = − 1. Find the point on y axis at which net $\bar{E} = 0$.* **[Ans. : (0, − 5.828, 0) or (0, − 0.1716, 0)]**
13. *A point charge of 20 nC is located at the origin. Determine the magnitude and direction of $\bar{E}$ at point P (1, 3, −4) m.* **[Ans. : $1.357\,\bar{a}_x + 4.073\,\bar{a}_y - 0.784\,\bar{a}_z$ V/m]**
14. *A circular disc of 10 cm radius is charged uniformly all over the surface with total charge of* $100\,\mu C$. *Find $\bar{E}$ at a point 20 cm away from the disc along its axis.*
[Hint : Find ρ_s = Q/surface area and then $\bar{E}$.] **[Ans. : $18.98 \times 10^6\,\bar{a}_n$ V/m]**
15. *Derive the expression for the electric field due to infinite sheet of charge placed in xy plane, having surface charge density of $\rho_S C/m^2$.* **[Ans. : $\dfrac{\rho_S}{2\varepsilon_0}\bar{a}_z$ V/m]**

16. *Calculate the force on a point charge of 50 μC placed at a point (0, 0, 5) m due to a charge of 500 μC that is uniformly distributed over a circular disc of radius 5 m and placed in the xy plane.*
[Ans. : 5.26 $\bar{a}_z$ N]

17. *Four point charges, each 20 μC are on the x and y axes at ± 4m. Fnd the force on a 200 μC point charge at (0, 0, 3) m.* **[Ans. : 3.456 $\bar{a}_z$ N]**

18. *A charge $Q_2 = 121 \times 10^{-9}$ C is located in vacuum at P_2 (– 0.03, 0.01, 0.04) m. Find the force on Q_2 due to $Q_1 = 110$ μC at P_1 (0.03, 0.08, – 0.02) m.* **[Ans. : 5.44 $\bar{a}_x$ – 6.33 $\bar{a}_y$ + 5.44 $\bar{a}_z$ N]**

19. *Eight 25 nC point charges in free space are located symmetrically on a circle of radius 0.2 m centered at the origin in the z = 0 plane. a) At what point on the z axis is $|\bar{E}|$ is maximum? b) What is the magnitude of maximum $\bar{E}$?* **[Ans. : (0, 0, ± 0.1414), 17.3 kV/m]**

20. *A ring of radius 6 m is placed in yz plane. It is centered at origin. Find electric field intensity at point (8, 0, 0) m. The line charge density is 18 nC/m.* **[Ans. : 48.75 $\bar{a}_x$ V/m]**

21. *A charge is distributed along z axis between ± 6 m with uniform charge density 25 nC/m. Calculate $\bar{E}$ at a point (2, 0, 0) m in free space.* **[Ans. : 213.16 $\bar{a}_x$ V/m]**

22. *A uniform line charge of infinite length with ρ_L = 20 nC/m lies along z axis. Find $\bar{E}$ at (6,8,3) m.*
[Ans. : 21.57 $\bar{a}_x$ + 28.76 $\bar{a}_y$ V/m]

23. *On the line x = 4 and y = – 4, there is a uniform charge distribution with density $\rho_L = 25$ nC/m. Determine $\bar{E}$ at (–2, –1, 4) m.* **[Ans. : – 59.92 $\bar{a}_x$ + 29.96 $\bar{a}_y$ V/m]**

24. *The infinite line charge parallel to z axis is at x = 6, y = 10. Find $\bar{E}$ at the general point $P(x,y,z)$ in cartesian system.*

[Ans. : $\dfrac{\rho_L}{2\pi\varepsilon_0\left[(x-6)^2+(y-10)^2\right]}\left[(x-6)\bar{a}_x+(y-10)\bar{a}_y\right]$ **V/m]**

25. *Find $\bar{E}$ at (10, 0, 0) due to a charge of 10 nC which is distributed uniformly along x axis between x = – 5 to + 5 m in free space.* **[Ans. : 1.8 $\bar{a}_x$ V/m]**

26. *A line charge density 24 nC/m is located in free space on the line y = 1, z = 2.*
a) Find $\bar{E}$ at P (6, –1, 3).
b) What point charge Q_A should be located at (–3, 4, 1) to cause y component of $\bar{E}$ to be zero at P ? **[Ans. : – 172.56 $\bar{a}_y$ + 86.28 $\bar{a}_z$ V/m, 4.43 μC]**

27. *Find $\bar{E}$ at P (0, 0, 2) m due to the infinite sheet of charge in xy plane with density 10 nC / m².*
[Ans. : 564.71 $\bar{a}_z$ V/m]

28. *Two infinite sheets of charge each with density ρ_S are located at $x = \pm 2$ m. Determine $\bar{E}$ in all directions.* **[Ans. : For x < –2 :** $-\dfrac{\rho_S}{\varepsilon_0}\bar{a}_x$**, For – 2 < x < 2 : 0, For x > 2 :** $\dfrac{\rho_S}{\varepsilon_0}\bar{a}_x$ **in V/m]**

29. *Four infinite sheets of charges with uniform charge densities 20 pC/m², – 8 pC/m², 6 pC/m² and –18 pC/m² are located at y = 6, y = 2, y = – 2 and y = – 5 respectively.*
Find $\bar{E}$ at a) (2, 5, –6) b) (0, 0, 0) c) (– 1, – 2.1, 6) d) $(10^6, 10^6, 10^7)$.
[Ans. : – 2.26 $\bar{a}_y$ V/m, – 1.355 $\bar{a}_y$ V/m, – 2.03 $\bar{a}_y$ V/m, 0 V/m]

30. A sheet of charge with $\rho_S = 2nC/m^2$ is in the plane x = 2 in free space and a line charge $\rho_l = 20$ nC/m is located at x = 1, z = 4.

a) Find $\bar{E}$ at P (0, 0, 0). b) $\bar{E}$ at (4, 5, 6).

c) What is the force per unit length on the line charge ?

[Ans. : $-134\,\bar{a}_x - 85\,\bar{a}_z$ V/m, $196\,\bar{a}_x + 55.31\,\bar{a}_z$ V/m, $-2.26\,\bar{a}_x$ μ N/m]

University Questions

1. If Q_1, Q_2 are carges located at distances r_1, r_2, ... from the point P, then the electric-intensity $\bar{E}$ at point P will be :

$$\bar{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{i=n}(Q_i / r_i^2)\bar{a}_{ri}$$

(UPTU : 2005-06, 5 Marks)

2. An infinite sheet with surface charge Q = 12 ε_0 Cm^{-2} is lying in the plane x − 2y + 3z = 4. Find an expression for the field-intensity on the side of the plane containing the origin. **(UPTU : 2005-06 , 5 Marks)**

3. State the word statement of Coulomb's law of forces. Three ponit charges $q_1 = 10^{-6}$C, $q_2 = -10^{-6}$C and $q_3 = 0.5 \times 10^{-6}$ C are located in air at the corners of an equilateral triangle of 50 cm side. Determine the magnitude and direction of the force on q_3. **(UPTU : 2006-07, 5 Marks)**

4. Two uniform line charges of density $\rho_l = 4$ nC/m lie in the x = 0 plane at y = ± 4m Find $\bar{E}$ at (4, 0, 10) m. **(UPTU : 2007-08, 5 Marks)**

5. A sphere of radius 2 cm is having volume charge density of ρ_v, given by $\rho_v = \cos^2\theta$. Find the total charge Q contained in the sphere. **(UPTU : 2007-08, 5 Marks)**

6. An infinite long line charge of uniform density ρ_L coulombs/cm is situated along the z-axis. Obtain electric field intensity due to this charge using Gauss's law. **(UPTU : 2007-08, 5 Marks)**

❑❑❑

3 Electric Flux Density and Gauss's Law

3.1 Introduction

Uptill now Coulomb's law and electric field intensity are discussed. The various possible charge distributions and corresponding electric field intensities are also discussed in the last chapter. Another important concept in electrostatics is electric flux. If a unit test charge is placed near a point charge, it experiences a force. The direction of this force can be represented by the lines, radially coming outward from a positive charge. These lines are called **streamlines** or **flux lines**. Thus the electric field due to a charge can be imagined to be present around it interms of a quantity called electric flux. The flux lines give the pictorial representation of distribution of electric flux around a charge. This chapter explains the concept of electric flux, electric flux density, Gauss's law, applications of Gauss's law and the divergence theorem.

3.2 Electric Flux

In 1837, **Michael Faraday** performed the experiment on electric field. He showed that the electric field around a charge can be imagined interms of presence of the lines of force around it. He suggested that the electric field should be assumed to be composed of very small bunches containing a fixed number of electric lines of force. Such a bunch or closed area is called a **tube of flux**. The total number of tubes of flux in any particular electric field is called as the **electric flux**.

> **Key Point:** *Thus the total number of lines of force in any particular electric field is called the electric flux. It is represented by the symbol ψ. Similar to the charge, unit of electric flux is also coulomb C.*

3.2.1 Properties of Flux Lines

The electric flux is nothing but the lines of force, around a charge. Such electric flux lines have following properties.

1. The flux lines start from positive charge and terminate on the negative charge as shown in the Fig. 3.1.

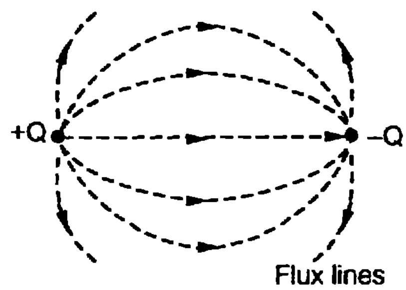


Fig. 3.1 Flux lines

2. If the negative charge is absent, then the flux lines terminate at infinity as shown in the Fig. 3.2. (a). While in absence of positive charge, the electric flux terminates on the negative charge from infinity. This is shown in the Fig. 3.2 (b).

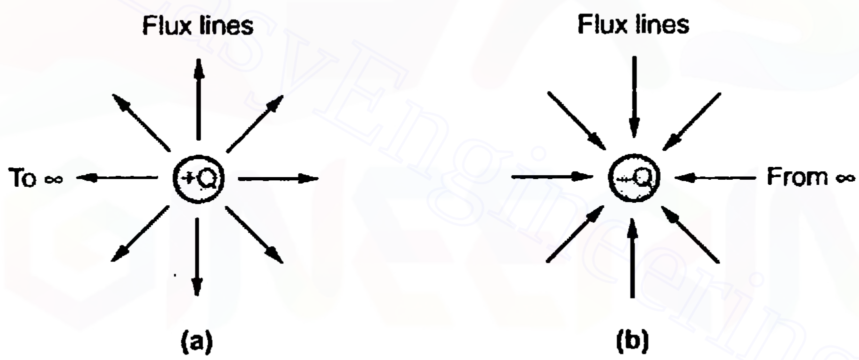


Fig. 3.2

3. There are more number of lines i.e. crowding of lines if electric field is stronger.

4. These lines are parallel and never cross each other.

5. The lines are independent of the medium in which charges are placed.

6. The lines always enter or leave the charged surface, normally.

7. If the charge on a body is $\pm$ Q coulombs, then the total number of lines originating or terminating on it is also Q. But the total number of lines is nothing but a flux.

$$\therefore \quad \boxed{\text{Electric flux } \psi \; = \; \text{Q coulombs (numerically)}}$$

This is according to SI units. Hence if Q is large then flux ψ is more surrounding the charge and vice versa.

The electric flux is also called **displacement flux.**

The flux is a scalar field. Let us define now a vector field associated with the flux called electric flux density.

3.3 Electric Flux Density ($\bar{D}$)

Consider the two point charges as shown in the Fig. 3.3. The flux lines originating from positive charge and terminating at negative charge are shown in the form of tubes.

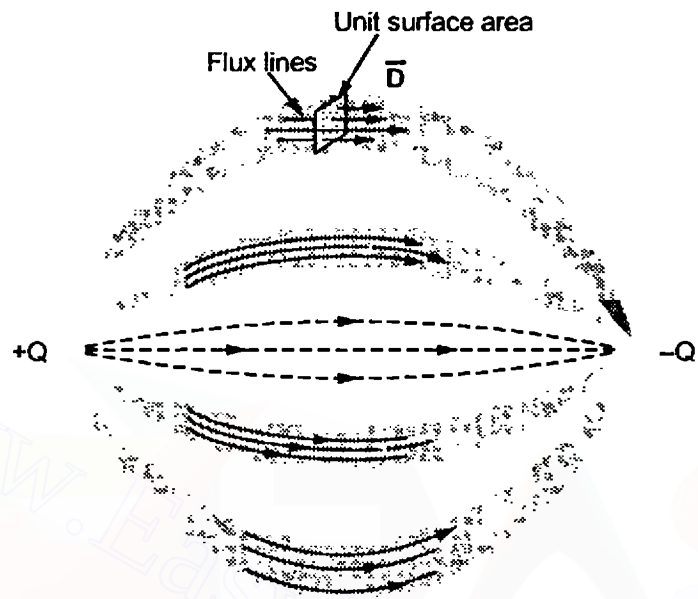


Fig. 3.3 Concept of electric flux density

Consider a unit surface area as shown in the Fig. 3.3. The number of flux lines are passing through this surface area.

The net flux passing normal through the unit surface area is called the **electric flux density**. It is denoted as $\bar{D}$. It has a specific direction which is normal to the surface area under consideration hence it is a vector field.

Consider a sphere with a charge Q placed at its centre. There are no other charges present around. The total flux distributes radially around the charge is $\psi = Q$. This flux distributes uniformly over the surface of the sphere.

Now, ψ = Total flux

While, S = Total surface area of sphere

then electric flux density is defined as,

$$\boxed{D = \frac{\psi}{S} \text{ in magnitude}} \qquad \text{... (1)}$$

As ψ is measured in coulombs and S in square metres, the units of D are C/m^2. This is also called **displacement flux density or displacement density.**

3.3.1 Vector Form of Electric Flux Density

Consider the flux distribution, due to a certain charge in the free space as shown in the Fig. 3.4.

Consider the differential surface area dS at point P. The flux crossing through this differential area is $d\psi$. The direction of $\overline{D}$ is same as that of direction of flux lines at that point. The differential area and flux lines are at right angles to each other at point P. Hence the direction of $\overline{D}$ is also normal to the surface area, in the direction of unit vector $\bar{a}_n$ which is normal to the surface area dS. Near point P, all the lines of flux $d\psi$ are having direction of that of $\bar{a}_n$ as the differential area dS is very small. Hence the flux density $\overline{D}$ at the point P can be represented in the vector form as,

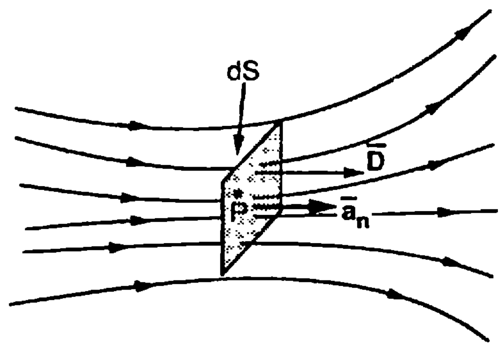


Fig. 3.4 Flux through dS

$$\boxed{\overline{D} = \frac{d\psi}{dS}\,\bar{a}_n \quad C/m^2} \quad \dots (2)$$

where $d\psi$ = Total flux lines crossing normal through the differential area dS

dS = Differential surface area

$\bar{a}_n$ = Unit vector in the direction normal to the differential surface area

3.4 $\overline{D}$ due to a Point Charge Q

Consider a point charge +Q placed at the centre of the imaginary sphere of radius r. This is shown in the Fig. 3.5.

The flux lines originating from the point charge + Q are directed radially outwards. The magnitude of the flux density at any point on the surface is,

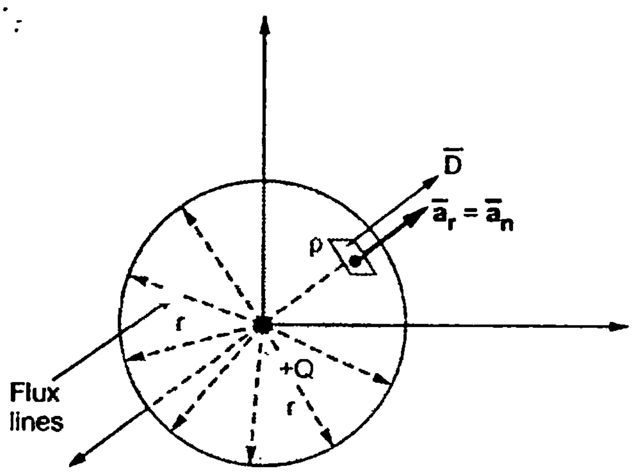


Fig. 3.5

$$|\bar{D}| = \frac{\text{Total flux } \psi}{\text{Total surface area S}} \quad \text{... (1)}$$

But $\psi = Q = \text{Total flux}$

and $S = 4\pi r^2 = \text{Total surface area}$

$\therefore$ $$|\bar{D}| = \frac{Q}{4\pi r^2} \quad \text{... (2)}$$

The unit vector directed radially outwards and normal to the surface at any point on the sphere is $\bar{a}_n = \bar{a}_r$.

Thus in the vector form, **electric flux density at a point which is at a distance of r, from the point charge + Q is given by,**

$$\boxed{\bar{D} = \frac{Q}{4\pi r^2}\bar{a}_r \text{ C/m}^2} \quad \text{... (3)}$$

3.5 Relationship between $\bar{D}$ and $\bar{E}$

In the last chapter, it has been derived that the electric field intensity $\bar{E}$ at a distance of r, from a point charge + Q is given by,

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r$$

Dividing the equations of $\bar{D}$ and $\bar{E}$ due to a point charge + Q we get,

$$\frac{\bar{D}}{\bar{E}} = \frac{\frac{Q}{4\pi r^2}\bar{a}_r}{\frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r} = \varepsilon_0$$

$\therefore$ $$\boxed{\bar{D} = \varepsilon_0 \bar{E}} \quad \text{... For free space} \quad \text{... (1)}$$

Thus $\bar{D}$ and $\bar{E}$ are related through the permittivity. If the medium in which charge is located is other than free space having relative permittivity ε_r then,

$$\bar{D} = \varepsilon_0 \varepsilon_r \bar{E}$$

i.e. $$\bar{D} = \varepsilon \bar{E} \quad \text{... (2)}$$

The following are the **important observations** :

1. The $\bar{D}$ and $\bar{E}$, both act in the same direction.
2. The $\bar{D}$ and $\bar{E}$ are related through the permittivity of the medium in which the charge is located.
3. Though the relationship is derived considering a point charge, **the result is equally applicable for any general charge distribution.**
4. The electric field $\bar{E}$ due to any charge configuration is a function of the permittivity ε, while the electric flux density $\bar{D}$ is not.

The relationship is very advantageous while solving the problems on multiple dielectrics.

3.6 Electric Flux Density for Various Charge Distributions

Let us obtain the expression for $\overline{D}$ considering various types of charge distributions.

3.6.1 Line Charge

Consider a line charge having uniform charge density of ρ_L C/m. Then the total charge along the line is given by,

$$Q = \int_L \rho_l \, dl \qquad \text{... (1)}$$

But

$$\overline{D} = \frac{Q}{4\pi r^2}\overline{a}_r = \frac{\int_L \rho_L \, dl}{4\pi r^2}\overline{a}_r \qquad \text{... (2)}$$

If the line charge is infinite then $\overline{E}$ is derived as,

$$\overline{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\overline{a}_r$$

and

$$\overline{D} = \varepsilon_0 \overline{E}$$

$\therefore$

$$\boxed{\overline{D} = \frac{\rho_L}{2\pi r}\overline{a}_r} \qquad \text{... Infinite line charge.}$$

3.6.2 Surface Charge

Consider a sheet of charge having uniform charge density of ρ_S C/m^2. Then the total charge on the surface is given by,

$$Q = \int_S \rho_S \, dS \qquad \text{... (3)}$$

$\therefore$

$$\overline{D} = \frac{Q}{4\pi r^2}\overline{a}_r$$

$$= \frac{\int_S \rho_S \, dS}{4\pi r^2}\overline{a}_r \qquad \text{... (4)}$$

The integration is over the surface S and is a double integral.

If the sheet of charge is infinite then $\overline{E}$ is derived as,

$$\overline{E} = \frac{\rho_S}{2\varepsilon_0}\overline{a}_n$$

and

$$\overline{D} = \varepsilon_0 \overline{E}$$

$\therefore$

$$\boxed{\overline{D} = \frac{\rho_S}{2}\overline{a}_n} \qquad \text{... Infinite sheet of charge}$$

3.6.3 Volume Charge

Consider a charge enclosed by a volume, with a uniform charge density of ρ_v C/m^3. Then the total charge enclosed by the volume is given by,

$$\boxed{Q = \int_{vol} \rho_v \, dv} \quad \text{... (5)}$$

and
$$\bar{E} = \frac{\int_{vol} \rho_v \, dv}{4\pi\varepsilon_0 r^2} \bar{a}_r \quad \text{... (6)}$$

and
$$\bar{D} = \varepsilon_0 \bar{E}$$

$$\therefore \quad \boxed{\bar{D} = \frac{\int_{vol} \rho_v \, dv}{4\pi r^2} \bar{a}_r} \quad \text{... (7)}$$

Example 3.1 : *Find $\bar{D}$ in cartesian co-ordinate system at point P (6, 8, – 10) due to a) a point charge of 40 mC at the origin, b) a uniform line charge of $\rho_L = 40 \mu C/m$ on the z-axis and c) a uniform surface charge of density $\rho_S = 57.2 \mu C/m^2$ on the plane $x = 12$ m.*

Solution : a) A point charge of 40 mC at the origin.

P(6, 8, – 10) and O(0, 0, 0)

$$\therefore \quad \bar{r} = (6-0)\bar{a}_x + (8-0)\bar{a}_y + (-10-0)\bar{a}_z$$

$$= 6\,\bar{a}_x + 8\bar{a}_y - 10\bar{a}_z$$

$$\therefore \quad |\bar{r}| = \sqrt{(6)^2 + (8)^2 + (-10)^2} = \sqrt{200}$$

$$\therefore \quad \bar{a}_r = \frac{6\,\bar{a}_x + 8\bar{a}_y - 10\bar{a}_z}{\sqrt{200}}$$

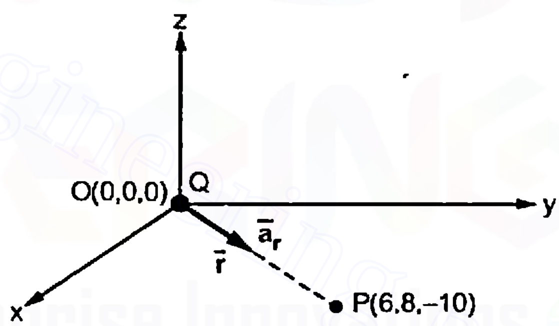


Fig. 3.6

$$\therefore \quad \bar{D} = \frac{Q}{4\pi r^2}\bar{a}_r = \frac{40\times10^{-3}}{4\pi\times(\sqrt{200})^2}\left\{\frac{6\bar{a}_x + 8\bar{a}_y - 10\bar{a}_z}{\sqrt{200}}\right\}$$

$$= 6.752\times10^{-6}\,\bar{a}_x + 9.003\times10^{-6}\bar{a}_y - 11.254\times10^{-6}\,\bar{a}_z \ \ C/m^2$$

b) ρ_L = 40 μC/m along z-axis

The charge is infinite hence,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

As the charge is along z-axis there can not be any component of $\bar{E}$ along z-direction

Consider a point on the line charge (0, 0, z) and P (6, 8, – 10). But while obtaining $\bar{r}$ do not consider z co-ordinate, as $\bar{E}$ and $\bar{D}$ have no $\bar{a}_z$ component.

$$\therefore \quad \bar{r} = (6-0)\,\bar{a}_x + (8-0)\,\bar{a}_y = 6\,\bar{a}_x + 8\,\bar{a}_y$$

$$\therefore \quad |\bar{r}| = \sqrt{(6)^2 + (8)^2} = 10$$

$$\therefore \quad \bar{a}_r = \frac{6\,\bar{a}_x + 8\bar{a}_y}{10}$$

$$\therefore \quad \bar{E} = \frac{\rho_L}{2\pi\varepsilon_0(10)}\left[\frac{6\,\bar{a}_x + 8\,\bar{a}_y}{10}\right]$$

$$\therefore \quad \bar{D} = \varepsilon_0\,\bar{E} = \frac{\rho_L}{2\pi\times 10}\left[\frac{6\bar{a}_x + 8\bar{a}_y}{10}\right]$$

$$= 3.819\times 10^{-7}\,\bar{a}_x + 5.092\times 10^{-7}\,\bar{a}_y \;\; C/m^2$$

c) $\rho_S = 57.2\,\mu\,C/m^2$ on the plane x = 12.

The sheet of charge is infinite over the plane x = 12 which is parallel to yz plane. The unit vector normal to this plane is $\bar{a}_n = \bar{a}_x$.

$$\therefore \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0}\bar{a}_n$$

The point P is on the back side of the plane hence $\bar{a}_n = -\bar{a}_x$, as shown in the Fig. 3.7.

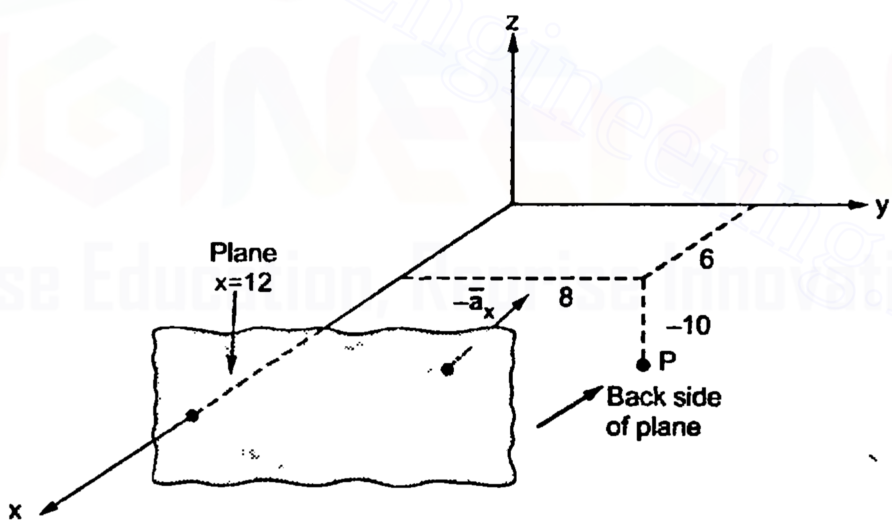


Fig. 3.7

$$\therefore \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0}(-\bar{a}_x)$$

But $\quad \bar{D} = \varepsilon_0\bar{E}$

$$\therefore \quad \bar{D} = \frac{\rho_S}{2}(-\bar{a}_x) = -28.6\times 10^{-6}\,\bar{a}_x \;\; C/m^2$$

Example 3.2 : *A point charge of 6 μC is located at the origin, a uniform line charge density of 180 nC/m lies along x-axis and uniform sheet of charge equal to 25 nC/m² lies in the z = 0 plane. Find i) $\bar{D}$ at A (0, 0, 4), ii) $\bar{D}$ at B (1, 2, 4) and iii) Total electric flux leaving the surface of the sphere of 4 m radius centered at the origin.*

Solution : i) Case 1 : Point charge Q = 6 μC at P (0, 0, 0).

While $\bar{D}$ to be obtained at A (0, 0, 4).

$$\therefore \quad \bar{r} = (4-0)\bar{a}_z = 4\bar{a}_z, \ |\bar{r}| = \sqrt{(4)^2} = 4, \ \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \bar{a}_z$$

$$\therefore \quad \bar{D}_1 = \frac{Q}{4\pi r^2}\bar{a}_r = \frac{6\times 10^{-6}}{4\pi\times(4)^2}\bar{a}_z = 2.984\times 10^{-8}\ \bar{a}_z \ \text{C/m}^2$$

Case 2 : Line Charge ρ_L = 180 nC/m along x-axis. So any point P on the charge is (x, 0, 0), while A (0, 0, 4). As charge is along x-axis, no component of $\bar{D}$ is along x-axis. So do not consider x co-ordinate while obtaining $\bar{r}$.

$$\therefore \quad \bar{r} = (4-0)\bar{a}_z = 4\bar{a}_z, \ |\bar{r}| = 4, \ \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \bar{a}_z$$

As charge is infinite,

$$\therefore \quad \bar{D}_2 = \frac{\rho_L}{2\pi r}\bar{a}_r = \frac{180\times 10^{-9}}{2\pi\times 4}\bar{a}_z = 7.161\times 10^{-9}\ \bar{a}_z \ \text{C/m}^2$$

Case 3 : Uniform sheet of charge lies in z = 0 plane. So the direction normal to it is z direction as plane is xy plane. Hence $\bar{a}_n = \bar{a}_z$ and ρ_S = 25 nC/m².

As sheet is infinite,

$$\bar{D}_3 = \frac{\rho_S}{2}\bar{a}_n = \frac{25\times 10^{-9}}{2}\bar{a}_z = 12.5\times 10^{-9}\bar{a}_z \ \text{C/m}^2$$

$$\therefore \quad \bar{D} = \bar{D}_1 + \bar{D}_2 + \bar{D}_3 = 49.501\times 10^{-9}\ \bar{a}_z \ \text{C/m}^2$$

ii) The point at which $\bar{D}$ is to be obtained is now B (1, 2, 4).

Case 1 : Point charge Q = 6 μC at P (0, 0, 0).

$$\therefore \quad \bar{r} = (1-0)\bar{a}_x + (2-0)\bar{a}_y + (4-0)\bar{a}_z = \bar{a}_x + 2\bar{a}_y + 4\bar{a}_z$$

$$\therefore \quad |\bar{r}| = \sqrt{(1)^2 + (2)^2 + (4)^2} = \sqrt{21}$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{a}_x + 2\bar{a}_y + 4\bar{a}_z}{\sqrt{21}}$$

$$\therefore \quad \bar{D}_1 = \frac{Q}{4\pi r^2}\bar{a}_r = \frac{6\times10^{-6}}{4\pi\times(\sqrt{21})^2}\left[\frac{\bar{a}_x+2\bar{a}_y+4\bar{a}_z}{\sqrt{21}}\right]$$

$$= 4.961\times10^{-9}\,\bar{a}_x + 9.923\times10^{-9}\,\bar{a}_y + 1.9845\times10^{-8}\,\bar{a}_z \ \ C/m^2$$

Case 2 : Line charge : The point on the charge is (x, 0, 0).

As charge is along x-axis, do not consider x co-ordinate.

$$\therefore \quad \bar{r} = (2-0)\bar{a}_y + (4-0)\,\bar{a}_z = 2\bar{a}_y + 4\bar{a}_z \qquad \text{... as B (1, 2, 4)}$$

$$\therefore \quad |\bar{r}| = \sqrt{(2)^2+(4)^2} = \sqrt{20}$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{2\bar{a}_y+4\bar{a}_z}{\sqrt{20}}$$

$$\therefore \quad \bar{D}_2 = \frac{\rho_L}{2\pi r}\bar{a}_r$$

$$= \frac{180\times10^{-9}}{2\pi\times\sqrt{20}}\left[\frac{2\bar{a}_y+4\bar{a}_z}{\sqrt{20}}\right]$$

$$= 2.8647\times10^{-9}\,\bar{a}_y + 5.7295\times10^{-9}\,\bar{a}_z \ C/m^2$$

Case 3 : Infinite sheet of charge in z = 0 plane.

The point B (1, 2, 4) is above z = 0 plane hence $\bar{a}_n = \bar{a}_z$ and $\bar{D}_3$ remains same as before.

$$\bar{D}_3 = \frac{\rho_S}{2}\bar{a}_n = \frac{25\times10^{-9}}{2}\bar{a}_z = 12.5\times10^{-9}\,\bar{a}_z \ \ C/m^2$$

$$\therefore \quad \bar{D} = \bar{D}_1 + \bar{D}_2 + \bar{D}_3$$

$$= 4.961\times10^{-9}\,\bar{a}_x + 1.2786\times10^{-8}\,\bar{a}_y + 3.807\times10^{-8}\,\bar{a}_z \ \ C/m^2$$

iii) Let us find the total charge enclosed by a sphere of radius 4 m.

Charge 1 : $Q_1 = 6\ \mu C$ at the origin.

Charge 2 : The charge on that part of the line which is enclosed by the sphere. The line charge intersects sphere at x = ± 4. Hence charge on the length of 8 m is enclosed by the sphere. This is shown in the Fig. 3.8.

$$\therefore \quad Q_2 = \rho_L \times \text{length enclosed} = 180\times10^{-9}\times8 = 1.44\,\mu C$$

Charge 3 : The intersection of z = 0 plane with a sphere is a circle with radius 4 m, in xy plane.

The surface area of this circle is πr^2.

$$\therefore \quad S = \pi\times(4)^2 = 50.2654\ \ m^2$$

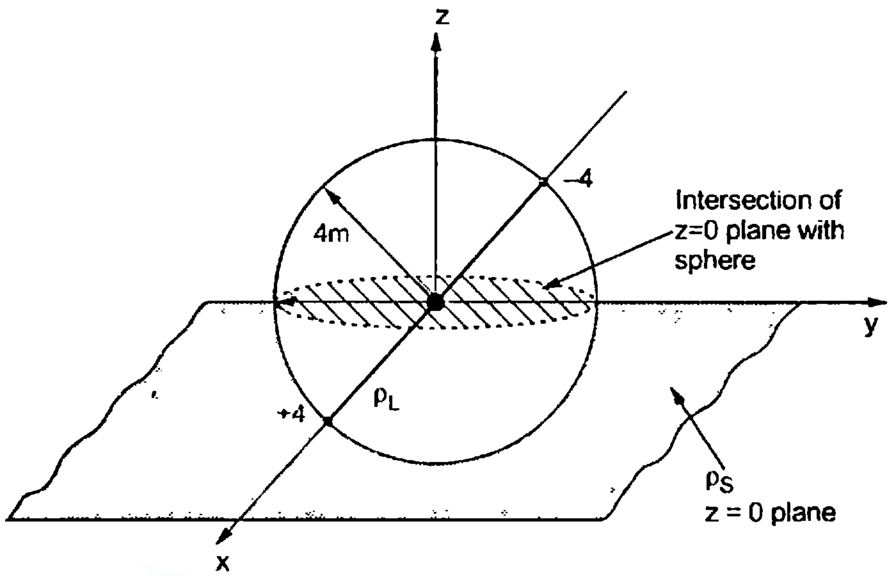


Fig. 3.8

Hence the total charge enclosed is,

$$\therefore \quad Q_3 = \rho_S \times S = 25 \times 10^{-9} \times 50.2654 = 1.2566\ \mu C$$

Hence the total charge enclosed by the sphere is,

$$Q_{total} = Q_1 + Q_2 + Q_3 = 8.6966\ \mu C$$

But $\psi = Q_{total}$ = Total electric flux leaving the surface of sphere

$= \mathbf{8.6966\ \mu C}$

3.7 Gauss's Law

It is seen that the charge Q emanates the flux ψ which is equal to the charge Q. This is provied by Faraday's experiment. Consider a sphere of radius r and a point charge + Q located at its centre. Then the total flux radiated outwards and passing through the total surface area of the sphere is same as the charge + Q, which is enclosed by the sphere.

Now replace the point charge by a line charge, such that the portion of the line charge enclosed by the sphere consists of same charge + Q as before. In this case too, the total flux radiating outwards remains same as Q which is the charge on the line enclosed by the sphere.

Similarly if the point charge + Q or a part of line charge carrying + Q are moved inside the sphere anywhere, still the total flux radiating outwards from the surface of the sphere remains same as Q.

Now instead of a sphere, any irregular closed surface is considered with total charge enclosed as + Q in any form i.e. either point, line or surface then the total flux crossing the surface of that irregular object remains same as Q, which is charge enclosed by that object.

These observations from Faraday's experiment lead to a law called Gauss's law. From the above discussion it is clear that irrespective of the shape of the closed surface and irrespective of the type of charge distribution, the total flux passing through the closed surface is the total charge enclosed by that surface.

Statement of Gauss's law :

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

3.7.1 Mathematical Representation of Gauss's Law

Consider any object of irregular shape as shown in the Fig. 3.9.

The total charge enclosed by the irregular closed surface is Q coulombs. It may be in any form of distribution. Hence the total flux that has to pass through the closed surface is Q. Consider a small differential surface dS at point P. As the surface is irregular, the direction of $\bar{D}$ as well as its magnitude is going to change from point to point on the surface. The surface dS under consideration can be represented in the vector form in terms of its area and direction normal to the surface at the point.

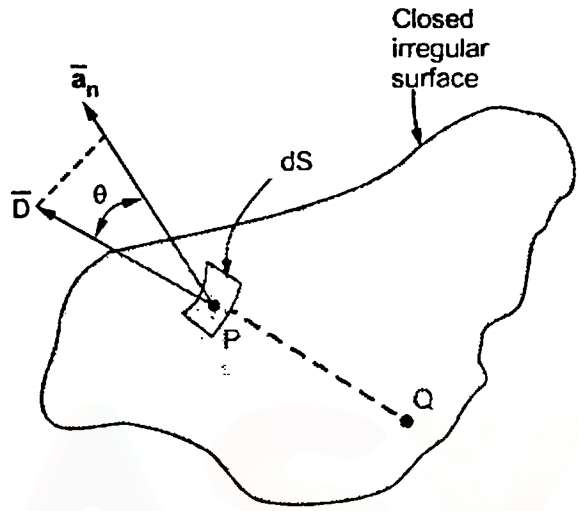


Fig. 3.9 Flux through irregular closed surface

$\therefore \quad d\bar{S} = dS\,\bar{a}_n$

where $\quad \bar{a}_n$ = Normal to the surface dS at point P

Key Point: *Note that the normal to the surface is in two directions but only directed outwards is considered as required. The normal going into the closed surface at point P is not required.*

The flux density at point P is $\bar{D}$ and its direction is such that it makes an angle θ with the normal direction at point P.

The flux dψ passing through the surface dS is the product of the component of $\bar{D}$ in the direction normal to the dS and dψ.

Mathematically this can be represented as,

$$d\psi = D_n\, dS \quad \text{... (1)}$$

where $\quad D_n$ = Component of $\bar{D}$ in the direction of normal to the surface dS

From Fig. 3.9 we can write,

$$D_n = |\bar{D}| \cos\theta \quad \text{... (2)}$$

$$\therefore \quad d\psi = |\bar{D}| \cos\theta\, dS \quad \text{... (3)}$$

From the definition of the dot product,

$$\bar{A} \bullet \bar{B} = |A|\,|B| \cos\theta_{AB}$$

We can write,

$$\left|\bar{D}\right| dS \cos 0 = \bar{D} \cdot d\bar{S} \qquad \dots (4)$$

$$\therefore \qquad d\psi = \bar{D} \cdot d\bar{S} \qquad \dots (5)$$

This is the flux passing through incremental surface area dS. Hence the total flux passing through the entire closed surface is to be obtained by finding the surface integration of the equation (5).

$$\therefore \qquad \boxed{\psi = \int d\psi = \oint_S \bar{D} \cdot d\bar{S}} \qquad \dots (6)$$

As seen earlier, $\oint$ sign indicates the integration over the closed surface and called closed surface integral. Though the integration sign is single, over the surface S it becomes double integration. Hence S is generally used along with the sign of closed surface integral.

Such a closed surface over which the integration in the equation (6) is carried out is called **Gaussian Surface.**

Now irrespective of the shape of the surface and the charge distribution, total flux passing through the surface is the total charge enclosed by the surface.

$$\therefore \qquad \boxed{\psi = \oint_S \bar{D} \cdot d\bar{S} = Q = \text{Charge enclosed}} \qquad \dots (7)$$

This is the mathematical representation of Gauss's law.

The charge enclosed may take any of the following forms :

1. If there are number of point charges $Q_1, Q_2, \dots, Q_n$ enclosed by the surface then

$$Q = Q_1 + Q_2 + \dots + Q_n = \Sigma Q_n$$

$$\therefore \qquad \boxed{\psi = Q = \Sigma Q_n} \qquad \dots (8)$$

2. If there is a line charge with line charge density ρ_l then,

$$\boxed{\psi = Q = \int_L \rho_L \, dl} \qquad \dots (9)$$

3. If there is a surface charge with surface charge density ρ_S then,

$$\boxed{\psi = Q = \int_S \rho_S \, dS} \qquad \dots (10)$$

4. If there is a volume charge with volume charge density ρ_v then,

$$\boxed{\psi = Q = \int_v \rho_v \, dv} \qquad \dots (11)$$

The common form used to represent Gauss's law mathematically is,

$$\boxed{\psi = Q = \oint_S \bar{D} \cdot d\bar{S} = \int_v \rho_v \, dv} \quad \text{... (12)}$$

If there are more than one charge distribution in Gaussian surface, the net charge is algebraic addition of all the individual charges.

If there is a closed surface such that there are no charges enclosed but there are charges around the surface then **net flux over the surface is zero.** This is because the flux from the charges outside, passes through the surface such that the flux entering is equal to flux leaving the surface.

3.7.2 Special Gaussian Surfaces

The surface over which is the Gauss's law is applied is called **Gaussian surface.** Obviously such a surface is a closed surface and it has to satisfy following conditions :

1. The surface may be irregular but should be sufficiently large so as to enclose the entire charge.
2. The surface must be closed.
3. At each point of the surface $\bar{D}$ is either normal or tangential to the surface.
4. The electric flux density D is constant over the surface at which $\bar{D}$ is normal.

3.8 Applications of Gauss's Law

The Gauss's law is infact the alternative statement of Coulomb's law. The Gauss's law can be used to find $\bar{E}$ or $\bar{D}$ for symmetrical charge distributions, such as point charge, an infinite line charge, an infinite sheet of charge and a spherical distribution of charge. The Gauss's law is also used to find the charge enclosed or the flux passing through the closed surface. Note that whether the charge distribution is symmetrical or not, Gauss's law holds for any closed surface but can be easily applied to the symmetrical distributions. But the Gauss's law cannot be used to find $\bar{E}$ or $\bar{D}$ if the charge distribution is not symmetric.

While selecting the closed Gaussian surface to apply the Gauss's law, following conditions must be satisfied,

1. $\bar{D}$ is every where either normal or tangential to the closed surface i.e. $\theta = \frac{\pi}{2}$ or π. So that $\bar{D} \cdot d\bar{S}$ becomes DdS or zero respectively.

2. $\bar{D}$ is constant over the portion of the closed surface for which $\bar{D} \cdot d\bar{S}$ is not zero.

Let us apply these ideas to the various charge distributions.

3.8.1 Point Charge

Let a point charge Q is located at the origin.

To determine $\bar{D}$ and to apply Gauss's law, consider a spherical surface around Q, with centre as origin. This spherical surface is Gaussian surface and it satisfies required condition. The $\bar{D}$ is always directed radially outwards along $\bar{a}_r$ which is normal to the spherical surface at any point P on the surface. This is shown in the Fig. 3.10.

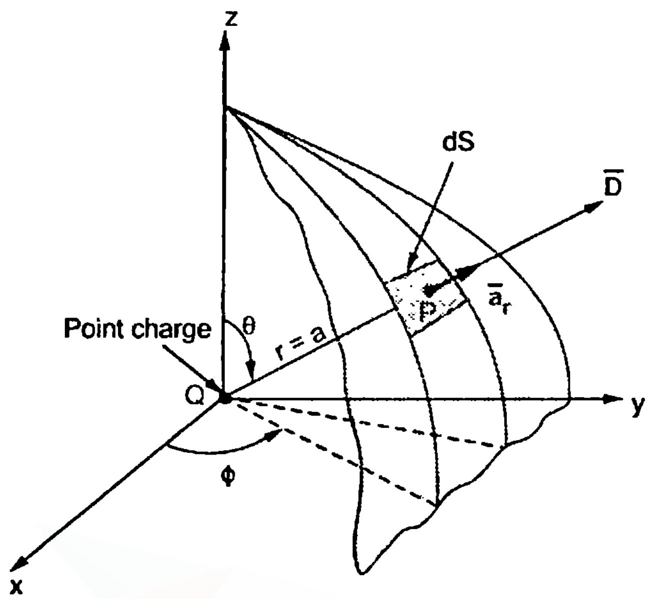


Fig. 3.10 Proof of Gauss's law

Consider a differential surface area dS as shown. The direction normal to the surface dS is $\bar{a}_r$, considering **spherical co-ordinate system**. The radius of the sphere is r = a.

The direction of $\bar{D}$ is along $\bar{a}_r$ which is normal to dS at any point P.

In spherical co-ordinate system, the dS normal to radial direction $\bar{a}_r$ is,

$$dS = r^2 \sin\theta\, d\theta\, d\phi = a^2 \sin\theta\, d\theta\, d\phi \quad (\text{as } r = a)$$

$$\therefore \quad d\bar{S} = dS\, \bar{a}_n = a^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r \qquad \text{... (1)}$$

Now $\bar{D}$ due to the point charge is given by,

$$\bar{D} = \frac{Q}{4\pi r^2}\bar{a}_r = \frac{Q}{4\pi a^2}\bar{a}_r \quad (\text{as } r = a) \qquad \text{... (2)}$$

$$\therefore \quad \bar{D} \bullet d\bar{S} = |\bar{D}||d\bar{S}| \cos\theta'$$

Note that θ' is the angle between $\bar{D}$ and $d\bar{S}$.

where $$|\bar{D}| = \frac{Q}{4\pi a^2}, \quad |d\bar{S}| = a^2 \sin\theta\, d\theta\, d\phi, \quad \theta' = 0°$$

The normal to $d\bar{S}$ is $\bar{a}_r$ while $\bar{D}$ also acts along $\bar{a}_r$ hence angle between $d\bar{S}$ and $\bar{D}$ i.e. $\theta' = 0°$.

$$\therefore \quad \bar{D} \bullet d\bar{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta\, d\theta\, d\phi \cos 0°$$

$$= \frac{Q}{4\pi} \sin\theta\, d\theta d\phi \qquad \text{... (3)}$$

Alternatively to avoid the confusion between the symbol θ we can write,

$$\therefore \quad \bar{D} \cdot d\bar{S} = \frac{Q}{4\pi a^2} \bar{a}_r \cdot a^2 \sin\theta \, d\theta \, d\phi \bar{a}_r$$

$$= \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi [\bar{a}_r \cdot \bar{a}_r]$$

But $\bar{a}_r \cdot \bar{a}_r = 1$

$$\therefore \quad \bar{D} \cdot d\bar{S} = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\therefore \quad \psi = \oint_S \bar{D} \cdot d\bar{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{Q}{4\pi} [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} = \frac{Q}{4\pi} [-(-1)-(-1)] [2\pi]$$

$$\therefore \quad \psi = Q \qquad \text{... Gauss's law is proved}$$

This proves the Gauss's law that Q coulombs of flux crosses the surface if Q coulombs of charge is enclosed by that surface.

> **Key Point:** *As $\bar{D}$ is obtained from the result of $\bar{E}$ which is obtained from Coulomb's law, it can be said that the above discussion is the proof of Gauss's law from the Coulomb's law.*

3.8.1.1 Use of Gauss's Law to Obtain $\bar{D}$ and $\bar{E}$

Alternatively Gauss's law can be used to obtain $\bar{D}$ and $\bar{E}$. Let us see how ?

From Gauss's law,

$$Q = \oint_S \bar{D} \cdot d\bar{S}$$

The steps to obtain $\bar{D}$ and $\bar{E}$ are,

1. Identify $|\bar{D}|$ and its direction.
2. Identify $|d\bar{S}|$ and direction normal to dS.
3. Take dot product, $\bar{D} \cdot d\bar{S}$.
4. Choose the Gaussian surface.
5. Integrate over the surface chosen as Gaussian surface, keeping $|\bar{D}|$ unknown as it is.
6. Find charge Q enclosed by Gaussian surface.
7. Equate the charge Q, to the integration obtained with $|\bar{D}|$ as unknown.
8. Determine $|\bar{D}|$ and express $\bar{D}$ with its direction. Then $\bar{E} = \bar{D}/\varepsilon_0$.

For a sphere of radius r, the flux density $\bar{D}$ is in radial direction $\bar{a}_r$ always. Let $|\bar{D}| = D_r$.

$$\therefore \quad \bar{D} = D_r \bar{a}_r$$

Let the Gaussian surface is a sphere of radius r enclosing charge Q.

While for the Gaussian surface i.e. sphere of radius r, dS normal to $\bar{a}_r$ is,

$$d\bar{S} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$\therefore \quad \bar{D} \cdot d\bar{S} = D_r \, r^2 \sin\theta \, d\theta \, d\phi \qquad \ldots (\bar{a}_r \cdot \bar{a}_r = 1)$$

Now integrate over the surface of sphere of constant radius 'r'.

$$\therefore \quad \oint_S \bar{D} \cdot d\bar{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, r^2 [-\cos\theta]_0^{\pi} \, [\phi]_0^{2\pi} = 4\pi r^2 D_r$$

But $\oint_S \bar{D} \cdot d\bar{S} = Q$

$$\therefore \quad Q = 4\pi r^2 D_r$$

$$\therefore \quad D_r = \frac{Q}{4\pi r^2} \quad \text{and hence}$$

$$\boxed{\bar{D} = D_r \bar{a}_r = \frac{Q}{4\pi r^2} \bar{a}_r}$$

and

$$\boxed{\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 r^2} \bar{a}_r}$$

The expressions are same as those obtained by Coulomb's law, earlier in the Chapter-2. This is the use of Gauss's law to obtain $\bar{D}$ and $\bar{E}$ for a given charge distribution.

> **Note** : Symmetry helps us to apply Gauss's law for the given situation. To understand symmetry, obtain the information,
>
> 1. With which co-ordinates does the $\bar{D}$ vary ?
> 2. Which components of $\bar{D}$ are present ?

This results into simpler integration to be solved to obtain the required result.

3.8.2 Infinite Line Charge

Consider an infinite line charge of density ρ_l C/m lying along z-axis from $-\infty$ to $+\infty$. This is shown in the Fig. 3.11.

Consider the Gaussian surface as the right circular cylinder with z-axis as its axis and radius r as shown in the Fig. 3.11. The length of the cylinder is L.

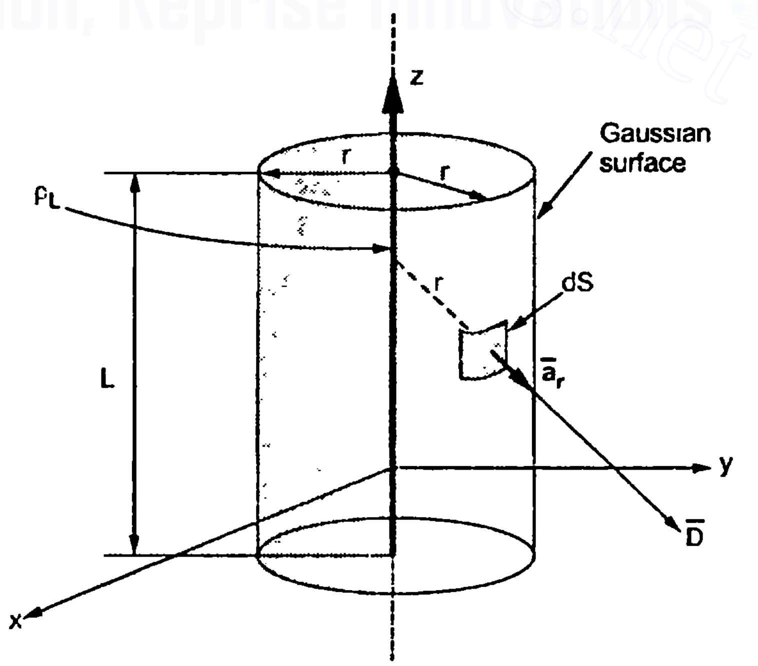


Fig. 3.11 Infinite line charge

The flux density at any point on the suface is directed radially outwards i.e in the $\bar{a}_r$ direction according to **cylindrical co-ordinate system.**

Consider differential surface area dS as shown which is at a radial distance r from the line charge. The direction normal to dS is $\bar{a}_r$.

As the line charge is along z-axis, there can not be any component of $\bar{D}$ in z direction. So $\bar{D}$ has only radial component.

Now $$Q = \oint_S \bar{D} \cdot d\bar{S}$$

The integration is to be evaluated for side surface, top surface and bottom surface.

$$\therefore \quad Q = \oint_{side} \bar{D} \cdot d\bar{S} + \oint_{top} \bar{D} \cdot d\bar{S} + \oint_{bottom} \bar{D} \cdot d\bar{S}$$

Now $\bar{D} = D_r \bar{a}_r$ as has only radial component

and $d\bar{S} = r\, d\phi dz\ \bar{a}_r$ normal to $\bar{a}_r$ direction.

$$\therefore \quad \bar{D} \cdot d\bar{S} = D_r\, r\, d\phi\, dz (\bar{a}_r \cdot \bar{a}_r) = D_r\, r\, d\phi\, dz \quad \text{... as } \bar{a}_r \cdot \bar{a}_r = 1$$

Now D_r is constant over the side surface.

As $\bar{D}$ has only radial component and no component along $\bar{a}_z$ and $-\bar{a}_z$ hence integrations over top and bottom surfaces is zero.

$$\therefore \quad \oint_{top} \bar{D} \cdot d\bar{S} = \oint_{bottom} \bar{D} \cdot d\bar{S} = 0$$

$$\therefore \quad Q = \oint_{side} \bar{D} \cdot d\bar{S} = \oint_{side} D_r\, r\, d\phi\, dz$$

$$= \int_{z=0}^{L} \int_{\phi=0}^{2\pi} D_r r\, d\phi\, dz = r D_r [z]_0^L [\phi]_0^{2\pi}$$

$$\therefore \quad Q = 2\pi r D_r L \quad \text{... (4)}$$

$$\therefore \quad D_r = \frac{Q}{2\pi r L}$$

$$\therefore \quad \bar{D} = D_r \bar{a}_r = \frac{Q}{2\pi r L} \bar{a}_r$$

But $$\frac{Q}{L} = \rho_L \ \text{C/m}$$

$$\therefore \quad \boxed{\bar{D} = \frac{\rho_L}{2\pi r} \bar{a}_r \ \text{C/m}^2} \quad \text{... Due to infinite line charge.}$$

and $$\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_L}{2\pi \varepsilon_0 r} \bar{a}_r \ \text{V/m}$$

The results are same as obtained from the Coulomb's law.

3.8.3 Coaxial Cable

Consider the two coaxial cylindrical conductors forming a coaxial cable. The radius of the inner conductor is 'a' while the radius of the outer conductor is 'b'. The coaxial cable is shown in the Fig. 3.12. The length of the cable is L.

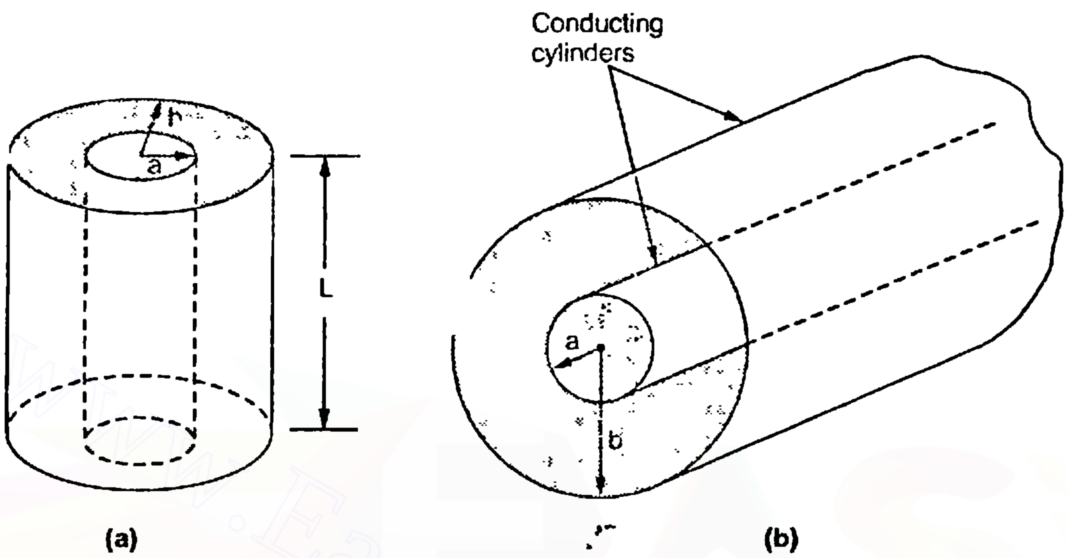


Fig. 3.12 Coaxial cable

The charge distribution on the outer surface of the inner conductor is having density $\rho_S \, C/m^2$. The total outer surface area of the inner conductor is $2\pi a L$.

Hence ρ_S can be expressed interms of ρ_l.

$$\therefore \qquad \rho_l = \frac{\rho_S \times \text{Surface area}}{\text{Total length}} = \frac{\rho_S \times 2\pi a L}{L}$$

$$\therefore \qquad \rho_L = 2\pi a \rho_S \ C/m$$

Thus the line charge density of inner conductor is ρ_L C/m.

Consider the right circular cylinder of length L as the Gaussian surface. Due to the symmetry, $\overline{D}$ has only radial component. From the discussion of line charge we can write,

$$Q = D_r \, 2\pi r L \qquad \dots (5)$$

where $a < r < b$ [Refer equation (4) in section 3.8.2]

The total charge on the inner conductor is to be obtained by evaluating the surface integral of the surface charge distribution.

$$\therefore \qquad Q = \oint_S \rho_S d\overline{S} \qquad \dots (6)$$

Now $dS = r \, d\phi \, dz$ but $r = a$

$$\therefore \qquad dS = a \, d\phi \, dz \qquad \dots (7)$$

$$\therefore \qquad Q = \oint_S \rho_S \, a \, d\phi \, dz = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_S \, a \, d\phi \, dz$$

$$= \rho_S \, a [z]_0^L \, [\phi]_0^{2\pi} = 2\pi a \, L\rho_S \qquad \dots (8)$$

Equating (5) and (8),

$$D_r \, 2\pi r L = 2\pi a L\rho_S$$

$$\therefore \qquad D_r = \frac{a\rho_S}{r} \qquad \dots (9)$$

This acts along radial direction i.e. $\bar{a}_r$.

$$\therefore \qquad \bar{D} = \frac{a\rho_S}{r} \bar{a}_r \qquad \dots (10)$$

But $\qquad \rho_S = \frac{\rho_L}{2\pi a}$

$$\therefore \qquad \bar{D} = \frac{a\rho_L}{2\pi a r} \bar{a}_r = \frac{\rho_L}{2\pi r} \bar{a}_r \ C/m^2 \qquad \dots (11)$$

$$\therefore \qquad \bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r} \bar{a}_r \quad (a < r < b) \quad V/m \qquad \dots (12)$$

This is same as obtained for infinite line charge. Every flux line starting from the positive charge on the inner cylinder must terminate on the negative charge on the inner surface of the outer cylinder. Hence the total charge on the inner surface of the outer cylinder is,

$$Q_{\text{outer cylinder}} = -2\pi \, a \, L\rho_{S(\text{inner})} \qquad \dots (13)$$

But $\qquad Q_{\text{outer cylinder}} = 2\pi b \, L\rho_{S(\text{outer})} \qquad \dots (14)$

$$\therefore \qquad 2\pi \, b \, L\rho_{S(\text{outer})} = -2\pi \, a \, L\rho_{S(\text{inner})}$$

$$\therefore \qquad \rho_{S(\text{outer})} = -\frac{a}{b} \rho_{S(\text{inner})} \qquad \dots (15)$$

If the Gaussian surface is considered such that r > b, then the total charge enclosed will be zero as equal and opposite charges on the cylinder will cancel each other.

Similarly inside the inner cylinder, r < a also the total charge enclosed will be zero.

3.8.4 Infinite Sheet of Charge

Consider the infinite sheet of charge of uniform charge density ρ_S C/m^2, lying in the z = 0 plane i.e. xy plane as shown in the Fig. 3.13.

Consider a rectangular box as a Gaussian surface which is cut by the sheet of charge to give dS = dx dy.

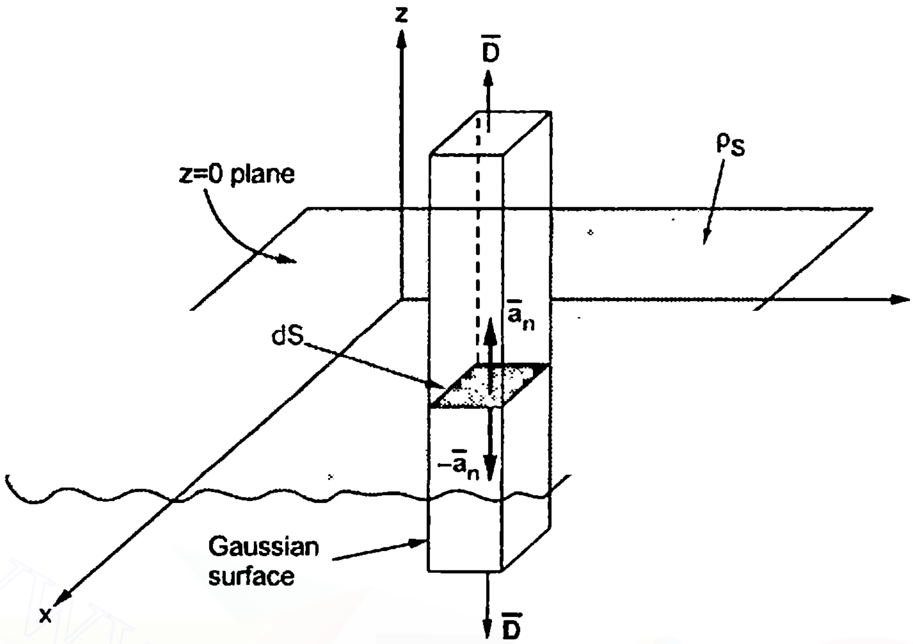


Fig. 3.13 Infinite sheet of charge

$\bar{D}$ acts normal to the plane i.e. $\bar{a}_n = \bar{a}_z$ and $-\bar{a}_n = -\bar{a}_z$ direction.

Hence $\bar{D} = 0$ in x and y directions.

Hence the charge enclosed can be written as,

$$Q = \oint_S \bar{D} \cdot d\bar{S} = \oint_{sides} \bar{D} \cdot d\bar{S} + \oint_{top} \bar{D} \cdot d\bar{S} + \oint_{bottom} \bar{D} \cdot d\bar{S}$$

But $\oint_{sides} \bar{D} \cdot d\bar{S} = 0$ as $\bar{D}$ has no component in x and y directions

Now $\bar{D} = D_z \bar{a}_z$ for top surface

and $d\bar{S} = dx\, dy\, \bar{a}_z$

$\therefore$ $\bar{D} \cdot d\bar{S} = D_z\, dx\, dy \left(\bar{a}_z \cdot \bar{a}_z\right) = D_z\, dx\, dy$

and $\bar{D} = D_z \left(-\bar{a}_z\right)$ for bottom surface.

and $d\bar{S} = dx\, dy \left(-\bar{a}_z\right)$

$\therefore$ $\bar{D} \cdot d\bar{S} = D_z\, dx\, dy \left(\bar{a}_z \cdot \bar{a}_z\right) = D_z\, dx\, dy$

$\therefore$ $$Q = \oint_{top} D_z\, dx\, dy + \oint_{bottom} D_z\, dx\, dy$$

Now $\oint_{top} dx\, dy = \oint_{bottom} dx\, dy = A$ = Area of surface

$\therefore$ $Q = 2 D_z A$

But $Q = \rho_S \times A$ as ρ_S = Surface charge density

$\therefore \quad \rho_S = 2D_z$

$\therefore \quad D_z = \frac{\rho_S}{2}$

$$\therefore \quad \bar{D} = D_z \, \bar{a}_z = \frac{\rho_S}{2} \bar{a}_z \; C/m^2 \qquad ... (16)$$

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_S}{2\varepsilon_0} \bar{a}_z \; V/m \qquad ... (17)$$

The results are same as obtained by the Coulomb's law for the infinite sheet of charge.

3.8.5 Spherical Shell of Charge

Consider an imaginary spherical shell of radius 'a'.

The charge is uniformly distributed over its surface with a density ρ_S C/m^2. Let us find $\bar{E}$ at a point P located at a distance r from the centre such that r > a and $r \le a$, using Gauss's law.

The shell is shown in the Fig. 3.14.

Case 1 : Point P outside the shell (r > a)

Consider a point P at a distance r from the origin such that r > a. The Gaussian surface passing through point P is a concentric sphere of radius r. Due to spherical Gaussian surface, the flux lines are directed radially outwards and are normal to the surface. Hence electric flux density $\bar{D}$ is also directed radially outwards at point P and has component only in $\bar{a}_r$ direction. Consider a differential surface area at P normal to $\bar{a}_r$ direction hence $dS = r^2 \sin\theta \, d\theta \, d\phi$ in spherical system.

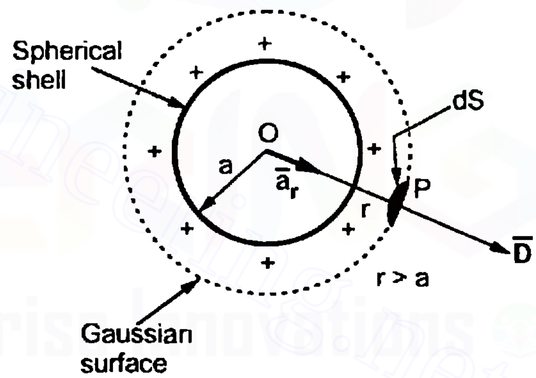


Fig. 3.14 Spherical shell of charge

$$\therefore \quad d\psi = \bar{D} \cdot d\bar{S} = [D_r \bar{a}_r] \cdot [r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r]$$

$$= D_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$$\therefore \quad \psi = \oint_S D_r \, r^2 \sin\theta \, d\theta \, d\phi = D_r \, r^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi$$

$$\therefore \quad \psi = D_r \, r^2 [-\cos\theta]_0^{\pi} \, [\phi]_0^{2\pi} = 4\pi r^2 D_r \qquad ... (18)$$

But $\psi = Q$ ... Gauss's law

$\therefore \quad Q = 4\pi r^2 D_r$

$$\therefore \qquad D_r = \frac{Q}{4\pi r^2}$$

$$\therefore \qquad \bar{D} = D_r\,\bar{a}_r = \frac{Q}{4\pi r^2}\bar{a}_r \ \ C/m^2 \qquad \text{... (19)}$$

$$\text{And} \qquad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r \ \ V/m \qquad \text{... (20)}$$

Thus for r > a, the field $\bar{E}$ is inversely proportional to the square of the distance from the origin.

If the surface charge density is ρ_S C/m^2 then

$$Q = \rho_S \times \text{Surface area of shell}$$

$$\therefore \qquad Q = \rho_S \times 4\pi a^2$$

$$\therefore \qquad \bar{E} = \frac{\rho_S\, 4\pi a^2}{4\pi\varepsilon_0 r^2}\bar{a}_r = \frac{\rho_S\, a^2}{\varepsilon_0 r^2}\bar{a}_r \ \ V/m \qquad \text{... (21)}$$

$$\text{and} \qquad \boxed{\bar{D} = \varepsilon_0\,\bar{E} = \frac{\rho_S a^2}{r^2}\,\bar{a}_r \ \ C/m^2} \qquad \text{... (22)}$$

Case 2 : Point P is on the shell (r = a)

On the shell, $r = a$

The Gaussian surface is same as the shell itself and $\bar{E}$ can be obtained using r = a in the equation (20).

$$\therefore \qquad \bar{E} = \frac{Q}{4\pi\varepsilon_0 a^2}\bar{a}_r \ \ V/m \qquad \text{... (23)}$$

Case 3 : Point P inside the shell (r < a)

The Gaussian surface, passing through the point P is again a spherical surface with radius r < a.

But it can be seen that the entire charge is on the surface and no charge is enclosed by the spherical shell. And when the Gaussian surface is such that no charge is enclosed, irrespective of any charges present outside, the total charge enclosed is zero.

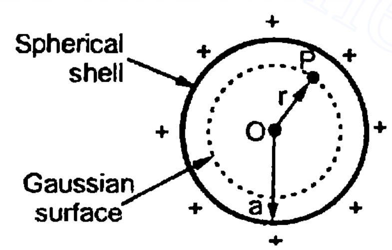


Fig. 3.15

$$\therefore \qquad \psi = Q = \oint_S \bar{D}\cdot d\bar{S} = 0 \qquad \text{... As per Gauss's law}$$

$$\text{Now} \qquad \oint_S d\bar{S} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta\, d\theta\, d\phi = 4\pi r^2$$

Thus $\oint_S d\bar{S} \neq 0$

Hence to satisfy that total charge enclosed is zero, inside the spherical shell.

$$\boxed{\bar{D} = 0 \quad \text{and} \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = 0} \qquad \ldots (24)$$

Thus electric flux density and electric field at any point **inside a spherical shell is zero.**

3.8.5.1 Variation of $\bar{E}$ against r

The variation of $\bar{E}$ against the radial distance r measured from the origin is shown in the Fig. 3.16.

For r < a,	$\bar{E} = 0$
For r = a,	$\bar{E} = \frac{Q}{4\pi\varepsilon_0 a^2}\,\bar{a}_r$
For r > a,	$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\bar{a}_r$

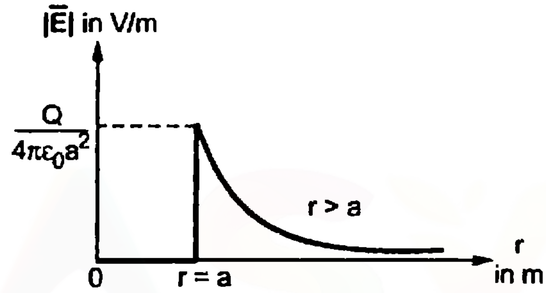


Fig. 3.16 Variation of $|\bar{E}|$ against r

After r = a, the $\bar{E}$ is inversely proportional to the square of the radial distance of a point from the origin. **The variation of $|\bar{D}|$ against r is also similar.** For the medium other than the free space, ε_0 must be replaced by $\varepsilon = \varepsilon_0 \varepsilon_r$.

3.8.6 Uniformly Charged Sphere

Consider a sphere of radius 'a' with a uniform charge density of ρ_v C/m^3. Let us find $\bar{E}$ at a point P located at a radial distance r from centre of the sphere such that r ≤ a and r > a, using Gauss's law.

The sphere is shown in the Fig. 3.17.

Case 1 : The point P is outside the sphere (r > a).

The Gaussian surface passing through point P is a spherical surface of radius r.

The flux lines and $\bar{D}$ are directed radially outwards along $\bar{a}_r$ direction.

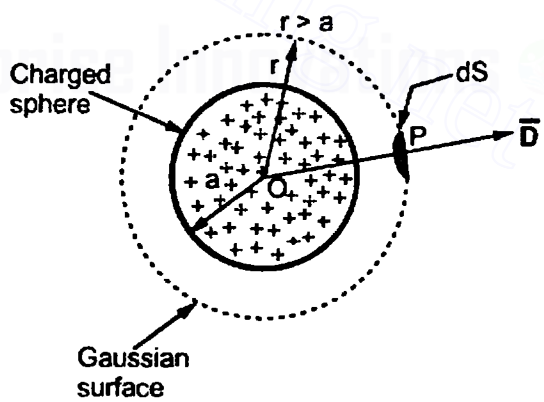


Fig. 3.17 Uniformly charged sphere

The differential area dS is considered at point P which is normal to $\bar{a}_r$ direction.

$$\therefore \quad dS = r^2 \sin\theta \, d\theta \, d\phi$$

$$\therefore \quad d\psi = \bar{D} \cdot d\bar{S} = D_r \bar{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$= D_r \, r^2 \sin\theta \, d\theta \, d\phi \qquad \ldots (\bar{a}_r \cdot \bar{a}_r = 1)$$

$$\therefore \quad \psi = Q = \oint_S \bar{D} \cdot d\bar{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= D_r\, r^2\, [-\cos\theta]_0^{\pi}\, [\phi]_0^{2\pi} = D_r\, r^2\, 4\pi$$

$$\therefore \quad D_r = \frac{Q}{4\pi r^2}$$

$$\therefore \quad \bar{D} = \frac{Q}{4\pi r^2}\, \bar{a}_r \quad C/m^2 \qquad \text{... (25)}$$

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 r^2}\, \bar{a}_r \quad V/m \qquad \text{... (26)}$$

The total charge enclosed can be obtained as,

$$Q = \int_v \rho_v\, dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} \rho_v\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho_v \left[\frac{r^3}{3}\right]_0^a [-\cos\theta]_0^{\pi}\, [\phi]_0^{2\pi}$$

$$= \frac{4}{3}\pi a^3 \rho_v \; C \qquad \text{... (27)}$$

$$\therefore \quad \bar{E} = \frac{\frac{4}{3}\pi a^3 \rho_v}{4\pi\varepsilon_0\, r^2}\, \bar{a}_r = \frac{a^3\, \rho_v}{3\,\varepsilon_0 r^2}\, \bar{a}_r \qquad \text{... (28)}$$

While $$\boxed{\bar{D} = \frac{a^3\, \rho_v}{3 r^2}\, \bar{a}_r} \qquad \text{... (29)}$$

These are the expressions for $\bar{D}$ and $\bar{E}$ outside the uniformly charged sphere.

Case 2 : The point P on the sphere (r = a).

The Gaussian surface is same as the surface of the charged sphere. Hence results can be obtained directly substituting r = a in the equation (28) and (26).

$$\therefore \quad \bar{E} = \frac{a^3 \rho_v}{3\,\varepsilon_0 a^2}\, \bar{a}_r = \frac{\rho_v\, a}{3\,\varepsilon_0}\, \bar{a}_r \qquad \text{... (30)}$$

and $$\boxed{\bar{D} = \varepsilon_0\, \bar{E} = \frac{\rho_v\, a}{3}\, \bar{a}_r} \qquad \text{... (31)}$$

Case 3 : The point P is inside the sphere (r < a) the Gaussian surface is a spherical surface of radius r where r < a.

Consider differential surface area dS as shown in the Fig. 3.18.

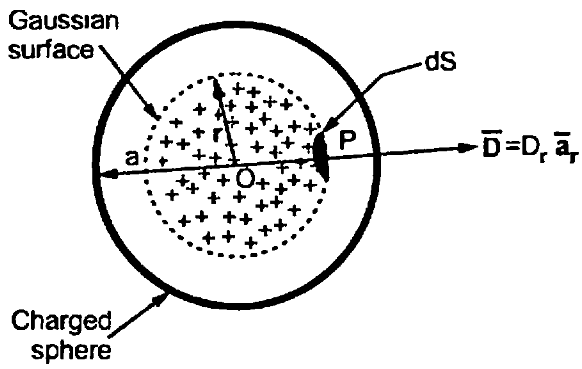


Fig. 3.18

Again $d\bar{S}$ and $\bar{D}$ are directed radially outwards.

$$\therefore \quad \bar{D} = D_r\,\bar{a}_r \quad \text{while} \quad d\bar{S} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

$$\therefore \quad d\psi = \bar{D}\cdot d\bar{S} = D_r r^2 \sin\theta\, d\theta\, d\phi \qquad \ldots (\bar{a}_r \cdot \bar{a}_r = 1)$$

$$\therefore \quad \psi = Q = \oint_S \bar{D}\cdot d\bar{S} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= D_r\, r^2\, [-\cos\theta]_0^{\pi}\, [\phi]_0^{2\pi} = 4\pi r^2 D_r$$

$$\therefore \quad D_r = \frac{Q}{4\pi r^2}$$

$$\therefore \quad \bar{D} = \frac{Q}{4\pi r^2}\bar{a}_r \ \text{C/m}^2 \qquad \ldots (32)$$

Now the charge enclosed is by the sphere of radius r only and not by the entire sphere. The charge outside the Gaussian surface will not affect $\bar{D}$.

$$\therefore \quad Q = \int_v \rho_v dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{4}{3}\pi r^3 \rho_v \qquad \text{where } r < a \qquad \ldots (33)$$

Using in equation (32) we get,

$$\bar{D} = \frac{\frac{4}{3}\pi r^3 \rho_v}{4\pi r^2}\bar{a}_r$$

$$\therefore \quad \boxed{\bar{D} = \frac{r}{3}\rho_v\, \bar{a}_r \qquad \ldots\ldots\ldots 0 < r \le a} \qquad \ldots (34)$$

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{r}{3\varepsilon_0}\rho_v\, \bar{a}_r \qquad 0 < r \le a \qquad \ldots (35)$$

Key Point: *The results obtained here can be used as the standard results while solving the problems.*

If the sphere is in a medium of permittivity ε_r then ε_0 must be replaced by $\varepsilon = \varepsilon_0 \varepsilon_r$.

3.8.6.1 Variation of $\bar{E}$ against r

From the equations (26), (28) and (33) it can seen that for r > a, the $\bar{E}$ is inversely proportional to square of the distance while for r < a it is directly proportional to the distance r. At r = a, $|\bar{E}| = \frac{\rho_v a}{3\varepsilon_0}$ depends on the radius of the charged sphere.

For r > a, the graph of $|\bar{E}|$ against r is parabolic while for r < a it is a straight line as shown in the Fig. 3.19.

The graph of $|\bar{D}|$ against r is exactly similar in nature as $|\bar{E}|$ against r.

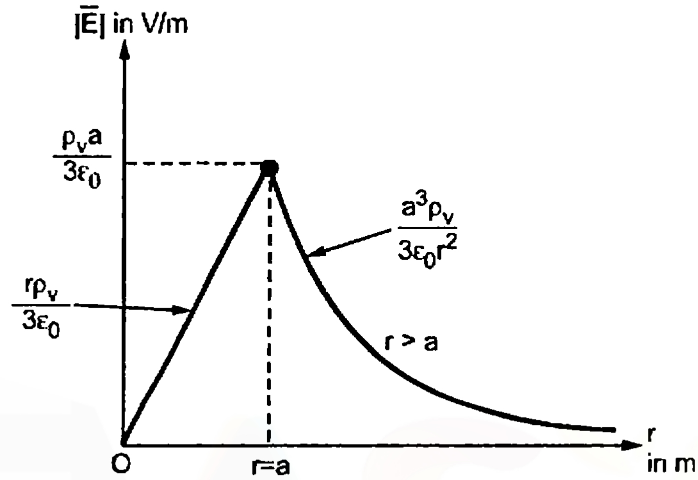


Fig. 3.19 Variation of $|\bar{E}|$ against r

3.9 Gauss's Law Applied to Differential Volume Element

Uptill now we have considered the various cases in which there exists a symmetry and component of $\bar{D}$ is normal to the surface and constant everywhere on the surface. But if there does not exist a symmetry and Gaussian surface can not be chosen such that normal component of $\bar{D}$ is constant or zero everywhere on the surface, Gauss's law can not be directly applied.

In such a case a differential closed Gaussian surface is considered. The closed surface is so small that $\bar{D}$ is almost constant everywhere on the surface. Finally results can be obtained by decreasing the volume enclosed by Gaussian surface to approach to zero.

Consider a cartesian co-ordinate system and a point P in it such that the electric flux density at P is given by,

$$\bar{D} = D_x \bar{a}_x + D_y \bar{a}_y + D_z \bar{a}_z \qquad \text{... (1)}$$

Consider the closed Gaussian differential surface in the form of rectangular box, which is a differential volume element. The sides of this element are $\Delta x, \Delta y$ and Δz. The position of this element is such that the point P is at the centre of the element and treated to be origin. Hence $\bar{D}$ at P can be denoted as $\bar{D}_0$. This is shown in the Fig. 3.20.

Let $\bar{D} = \bar{D}_0 = D_{x0} \bar{a}_x + D_{y0} \bar{a}_y + D_{z0} \bar{a}_z$ at point P

The components D_{x0}, D_{y0} and D_{z0} vary with distance in the respective directions.

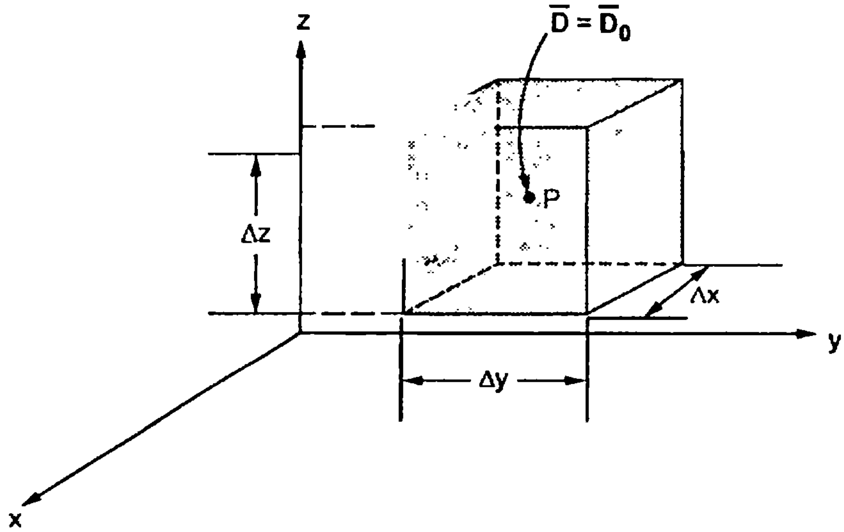


Fig. 3.20 Differential volume element

According to Gauss's law,

$$Q = \oint_S \bar{D} \cdot d\bar{S} \quad \text{... (2)}$$

The total surface integral is to be evaluated over six surfaces front, back, leftside, rightside, top and bottom.

$$\therefore \quad \oint_S \bar{D} \cdot d\bar{S} = \left\{ \int_{\text{front}} + \int_{\text{back}} + \int_{\text{leftside}} + \int_{\text{rightside}} + \int_{\text{top}} + \int_{\text{bottom}} \right\} \bar{D} \cdot d\bar{S} \quad \text{... (3)}$$

Consider the front surface of the differential element. Though $\bar{D}$ is varying with distance, for small surface like front surface it can be assumed constant.

And $\quad d\bar{S} = \Delta y \Delta z \, \bar{a}_x \quad$ ... as $\bar{a}_x$ is normal to front

while $\quad \bar{D} = \bar{D}_{front}$ constant

$$\therefore \quad \int_{\text{front}} \bar{D} \cdot d\bar{S} = \bar{D}_{front} \cdot (\Delta y \Delta z) \, \bar{a}_x \quad \text{... (4)}$$

But $\quad \bar{D}_{front} = D_{x,\,front} \, \bar{a}_x \quad$ ... (5)

$$\therefore \quad \int_{\text{front}} \bar{D} \cdot d\bar{S} = D_{front} \Delta y \Delta z \quad \text{as } \bar{a}_x \cdot \bar{a}_x = 1 \quad \text{... (6)}$$

It has been mentioned that $D_{x,\,front}$ is changing in x direction. At P, it is D_{x0} while on the front surface it will change and given by,

$$D_{x,\,front} = D_{x0} + \begin{bmatrix} \text{Rate of change} \\ \text{of } D_x \text{ with } x \end{bmatrix} \times \begin{bmatrix} \text{Distance of surface} \\ \text{from P} \end{bmatrix}$$

$$D_{x,\,front} = D_{x0} + \frac{\partial D_x}{\partial x} \frac{\Delta x}{2} \quad \text{... (7)}$$

- The point P is at the centre so distance of surface in x direction from P is $\frac{\Delta x}{2}$.
- The rate of change is expressed as **partial derivative** as Dx varies with y and z co-ordinates also.

$$\therefore \quad \int_{\text{front}} \bar{D} \cdot d\bar{S} \approx \left[D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right]\Delta y \Delta z \qquad \text{... (8)}$$

Consider the integral over the back surface,

$$\therefore \quad \int_{\text{back}} \bar{D} \cdot d\bar{S} \approx \bar{D}_{\text{back}} \cdot d\bar{S}$$

where $\quad \bar{D}_{\text{back}} = D_{x,\text{back}}\,(\bar{a}_x)$

$$\therefore \quad d\bar{S} = \Delta y \Delta z(-\bar{a}_x)$$

Key Point: *Note that the flux is entering from back side and leaving from front in positive x direction hence $\bar{a}_x$ is used positive for $\bar{D}_{back}$. While the surface considered from point P is in negative x direction hence $-\bar{a}_x$ is used for expressing $d\bar{S}$.*

$$\therefore \quad \bar{D}_{\text{back}} \cdot d\bar{S} = -D_{x,\text{back}}\,\Delta y \Delta z \qquad \text{... } (a_x \cdot a_x = 1)$$

$$\therefore \quad \int_{\text{back}} \bar{D} \cdot d\bar{S} \approx D_{x,\text{back}}\,\Delta y \Delta z \qquad \text{... (9)}$$

Now $D_{x,\text{back}}$ is changing with x. At P it is D_{x0} while on the back side it will be different and can be obtained as,

$$\therefore \quad D_{x,\text{back}} = D_{x0} - \begin{bmatrix}\text{Rate of change} \\ \text{of } D_x \text{ with x}\end{bmatrix} \times \begin{bmatrix}\text{Distance of surface} \\ \text{from P}\end{bmatrix}$$

$$\therefore \quad D_{x,\text{back}} = D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x} \qquad \text{... (10)}$$

The negative sign is used as the surface is in negative direction of x from P.

Substituting in (9) we get,

$$\int_{\text{back}} \bar{D} \cdot d\bar{S} = -\left[D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right]\Delta y \Delta z$$

$$\therefore \quad \int_{\text{back}} \bar{D} \cdot d\bar{S} = \left[-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right]\Delta y \Delta z \qquad \text{... (11)}$$

Combining equation (8) and equation (11),

$$\int_{\text{front}} + \int_{\text{back}} = 2 \times \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\,\Delta y \Delta z$$

$$\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z \quad \text{... (12)}$$

Similarly we can write,

$$\int_{left} + \int_{right} = \frac{\partial D_y}{\partial y} \Delta x \Delta y \Delta z \quad \text{... (13)}$$

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z} \Delta x \Delta y \Delta z \quad \text{... (14)}$$

$$\therefore \quad \oint_S \bar{D} \cdot d\bar{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

But $\Delta x \Delta y \Delta z$ = Differential volume Δv

$$\therefore \quad \oint_S \bar{D} \cdot d\bar{S} = Q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v \quad \text{... (15)}$$

Thus the charge enclosed in volume Δv is given by,

$$\boxed{Q = \text{Charge enclosed in volume } \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v} \quad \text{... (16)}$$

This result leads to the concept of divergence.

Example 3.3 : *The flux density $D = \frac{r}{3}\, \bar{a}_r$ nC/m² is in the free space :*

a) Find $\bar{E}$ at r = 0.2 m.
b) Find the total electric flux leaving the sphere of r = 0.2 m.
c) Find the total charge within the sphere of r = 0.3 m.

Solution : a) $\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{r}{3\,\varepsilon_0} \bar{a}_r$ and r = 0.2 m

$$\therefore \quad \bar{E} = \frac{0.2 \times 10^{-9}}{3 \times 8.854 \times 10^{-12}} = 7.5295\ \bar{a}_r \text{ V/m}$$

b) $Q = \psi = \oint_S \bar{D} \cdot d\bar{S}$

Consider a differential area dS normal to $\bar{a}_r$ which is $r^2 \sin\theta\, d\theta\, d\phi$

$$\therefore \quad d\bar{S} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

and $\bar{D} = \frac{r}{3} \bar{a}_r$

$$\therefore \quad \bar{D} \cdot d\bar{S} = \frac{r^3}{3} \sin\theta\, d\theta\, d\phi \quad \text{... } (\bar{a}_r \cdot \bar{a}_r = 1)$$

$$\therefore \quad Q = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{r^3}{3} \sin\theta \, d\theta \, d\phi = \frac{r^3}{3} [-\cos\theta]_0^{\pi} \, [\phi]_0^{2\pi}$$

$$= \frac{4}{3} \pi r^3 \text{ nC} \qquad \text{... Note that } \overline{D} \text{ is in nC/m}^3$$

$$\therefore \text{At } r = 0.2 \text{ m}, \quad Q = \frac{4}{3} \pi \times (0.2)^3 = 0.0335 \text{ nC} = \mathbf{33.51 \text{ pC}}$$

$$\text{c) At } r = 0.3 \text{ m}, \quad Q = \frac{4}{3} \pi \times (0.3)^3 = 0.113 \text{ nC} = \mathbf{113.097 \text{ pC}}$$

Example 3.4 : *Three concentric spherical surfaces have radii r = 3, 5 and 7 cm respectively and have uniform charge densities of 200, – 50 and ρ_x μC/m² respectively. Find.*

a) $\overline{D}$ and $\overline{E}$ at r = 2 cm, 4 cm and 6 cm.

b) Find ρ_x if $\overline{D} = 0$ at r = 7.32 cm.

Solution : a) At r = 2 cm, it is inner side of inner sphere. It is seen that inside a spherical shell with surface charge $\overline{E}$ and $\overline{D}$ = 0. Now r = 2 cm is inside of all three spheres hence $\overline{E} = \overline{D} = 0$.

At r = 4 cm which is exterior to innermost sphere but inside of spheres having radii 5 and 7 cm. Hence at r = 4 cm, $\overline{D}$ and $\overline{E}$ exist due to sphere of r = 3 cm with $\rho_S = 200 \ \mu\text{C/m}^2$.

$$\overline{E} = \frac{\rho_S a^2}{\varepsilon_0 r^2} \bar{a}_r \qquad \text{... (Refer section 3.8.5)}$$

$$= \frac{200 \times 10^{-6} \times (3 \times 10^{-2})^2}{8.853 \times 10^{-12} \times (4 \times 10^{-2})^2} \bar{a}_r = 12.706 \times 10^6 \ \bar{a}_r \ \text{V/m}$$

Here a = Radius of sphere = 3 cm and r = 4 cm is distance.

and $\overline{D} = \varepsilon_0 \overline{E} = \mathbf{112.5 \ \bar{a}_r \ \mu C/m^2}$

At r = 6 cm, the $\overline{E}$ and $\overline{D}$ will be due to the two spherical shells having radii 3 and 5 cm. While due to sphere of r = 7 cm, $\overline{D}$ and $\overline{E}$ are zero at r = 6 cm.

$\therefore \quad a_1 = 3$ cm, $\quad \rho_{S1} = 200 \ \mu\text{C/m}^2$

$$\therefore \quad \overline{E}_1 = \frac{\rho_{S1} (a_1)^2}{\varepsilon_0 (r)^2} \bar{a}_r = \frac{200 \times 10^{-6} \times (3 \times 10^{-2})^2}{8.854 \times 10^{-12} \times (6 \times 10^{-2})^2} \bar{a}_r$$

$$= \mathbf{5.6471 \times 10^6 \ \bar{a}_r \ V/m}$$

$$\therefore \quad \overline{D}_1 = \varepsilon_0 \overline{E} = 50 \ \bar{a}_r \ \mu\text{C/m}^2$$

And $a_2 = 5$ cm, $\quad \rho_{S2} = -50 \ \mu\text{C/m}^2$

$$\therefore \quad \bar{E}_2 = \frac{\rho_{S2}\,(a_2)^2}{\varepsilon_0\,(r)^2}\bar{a}_r = \frac{-50\times10^{-6}\times\left(5\times10^{-2}\right)^2}{8.854\times10^{-12}\times\left(6\times10^{-2}\right)^2}\bar{a}_r$$

$$= -3.9216\times10^{6}\,\bar{a}_r \text{ V/m}$$

$$\therefore \quad \bar{D}_2 = \varepsilon_0\bar{E} = -34.722\,\bar{a}_r\ \mu\text{C/m}^2$$

$$\therefore \quad \bar{E} = \bar{E}_1+\bar{E}_2 = 1.7255\times10^{6}\,\bar{a}_r \text{ V/m}$$

$$\text{and} \quad \bar{D} = \bar{D}_1+\bar{D}_2 = 15.278\,\bar{a}_r\ \mu\text{C/m}^2$$

Note that radial distance r is measured from the centre i.e. origin of the spheres.

b) The spheres are shown in the Fig. 3.21.

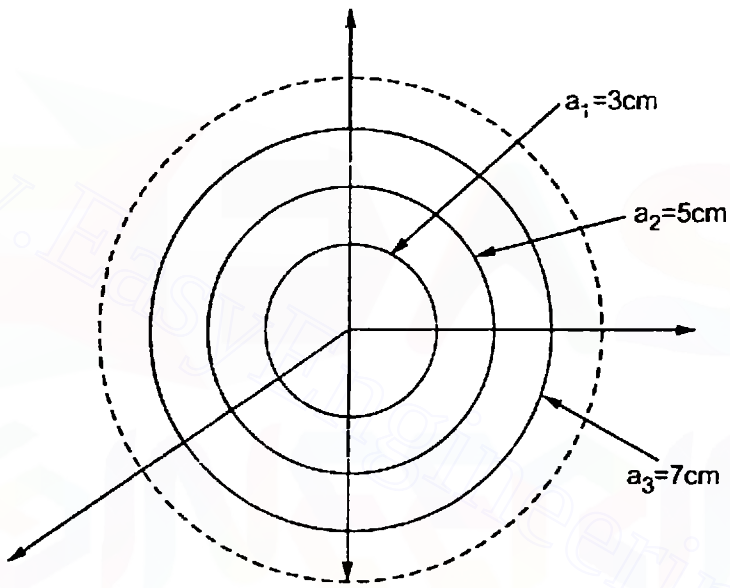


Fig. 3.21

At **r** = 7.32 cm, all three shells produce $\bar{D}$.

$$\therefore \quad \bar{D}_1 = \frac{\rho_{S1}\,(a_1)^2}{(r)^2}\bar{a}_r$$

$$\therefore \quad \bar{D}_2 = \frac{\rho_{S2}\,(a_2)^2}{(r)^2}\bar{a}_r$$

$$\therefore \quad \bar{D}_3 = \frac{\rho_x\,(a_3)^2}{(r)^2}\bar{a}_r$$

But $\bar{D}$ = 0 at r = 7.32 cm as given.

$$\therefore \quad \bar{D} = \bar{D}_1+\bar{D}_2+\bar{D}_3 = 0$$

$$= \frac{\rho_{S1}\,(a_1)^2+\rho_{S2}\,(a_2)^2+\rho_x\,(a_3)^2}{(r)^2}\bar{a}_r = 0$$

But $r \neq 0$ and $\bar{a}_r \neq 0$

$$\therefore \quad \rho_{S1}(a_1)^2 + \rho_{S2}(a_2)^2 + \rho_x (a_3)^2 = 0$$

$$\therefore \quad \rho_x = -\left[\frac{200\times10^{-6}\times(3\times10^{-2})^2 + (-50\times10^{-6})\times(5\times10^{-2})^2}{(7\times10^{-2})^2}\right]$$

$$= -11.2244\ \mu C/m^2$$

Example 3.5 : *A uniform line charge $\rho_{L1} = 2.5\ \mu C/m^2$ lies along the z-axis and a concentric circular cylinder of radius 3 m has a surface charge density of $\rho_S = -0.12\ \mu C/m^2$. Both the distributions are infinite in extent with respect to z-axis. Using Gauss's law, find $\bar{D}$ in all the regions. The region is free space.*

Solution : The arrangement is shown in the Fig. 3.22.

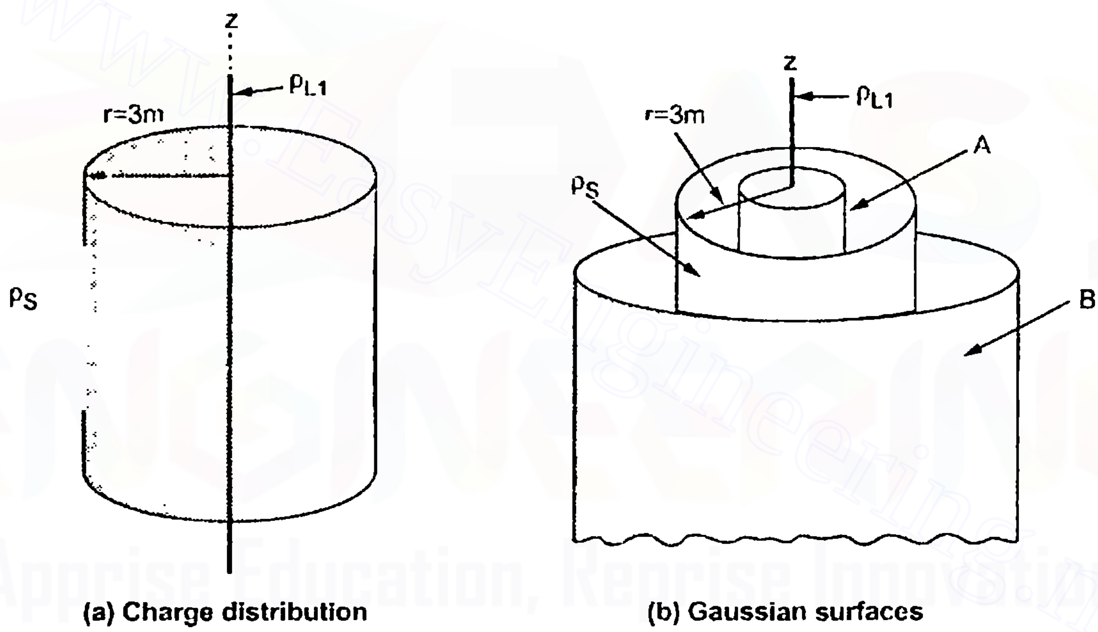


Fig. 3.22

The spherical surface A shown in the Fig. 3.22 (b) is the Gaussian surface for the line charge. Let the differential surface area is $dS = r\,d\phi\,dz$ to which $\bar{a}_r$ is normal. The $\bar{D}$ is directed radially outwards. The length of the Gaussian surface is L.

$$\therefore \quad \bar{D} = D_r\,\bar{a}_r \quad \text{and} \quad d\bar{S} = r\,d\phi\,dz\,\bar{a}_r$$

The radius r of Gaussian surface A is $0 < r < 3$.

$$\therefore \quad Q = \oint_S \bar{D}\cdot d\bar{S} = \oint_S D_r\, r\,d\phi\,dz \qquad \ldots(\bar{a}_r \cdot \bar{a}_r = 1)$$

$$= \int_{z=0}^{L}\int_{\phi=0}^{2\pi} D_r\, r\,d\phi\,dz = D_r r[\phi]_0^{2\pi}[z]_0^L = D_r r 2\pi L$$

But charge on the line of length L is $Q = \rho_{L1} \times L$

$$\therefore \quad \rho_{L1} L = D_r \, r \, 2\pi L$$

$$\therefore \quad D_r = \frac{\rho_{L1}}{2\pi r} \quad \text{and} \quad \bar{D} = \frac{\rho_{L1}}{2\pi r} \bar{a}_r$$

$$\therefore \quad \bar{D} = \frac{2.5 \times 10^{-6}}{2\pi r} \bar{a}_r = \frac{0.3978}{r} \bar{a}_r \ \mu C/m^2$$

This is in the region $0 < r < 3$m.

The spherical surface B is the Gaussian surface enclosing both the charge distributions.

Due to the line charge, $\bar{D}_1 = \frac{\rho_{L1}}{2\pi r} \bar{a}_r$ remains same.

And due to cylinder of radius 3 m, let it be $\bar{D}_2$. The direction of $\bar{D}_2$ is radially outwards. Consider differential surface area normal to $\bar{a}_r$ which is $r\, d\phi\, dz$. The length of Gaussian surface is L.

$$\therefore \quad \bar{D}_2 = D_{2r} \bar{a}_r \quad \text{and} \quad d\bar{S} = r\, d\phi\, dz\, \bar{a}_r$$

$$\therefore \quad Q = \oint_S \bar{D}_2 \cdot d\bar{S} = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} D_{2r}\, r\, d\phi\, dz \qquad \dots (\bar{a}_r \cdot \bar{a}_r = 1)$$

$$= D_{2r}\, r\, 2\pi L$$

Now charge on the surface of length L and radius r is,

$$Q = \rho_S \times \text{Surface area} = \rho_S \times 2\pi r L$$

where r = 3 m = Radius of charge distribution

$$= 2\pi \times (-0.12 \times 10^{-6}) \times (3) \times L = -2.2619 \times 10^{-6} \text{ L C}$$

$$\therefore \quad -2.2619 \times 10^{-6} L = D_{2r} \times r \times 2\pi L$$

$$\therefore \quad D_{2r} = \frac{-2.2619 \times 10^{-6}}{2\pi r} = \frac{-0.36}{r} \times 10^{-6}$$

$$\therefore \quad \bar{D}_2 = \frac{-0.36}{r} \bar{a}_r \ \mu C/m^2 \qquad \text{for } r > 3$$

$$\therefore \quad \bar{D} = \bar{D}_1 + \bar{D}_2 = \frac{0.0378}{r} \bar{a}_r \ \mu C/m^2 \qquad \text{for } r > 3$$

3.10 Divergence

Applying Gauss's law to the differential volume element, we have obtained the relation,

$$Q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v \qquad \dots (1)$$

This is the charge enclosed in the volume Δv.

But $\quad Q = \oint_S \bar{D} \cdot d\bar{S}$ by Gauss's law $\qquad \dots (2)$

To apply Gauss's law, we have assumed a differential volume element as the Gaussian surface, over which $\bar{D}$ is constant. Hence equations (1) and (2) can be equated in limiting case as $\Delta v \to 0$.

$$\therefore \quad \oint_S \bar{D} \cdot d\bar{S} = \lim_{\Delta v \to 0} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

$$\therefore \quad \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{D} \cdot d\bar{S}}{\Delta v} \qquad \text{... (3)}$$

Thus in general if $\bar{A}$ is any vector say force, velocity, temperature gradient etc. then,

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{A} \cdot d\bar{S}}{\Delta v} \qquad \text{... (4)}$$

This mathematical operation on $\bar{A}$ is called a **divergence.** It is denoted as **divergence** $\bar{A}$. Hence mathematically divergence is given by,

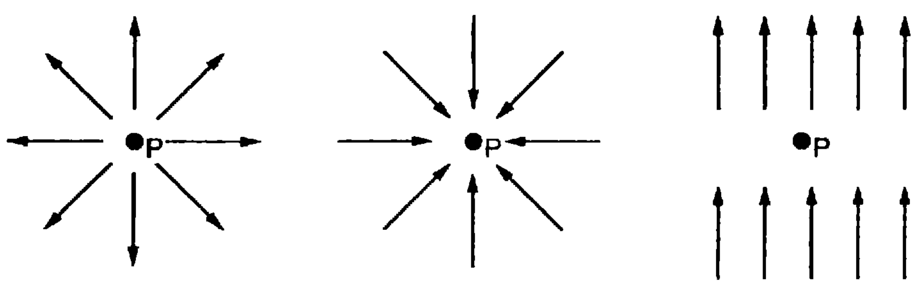
$$\boxed{\text{div } \bar{A} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{A} \cdot d\bar{S}}{\Delta v} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \qquad \text{... (5)}$$

3.10.1 Physical Meaning of Divergence

From the equation (5), the physical meaning of divergence can be obtained. Let $\bar{A}$ be the flux density vector then,

the divergence of the vector flux density $\bar{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

Hence the divergence of $\bar{A}$ at a given point is a measure of how much the field represented by $\bar{A}$ diverges or converges from that point. If the field is diverging at point P of vector field $\bar{A}$ as shown in the Fig. 3.23 (a), then divergence of $\bar{A}$ at point P is positive. The field is spreading out from point P. If the field is converging at the point P as shown in the Fig. 3.23 (b), then the divergence of $\bar{A}$ at the point P is negative. It is practically a convergence i.e. negative of divergence. If the field at point P is as shown in the Fig. 3.23 (c), so whatever field is converging, same is diverging then the divergence of $\bar{A}$ at point P is zero.



(a) Positive (b) Negative (c) Zero

Fig. 3.23 Divergence at P

Practically consider a tube of a vehicle in which air is filled at a pressure. If it is punctured, then air inside tries to rush out from a tube through a small hole. Thus the velocity of air at the hole is greatest while away from the hole it is less. If now any closed surface is considered inside the tube, at one end velocity field is less while from other end it has higher value, as air rushes towards the hole. Hence the divergence of such velocity inside is positive. This is shown in the Fig. 3.24 (a) and (b).

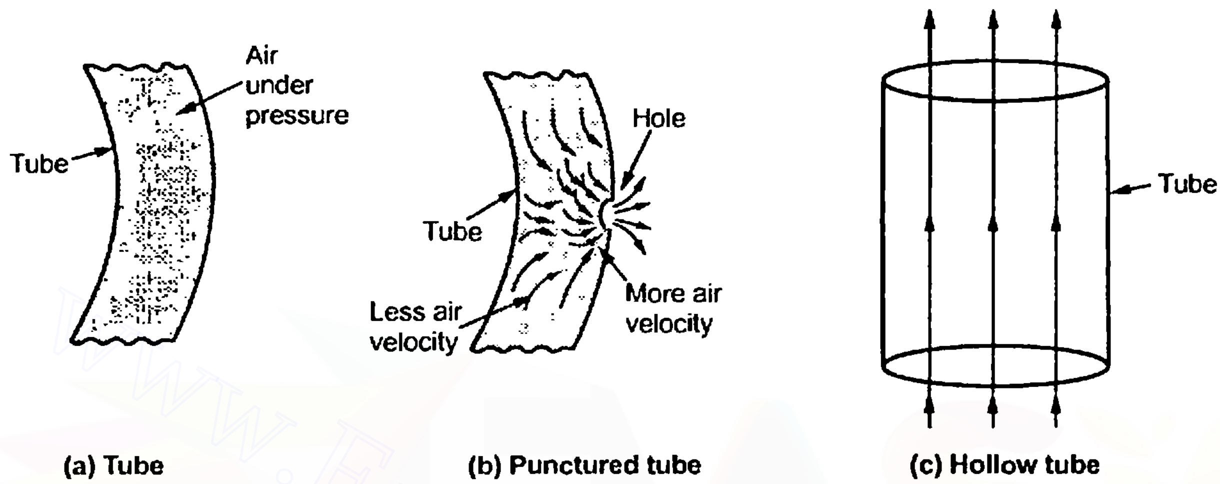


Fig. 3.24 Concept of divergence

As seen from the Fig. 3.24 (b), the air velocity is a function of distance and hence divergence of velocity is positive. The density of lines near hole is high showing higher air velocity. The source of such velocity lines is throughout the tube and hence anywhere inside the tube, at any point the divergence is positive.

If there is a hollow tube open from both ends then air enters from one end and passes through the tube and leaves from other end. This is shown in the Fig. 3.24 (c). The velocity of air is constant everywhere inside the tube. In such a case the divergence of the velocity field is zero, inside the tube.

A positive divergence for any vector quantity indicates a **source** of that vector quantity at that point. A negative divergence for any vector quantity indicates a **sink** of that vector quantity at that point. A zero divergence indicates there is no source or sink exists at that point.

In short, if more lines enter a small volume than the lines leaving it, there is positive divergence. If more lines leave a small volume than the lines entering it, there is negative divergence. If the same number of lines enter and leave a small volume, the field has zero divergence. Note that the volume must be infinitesimally small, shrinking to zero at that point, where divergence is obtained.

As the result of divergence of a vector field is a scalar, the divergence indicates how much flux lines are leaving a small volume, per unit volume and there is no direction associated with the divergence.

3.10.2 The Vector Operator ∇

The divergence of the vector field $\bar{A}$ is given by,

$$\text{div } \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The **divergence of a vector is a scalar quantity.**

The divergence operation can be represented by the use of mathematical operator called **del operator** ∇ which is a **vector operator**. It is given by,

$$\boxed{\nabla = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z} \quad \text{... (6)}$$

Now the $\bar{A}$ is a vector field and ∇ is also a vector. The result of divergence is a scalar. Thus to get the scalar from the two vectors, it is necessary to take **dot product** of the two.

If $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$ then

$$\nabla \bullet \bar{A} = \left[\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right] \bullet [A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z]$$

Now $\bar{a}_x \bullet \bar{a}_x = \bar{a}_y \bullet \bar{a}_y = \bar{a}_z \bullet \bar{a}_z = 1$

While other dot products such as $\bar{a}_x \bullet \bar{a}_y$ etc. are zero.

$$\therefore \quad \nabla \bullet \bar{A} = \frac{\partial (A_x)}{\partial x} + \frac{\partial (A_y)}{\partial y} + \frac{\partial (A_z)}{\partial z} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\therefore \quad \nabla \bullet \bar{A} = \text{div } \bar{A} \quad \text{... (7)}$$

Note the following observations regarding ∇ :

1. ∇ is a mathematical operator and need not be involved always in the dot product.

2. It may be operated on a scalar field to obtain vector result. Thus if m is a scalar field then,

$$\nabla m = \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right) m = \frac{\partial m}{\partial x}\bar{a}_x + \frac{\partial m}{\partial y}\bar{a}_y + \frac{\partial m}{\partial z}\bar{a}_z$$

3. The ∇ operator does not have any other specific form in different coordinate systems. Whatever may be the coordinate system in which $\bar{A}$ is represented, $\nabla \bullet \bar{A}$ represents a divergence of $\bar{A}$.

3.10.3 Divergence in Different Co-ordinate Systems

In a cartesian system, the differential volume unit is given by dv = dx dy dz while in cylindrical system it is given by dv = r dr dϕ dz. In the spherical system it is given by $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$ Thus the expressions for divergence in different co-ordinate systems are different.

These expressions of divergence, in different co-ordinate systems are given by,

$$\nabla \bullet \bar{A} = \text{div } \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{... Cartesian}$$

$$\nabla \bullet \bar{A} = \text{div } \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad \text{... Cylindrical}$$

$$\nabla \bullet \bar{A} = \text{div } \bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \quad \text{... Spherical}$$

The relations are frequently required in the engineering electromagnetics.

3.10.4 Properties of Divergence of Vector Field

The various properties of divergence of a vector field are,

1. The divergence produces a scalar field as the dot product is involved in the operation. The result does not have direction associated with it.
2. The divergence of a scalar has no meaning. Thus if m is a scalar field then $\nabla \bullet m$ has no meaning. Note that ∇ operator can operate on scalar field but dot product i.e. divergence of a scalar has no meaning.
3. $\nabla \bullet \left(\bar{A} + \bar{B}\right) = \nabla \bullet \bar{A} + \nabla \bullet \bar{B}$

3.11 Maxwell's First Equation

The divergence of electric flux density $\bar{D}$ is given by,

$$\text{div } \bar{D} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{D} \bullet d\bar{S}}{\Delta v} \quad \text{... (1)}$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

According to Gauss's law, it is known that

$$\psi = Q = \oint_S \bar{D} \bullet d\bar{S} \quad \text{... (2)}$$

Expressing Gauss's law per unit volume basis

$$\frac{Q}{\Delta v} = \frac{\oint_S \bar{D} \bullet d\bar{S}}{\Delta v} \quad \text{... (3)}$$

Taking lim $\Delta v \to 0$ i.e. volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{D} \bullet d\bar{S}}{\Delta v} \quad \text{... (4)}$$

But $$\lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v \text{ at that point} \quad \text{... (5)}$$

The equation (5) gives the volume charge density at the point where divergence is obtained.

Equating (1) and (5),

$$\text{div } \overline{D} = \rho_v \qquad \ldots (6)$$

$$\text{i.e. } \nabla \bullet \overline{D} = \rho_v$$

This is volume charge density around a point. The equation (6) is called **Maxwell's first equation applied to electrostatics.** This is also called the **point form of Gauss's law or Gauss's law in differential form.**

> The statement of Gauss's law in point form is,
>
> The divergence of electric flux density in a medium at a point (differential volume shrinking to zero), is equal to the volume charge density (charge per unit volume) at the same point.

➠ **Example 3.6 :** *Find the divergence of* $\overline{A}$ *at* $P(5, \pi/2, 1)$, *where* $\overline{A} = rz\sin\phi\,\overline{a}_r + 3rz^2\cos\phi\,\overline{a}_\phi$.

Solution : Given $\overline{A}$ in cylindrical system,

$$\therefore \quad \text{div } \overline{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

where $A_r = r\,z\sin\phi$, $A_\phi = 3\,r\,z^2\cos\phi$, $A_z = 0$

$$\therefore \quad \text{div } \overline{A} = \frac{1}{r}\frac{\partial}{\partial r}[r^2 z\sin\phi] + \frac{1}{r}\frac{\partial}{\partial\phi}[3\,r\,z^2\cos\phi] + 0$$

$$= \frac{1}{r}\cdot z\sin\phi\cdot 2r + \frac{1}{r}\cdot 3r\,z^2[-\sin\phi]$$

$$= 2z\sin\phi - 3z^2\sin\phi$$

At point P, $r = 5$, $\phi = \frac{\pi}{2}$, $z = 1$

$$\therefore \quad \text{div } \overline{A} = 2\times1\times\sin\frac{\pi}{2} - 3\times1\times\sin\frac{\pi}{2}$$

$$= -1 \text{ at P.}$$

3.12 Divergence Theorem

From the Gauss's law we can write,

$$Q = \oint_S \overline{D}\bullet d\overline{S} \qquad \ldots (1)$$

While the charge enclosed in a volume is given by,

$$Q = \int_v \rho_v\, dv \qquad \ldots (2)$$

But according to Gauss's law in the point form,

$$\nabla \bullet \bar{D} = \rho_v \quad \text{... (3)}$$

Using in (2), $$Q = \int_v (\nabla \bullet \bar{D})\, dv \quad \text{... (4)}$$

Equating (1) and (4),

$$\boxed{\oint_S \bar{D} \bullet d\bar{S} = \int_v (\nabla \bullet \bar{D})\, dv} \quad \text{... (5)}$$

The equation (5) is called **divergence theorem.** It is also called the **Gauss-Ostrogradsky theorem.** The theorem can be stated as,

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface.

The theorem can be applied to any vector field but partial derivatives of that vector field must exist. The equation (5) is the divergence theorem as applied to the flux density. Both sides of the divergence theorem give the net charge enclosed by the closed surface i.e. net flux crossing the closed surface.

With the help of the divergence theorem, the surface integral can be converted into a volume integral, provided that the closed surface encloses certain volume. Thus volume integral on right hand side of the theorem must be calculated over a volume which must be enclosed by the closed surface on left handside. The theorem is applicable only under this condition.

Points to remember while solving problems.

1. Draw the sketch of the surface enclosed by the given conditions.

2. $\bar{D}$ acts within the region bounded by given conditions towards the various surfaces. Thus note the direction of surface with respect to region in which $\bar{D}$ is given to give proper sign to the unit vector while defining $d\bar{S}$. For example, consider the region

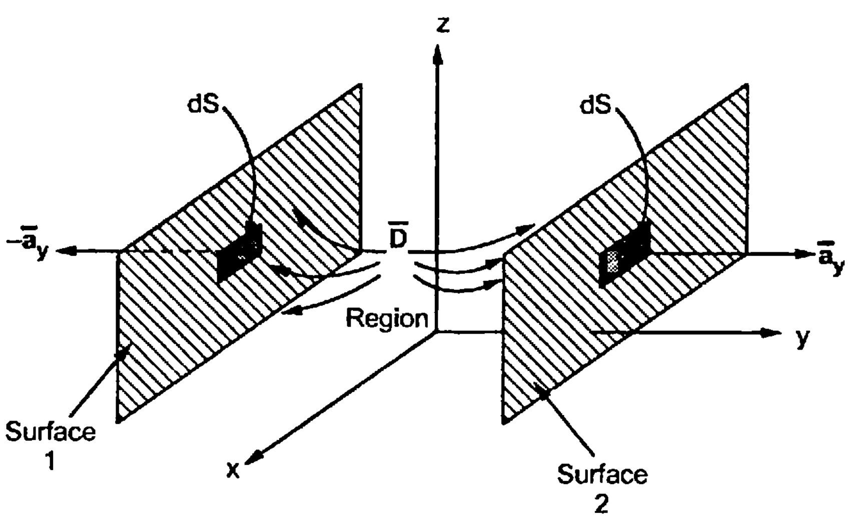


Fig. 3.25

bounded by two planes as shown in the Fig. 3.25. For surface 1, with respect to $\bar{D}$ in the region, dS is in $-\bar{a}_y$ direction. While for surface 2, with respect to $\bar{D}$ in the region, dS is in $+\bar{a}_y$ direction.

3. Then evaluate $\oint_S \bar{D} \cdot d\bar{S}$ over all the possible surfaces.

4. Evaluate $\int_v (\nabla \cdot \bar{D})\, dv$ to verify the divergence theorem. Take care of variables in the partial derivatives.

Example 3.7 : *Given that $\bar{A} = 30 e^{-r}\, \bar{a}_r - 2z\, \bar{a}_z$ in the cylindrical co-ordinates. Evaluate both sides of the divergence theorem for the volume enclosed by r = 2, z = 0 and z = 5.*

Solution : The divergence theorem states that

$$\oint_S \bar{A} \cdot d\bar{S} = \int_v (\nabla \cdot \bar{A})\, dv$$

Now

$$\oint_S \bar{A} \cdot d\bar{S} = \left[\oint_{side} + \oint_{top} + \oint_{bottom} \right] \bar{A} \cdot d\bar{S}$$

Consider $d\bar{S}$ normal to $\bar{a}_r$ direction which is for the side surface.

$$\therefore \quad d\bar{S} = r\, d\phi\, dz\, \bar{a}_r$$

$$\therefore \quad \bar{A} \cdot d\bar{S} = (30 e^{-r}\, \bar{a}_r - 2z\bar{a}_z) \cdot r\, d\phi\, dz\, \bar{a}_r$$

$$= 30\, r\, e^{-r} (\bar{a}_r \cdot \bar{a}_r)\, d\phi\, dz = 30\, r\, e^{-r}\, d\phi\, dz$$

$$\therefore \quad \oint_{side} \bar{A} \cdot d\bar{S} = \int_{\phi=0}^{2\pi} \int_{z=0}^{5} 30\, r\, e^{-r}\, d\phi\, dz \quad \text{with} \quad r = 2$$

$$= 30 \times 2 \times e^{-2} \times [\phi]_0^{2\pi} \times [z]_0^5 = 255.1$$

The $d\bar{S}$ on top has direction $\bar{a}_z$ hence for top surface,

$$d\bar{S} = r\, dr\, d\phi\, \bar{a}_z$$

$$\therefore \quad \bar{A} \cdot d\bar{S} = (30 e^{-r}\, \bar{a}_r - 2z\bar{a}_z) \cdot r\, dr\, d\phi\, \bar{a}_z$$

$$= -2\, z\, r\, dr\, d\phi \qquad \ldots (\bar{a}_z \cdot \bar{a}_z = 1)$$

$$\therefore \quad \oint_{top} \bar{A} \cdot d\bar{S} = \int_{\phi=0}^{2\pi} \int_{r=0}^{2} -2\, z\, r\, dr\, d\phi \quad \text{with} \quad z = 5$$

$$= -2 \times 5 \times \left[\frac{r^2}{2} \right]_0^2 \times [\phi]_0^{2\pi} = -40\pi$$

While $d\bar{S}$ for bottom has direction $-\bar{a}_z$ hence for bottom surface,

$$d\bar{S} = r\, dr\, d\phi\, (-\bar{a}_z)$$

$$\therefore \quad \bar{A} \cdot d\bar{S} = (30\, e^{-r}\, \bar{a}_r - 2z\bar{a}_z) \cdot r\, dr\, d\phi\, (-\bar{a}_z)$$

$$= 2\, z\, r\, dr\, d\phi \qquad \ldots (\bar{a}_z \cdot \bar{a}_z = 1)$$

But $z = 0$ for the bottom surface, as shown in the Fig. 3.26.

$$\therefore \quad \oint_S \bar{A} \cdot d\bar{S} = 255.1 - 40\,\pi + 0$$

$$= \mathbf{129.4363}$$

This is the **left hand side of divergence theorem.**

Now evaluate $\int_v (\nabla \cdot \bar{A})\, dv$

$$\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

and $A_r = 30\, e^{-r}$, $A_\phi = 0$, $A_z = -2z$

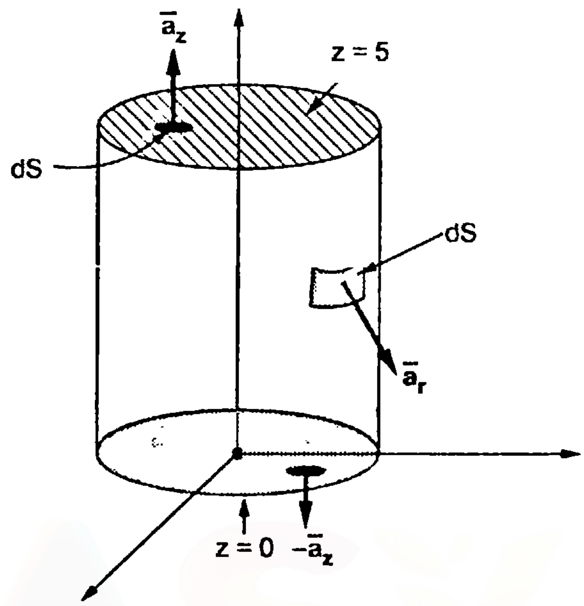


Fig. 3.26

$$\therefore \quad \nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(30\, r\, e^{-r}) + 0 + \frac{\partial}{\partial z}(-2\, z)$$

$$= \frac{1}{r}\left\{30\, r\, (-e^{-r}) + 30\, e^{-r}\, (1)\right\} + (-2)$$

$$= -30\, e^{-r} + \frac{30}{r}\, e^{-r} - 2$$

$$\therefore \quad \int_v (\nabla \cdot \bar{A})\, dv = \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{r=0}^{2}\left(-30\, e^{-r} + \frac{30}{r}\, e^{-r} - 2\right) r\, dr\, d\phi\, dz$$

$$= \int_{z=0}^{5}\int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(-30\, r\, e^{-r} + 30\, e^{-r} - 2r\right)\, dr\, d\phi\, dz$$

$$= \left\{-30\, r\left[\frac{e^{-r}}{-1}\right] - \int (-30)\left[\frac{e^{-r}}{-1}\right] dr + 30\left[\frac{e^{-r}}{-1}\right] - \left[2\frac{r^2}{2}\right]\right\} [z]_0^5\, [\phi]_0^{2\pi}$$

Obtained using integration by parts.

$$= \left[30\, r\, e^{-r} + 30\, e^{-r} - 30\, e^{-r} - r^2\right]_0^2 [5][2\,\pi]$$

$$= \left[60\, e^{-2} - 2^2\right][10\,\pi] = \mathbf{129.437}$$

This is same as obtained from the left hand side.

Example 3.8 : *Given that $\bar{D} = \frac{5r^2}{4}\,\bar{a}_r\ C/m^2$. Evaluate both the sides of divergence theorem for the volume enclosed by r = 4m and $\theta = \pi/4$.*

Solution : The given $\bar{D}$ is in spherical co-ordinates. The volume enclosed is shown in the Fig. 3.27.

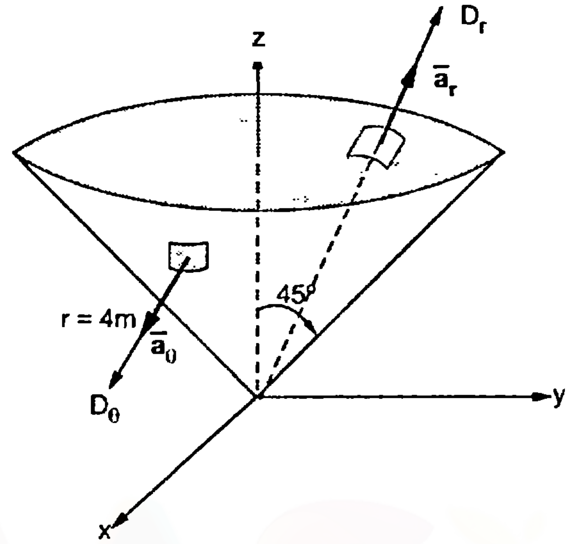


Fig. 3.27

According to divergence theorem,

$$\oint_S \bar{D} \cdot d\bar{S} = \int_v (\nabla \cdot \bar{D})\, dv$$

The given $\bar{D}$ has only radial component as given. Hence $D_r = \frac{5\,r^2}{4}$ while $D_\theta = D_\phi = 0$.

Hence $\bar{D}$ has a value only on the surface r = 4m.

Consider dS normal to the $\bar{a}_r$ direction i.e. $r^2 \sin\theta\, d\theta\, d\phi$

$$\therefore \quad d\bar{S} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

$$\therefore \quad \bar{D} \cdot d\bar{S} = \left(r^2 \sin\theta\, d\theta\, d\phi\right)\left(\frac{5r^2}{4}\right) = \frac{5}{4}\, r^4 \sin\theta\, d\theta\, d\phi \qquad \ldots (\bar{a}_r \cdot \bar{a}_r = 1)$$

$$\therefore \quad \oint_S \bar{D} \cdot d\bar{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \frac{5}{4}\, r^4 \sin\theta\, d\theta\, d\phi$$

$$= \frac{5}{4}\, r^4\, [-\cos\theta]_0^{\pi/4}\, [\phi]_0^{2\pi} \quad \text{and} \quad r = 4m$$

$$= \frac{5}{4}\,(4)^4 \left[-\cos\frac{\pi}{4} - (-\cos 0)\right] [2\pi]$$

$$= \mathbf{588.896\ C}$$

To evaluate right hand side, find $\nabla \cdot \bar{D}$.

$$\nabla \cdot \bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{5}{4}r^2\right)\right] + 0 + 0 = \frac{5}{4r^2}\frac{\partial}{\partial r}(r^4)$$

$$= \frac{5}{4r^2}(4r^3) = 5r$$

In spherical coordinates, $dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$\therefore \quad \int_v (\nabla \cdot \bar{D})\, dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \int_{r=0}^{4} (5\,r)(r^2 \sin\theta\, dr\, d\theta\, d\phi)$$

$$= 5\left[\frac{r^4}{4}\right]_0^4 [-\cos\theta]_0^{\pi/4} [\phi]_0^{2\pi} = 5\times\frac{4^4}{4}\times\left[-\cos\frac{\pi}{4}-(-\cos 0)\right]\times 2\pi$$

$$= 588.896 \text{ C}$$

Example 3.9 : *Find the total charge in a volume defined by the six planes for which* $1 \le x \le 2,\ 2 \le y \le 3,\ 3 \le z \le 4$ *if,*
$\bar{D} = 4x\ \bar{a}_x + 3y^2\ \bar{a}_y + 2z^3\ \bar{a}_z\ C/m^2$.

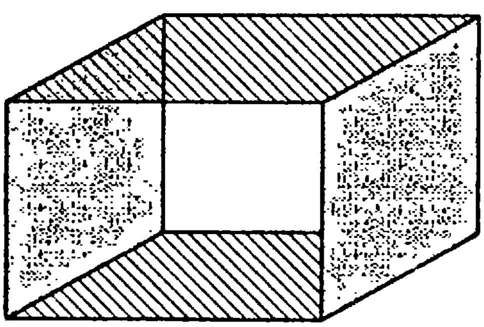
Solution : The volume bounded by the given planes is a cube. To evaluate total charge use Gauss's law.

$$Q = \oint_S \bar{D} \cdot d\bar{S}$$

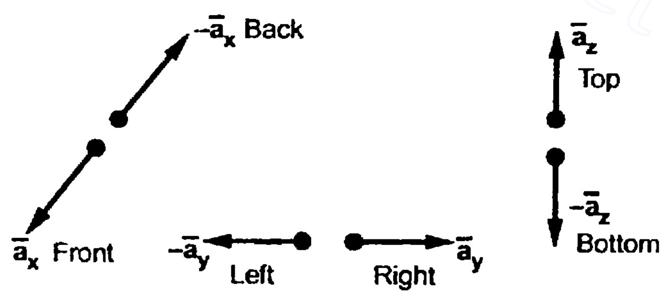
But to evaluate $\bar{D} \cdot d\bar{S}$, it is necessary to consider all six faces of the cube. Let us find $d\bar{S}$ for each surface.

1) Front surface (x = 2), dS = dy dz, direction = $\bar{a}_x$, $d\bar{S}$ = dy dz $\bar{a}_x$
2) Back surface (x = 1), dS = dy dz, direction = $-\bar{a}_x$, $d\bar{S}$ = – dy dz $\bar{a}_x$
3) Right side (y = 3), dS = dx dz, direction = $\bar{a}_y$, $d\bar{S}$ = dx dz $\bar{a}_y$
4) Left side (y = 2), dS = dx dz, direction = $-\bar{a}_y$, $d\bar{S}$ = – dx dz $\bar{a}_y$
5) Top side (z = 4), dS = dx dy, direction = $\bar{a}_z$, $d\bar{S}$ = dx dy $\bar{a}_z$
6) Bottom side (z = 3), dS = dx dy, direction = $-\bar{a}_z$, $d\bar{S}$ = – dx dy $\bar{a}_z$

Key Point: *Remember that though the co-ordinates of x, y and z are positive, the directions of unit vectors are with respect to region bounded by the planes, as shown in the Fig. 3.28 (b).*



(a) Cube



(b) Directions of $d\bar{S}$

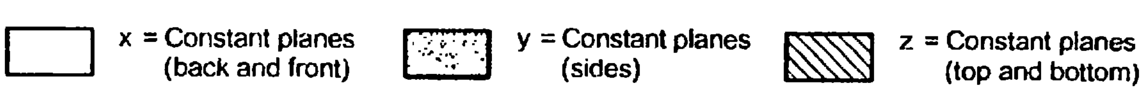


Fig. 3.28

For front $\bar{D} \cdot d\bar{S} = 4x \, dy \, dz, \quad x = 2 \quad \ldots \bar{a}_x \cdot \bar{a}_x = 1$

For back $\bar{D} \cdot d\bar{S} = -4x \, dy \, dz, \quad x = 1 \quad \ldots \bar{a}_x \cdot \bar{a}_x = 1$

For right $\bar{D} \cdot d\bar{S} = 3y^2 \, dx \, dz, \quad y = 3 \quad \ldots \bar{a}_y \cdot \bar{a}_y = 1$

For left $\bar{D} \cdot d\bar{S} = -3y^2 \, dx \, dz, \quad y = 2 \quad \ldots \bar{a}_y \cdot \bar{a}_y = 1$

For top $\bar{D} \cdot d\bar{S} = 2z^3 \, dx \, dy, \quad z = 4 \quad \ldots \bar{a}_z \cdot \bar{a}_z = 1$

For bottom $\bar{D} \cdot d\bar{S} = -2z^3 \, dx \, dy, \quad z = 3 \quad \ldots \bar{a}_z \cdot \bar{a}_z = 1$

$$\therefore \oint_S \bar{D} \cdot d\bar{S} = \underbrace{\int_{z=3}^{4} \int_{y=2}^{3} 4x \, dy \, dz}_{\text{Front, } x = 2} + \underbrace{\int_{z=3}^{4} \int_{y=2}^{3} -4x \, dy \, dz}_{\text{Back, } x = 1}$$

$$+ \underbrace{\int_{z=3}^{4} \int_{x=1}^{2} 3y^2 \, dx \, dz}_{\text{Right, } y = 3} + \underbrace{\int_{z=3}^{4} \int_{x=1}^{2} -3y^2 \, dx \, dz}_{\text{Left, } y = 2}$$

$$+ \underbrace{\int_{y=2}^{3} \int_{x=1}^{2} 2z^3 \, dx \, dz}_{\text{Top, } z = 4} + \underbrace{\int_{y=2}^{3} \int_{x=1}^{2} -2z^3 \, dx \, dz}_{\text{Bottom, } z = 3}$$

$$= (4)(2)[y]_2^3 [z]_3^4 - (4)(1)[y]_2^3 [z]_3^4 + (3)(3)^2 [x]_1^2 [z]_3^4$$

$$= -(3)(2)^2 [x]_1^2 [z]_3^4 + (2)(4)^3 [x]_1^2 [y]_2^3 - (2)(3)^3 [x]_1^2 [y]_2^3$$

$$= 8 - 4 + 27 - 12 + 128 - 54 = 93 \text{ C}$$

This is the total charge enclosed.

Let us verify by divergence theorem.

$$\nabla \cdot \bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 4 + 6y + 6z^2$$

$$\int_v (\nabla \cdot \bar{D}) \, dv = \int_{z=3}^{4} \int_{y=2}^{3} \int_{x=1}^{2} (4 + 6y + 6z^2) \, dx \, dy \, dz \quad \ldots \text{Integrate w.r.t. } x$$

$$= \int_{z=3}^{4} \int_{y=2}^{3} (4 + 6y + 6z^2)[x]_1^2 \, dy \, dz \quad \ldots \text{Integrate w.r.t. } y$$

$$= \int_{z=3}^{4} \left[4y + 6\frac{y^2}{2} + 6z^2 y \right]_2^3 dz$$

$$= \int_{z=3}^{4} \left[4(3-2) + \frac{6}{2}(3^2 - 2^2) + 6z^2 (3-2) \right] dz$$

$$= \int_{z=3}^{4} \left(4+15+6z^2\right) dz = \left[19z+6\frac{z^3}{3}\right]_3^4$$

$$= 19(4-3)+2\left(4^3-3^3\right) = \mathbf{93\ C}$$

Thus divergence theorem is verified.

Examples with Solutions

Example 3.10 : *The flux density within the cylindrical volume bounded by r = 5m, z = 0 and z = 2m is given by,*

$\overline{D} = 30e^{-r}\,\overline{a}_r - 2z\overline{a}_z\ C/m^2$

What is the total outward flux crossing the surface of the cylinder ?

Solution : The cylinder is shown in the Fig. 3.29.

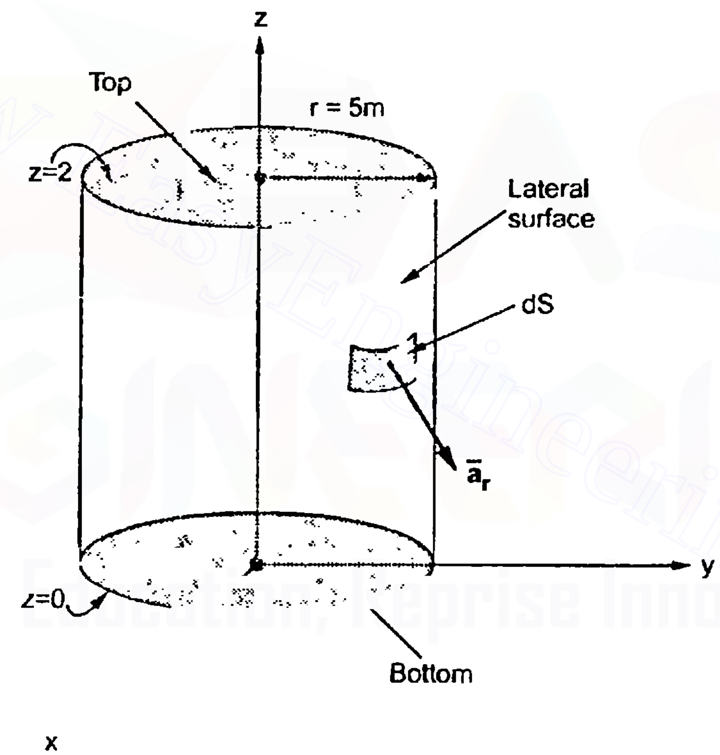


Fig. 3.29

As the total outward flux is asked all surfaces, lateral, top and bottom must be considered.

Case 1 : Consider the lateral surface, the normal direction to which is $\bar{a}_r$.

Consider differential surface area normal to $\bar{a}_r$ which is

$$dS = r\,d\phi\,dz.$$

$$\therefore \quad d\overline{S} = r\,d\phi\,dz\,\bar{a}_r$$

$$\therefore \quad \bar{D}\cdot d\bar{S} = \left[30e^{-r}\,\bar{a}_r - 2z\bar{a}_z\right]\cdot\; r\,d\phi\,dz\,\bar{a}_r$$

$$= 30\,r\,e^{-r}\,d\phi\,dz \qquad \ldots\left(\bar{a}_r\cdot\bar{a}_r = 1,\ \bar{a}_r\cdot\bar{a}_z = 0\right)$$

According to Gauss's law,

$$\psi_1 = \oint_{lateral} \bar{D}\cdot d\bar{S} = \int_{z=0}^{2}\int_{\phi=0}^{2\pi} 30\,r\,e^{-r}\,d\phi\,dz \qquad \ldots\ r = 5 \text{ constant}$$

$$= 30\,r\,e^{-r}\,[\phi]_0^{2\pi}\;[z]_0^2 \qquad \ldots\ r = 5 \text{ constant}$$

$$= 30\times5\times e^{-5}\times2\pi\times2 = \mathbf{12.7\ C}$$

Case 2 : Top surface, for which normal direction is $\bar{a}_z$. The differential area $dS = r\,dr\,d\phi$ normal to $\bar{a}_z$.

$$\therefore \quad d\bar{S} = r\,dr\,d\phi\;\bar{a}_z \quad \text{and} \quad z = 2 \text{ for top surface}$$

$$\therefore \quad \bar{D}\cdot d\bar{S} = \left(30e^{-r}\,\bar{a}_r - 2z\bar{a}_z\right)\cdot\left(r\,dr\,d\phi\bar{a}_z\right)$$

$$= -\,2\,zr\,dr\,d\phi \qquad \ldots\left(\bar{a}_z\cdot\bar{a}_z = 1,\ \bar{a}_r\cdot\bar{a}_z = 0\right)$$

$$\therefore \quad \psi_2 = \oint_{top} \bar{D}\cdot d\bar{S}$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{5} -\,2\,z\,r\,dr\,d\phi \quad \text{with } z = 2$$

$$= -2\,z\left[\frac{r^2}{2}\right]_0^5\;[\phi]_0^{2\pi} \qquad \ldots\ z = 2 \text{ constant}$$

$$= -2\times2\times12.5\times2\pi$$

$$= \mathbf{-\,314.1592\ C}$$

Case 3 : Bottom surface, for which normal direction is $-\bar{a}_z$ with respect to region. The differential area $dS = r\,dr\,d\phi$ normal to $\bar{a}_z$.

$$\therefore \quad d\bar{S} = r\,dr\,d\phi\left(-\bar{a}_z\right) \quad \text{and} \quad z = 0 \text{ for bottom}$$

$$\therefore \quad \bar{D}\cdot d\bar{S} = \left(30e^{-r}\,\bar{a}_r - 2z\bar{a}_z\right)\cdot\; r\,dr\,d\phi\;\left(-\bar{a}_z\right)$$

$$= 2\,zr\,dr\,d\phi \quad \text{with} \quad z = 0$$

$$= 0$$

$$\therefore \quad \psi_3 = \oint_{bottom} \bar{D}\cdot d\bar{S} = 0 \quad \text{as} \quad z = 0 \text{ for bottom}$$

$$\therefore \quad \psi_{net} = \psi_1 + \psi_2 + \psi_3$$

$$= \mathbf{-\,301.4592\ C}$$

Example 3.11 : *If $\bar{D} = 12x^2\bar{a}_x - 3z^3\bar{a}_y - 9yz^2\bar{a}_z\ C/m^3$ in free space, specify the point within the cube $1 \le x, y, z \le 2$ at which the following quantity is maximum and give that maximum value.*

a) $|\bar{D}|$ b) $|\rho_v|$ c) ρ_v

Solution : a) From given $\bar{D}$

$$|\bar{D}| = \sqrt{(12x^2)^2 + (-3z^3)^2 + (-9yz^2)^2}$$

$$= \sqrt{144x^4 + 9z^6 + 81y^2z^4}$$

The $|\bar{D}|$ is maximum, when x, y and z are maximum in the given region.

$\therefore$ $x = y = z = 2$ ... maximum values

$\therefore$ At P (2, 2, 2), $|\bar{D}|$ will be maximum.

$$|\bar{D}|_{max} = \sqrt{144\times2^4 + 9\times2^6 + 81\times2^2\times2^4} = 89.8\ C/m^2$$

b) According to Gauss's law in point form,

$$\nabla \cdot \bar{D} = \rho_v$$

$$\therefore \quad \nabla \cdot \bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 24x + 0 - 18yz$$

$$\therefore \quad \rho_v = 24x - 18yz$$

$|\rho_v|$ will be maximum when x is minimum and yz are maximum. i.e. x = + 1 and y = z= 2.

$$\therefore \quad |\rho_v|_{max} = |24\times(+1) - 18\times2\times2| = |+24-72| = 48\ C/m^3$$

c) ρ_v is maximum when x is maximum i.e. 2 and y, z are minimum i.e. y = z = 1. Thus ρ_v is maximum at P (2, 1, 1).

$$\rho_v \max = 24\times2 - 18\times1\times(+1) = 30\ C/m^3$$

Example 3.12 : *Determine the net flux of the vector field $\bar{D}(x, y, z) = 2x^2y\bar{a}_x + 2\bar{a}_y + y\bar{a}_z$ emerging from the unit cube $0 \le x, y, z \le 1$.*

[M.U. May-2002]

Solution : The x, y and z coordinates are all positive for the cube Hence the cube is as shown in the Fig. 3.30.

From the Gauss's law,

$$\psi = \oint_S \bar{D} \cdot d\bar{S}$$

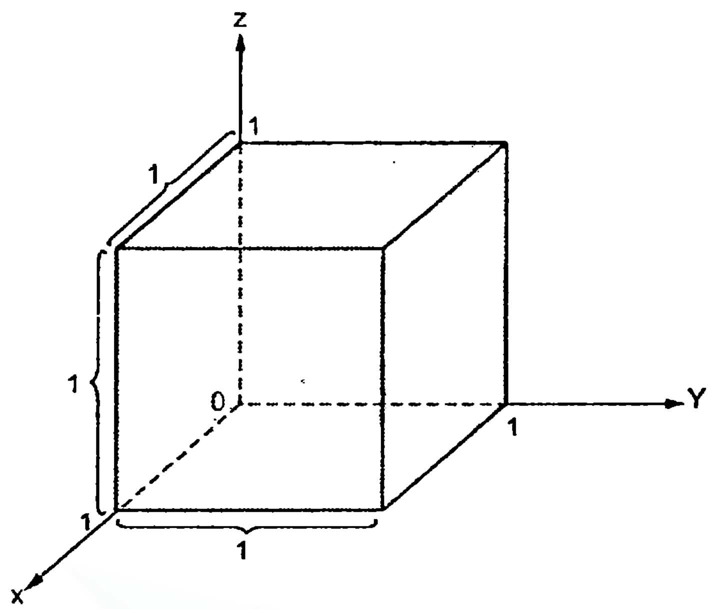


Fig. 3.30

The integral is to be evaluated on the six surfaces of the cube. Let us find $d\bar{S}$ for each surface.

1) Front surface (x = 1), dS = dy dz, direction = $\bar{a}_x$, $d\bar{S}$ = dy dz $\bar{a}_x$

2) Back surface (x = 0), dS = dy dz, direction = $-\bar{a}_x$, $d\bar{S}$ = – dy dz $\bar{a}_x$

3) Right surface (y = 1), dS = dx dz, direction = $\bar{a}_y$, $d\bar{S}$ = dx dz $\bar{a}_y$

4) Left surface (y = 0), dS = dx dz, direction = $-\bar{a}_y$, $d\bar{S}$ = – dx dz a_y

5) Top surface (z = 1), dS = dx dy, direction = $\bar{a}_z$, $d\bar{S}$ = dx dy $\bar{a}_z$

6) Bottom surface (z = 0), dS = dx dy, direction = $-\bar{a}_z$, $d\bar{S}$ = – dx dy a_z

The directions of $d\bar{S}$ are with respect to region enclosed.

Let us obtain $\bar{D} \bullet d\bar{S}$ for all the surfaces,

For front	$\bar{D} \bullet d\bar{S} = 2x^2y\,dy\,dz,$	$x = 1$	... $\bar{a}_x \bullet \bar{a}_x = 1$
For back	$\bar{D} \bullet d\bar{S} = -2x^2y\,dy\,dz,$	$x = 0$	... $\bar{a}_x \bullet \bar{a}_x = 1$
For right	$\bar{D} \bullet d\bar{S} = z\,dx\,dz,$	$y = 1$	... $\bar{a}_y \bullet \bar{a}_y = 1$
For left	$D \bullet dS = -z\,dx\,dz,$	$y = 0$	... $\bar{a}_y \bullet \bar{a}_y = 1$
For top	$\bar{D} \bullet d\bar{S} = y\,dx\,dy,$	$z = 1$	... $\bar{a}_z \bullet \bar{a}_z = 1$
For bottom	$\bar{D} \bullet d\bar{S} = -y\,dx\,dy,$	$z = 0$	... $\bar{a}_z \bullet \bar{a}_z = 1$

$$\therefore \quad \oint_S \bar{D} \bullet d\bar{S} = \underbrace{\int_{z=0}^{1}\int_{y=0}^{1} 2x^2y\,dy\,dz}_{\text{Front, } x=1} + \underbrace{\int_{z=0}^{1}\int_{y=0}^{1} -2x^2y\,dy\,dz}_{\text{Bottom, } x=0}$$

$$+\underbrace{\int_{z=0}^{1}\int_{x=0}^{1} z\,dx\,dz}_{\text{Right, } y=1}+\underbrace{\int_{z=0}^{1}\int_{x=0}^{1} -z\,dx\,dz}_{\text{Left, } y=-1}$$

$$+\underbrace{\int_{y=0}^{1}\int_{x=0}^{1} y\,dx\,dy}_{\text{Top, } z=1}+\underbrace{\int_{y=0}^{1}\int_{x=0}^{1} -y\,dx\,dy}_{\text{Bottom, } z=0}$$

$$= (2)(1)^2\left[\frac{y^2}{2}\right]_0^1 [z]_0^1+(0)+\left[\frac{z^2}{2}\right]_0^1 [x]_0^1+\left[\frac{-z^2}{2}\right]_0^1 [x]_0^1$$

$$+\left[\frac{y^2}{2}\right]_0^1 [x]_0^1+\left[\frac{-y^2}{2}\right]_0^1 [x]_0^1$$

$$= 1+0+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}=1\text{ C}$$

$\therefore \qquad \psi = 1\text{ C}$

The students can verify the result using divergence theorem by obtaining $\int_v (\nabla \cdot \bar{D})\,dv$.

Example 3.13 : *Prove that the divergence of the electric field and that of electric flux density in a charge free region is zero.*

Solution : From point form of Gauss's law we can write,

$$\nabla \cdot \bar{D} = \text{div } \bar{D} = \rho_v \qquad \text{... (1)}$$

While $\qquad \bar{D} = \varepsilon_0 \bar{E}$

$\therefore \qquad \nabla \cdot (\varepsilon_0 \bar{E}) = \rho_v$

The ε_0 is scalar and constant hence can be taken outside.

$\therefore \qquad \varepsilon_0 \nabla \cdot \bar{E} = \rho_v$

$$\therefore \qquad \nabla \cdot \bar{E} = \text{div } \bar{E} = \frac{\rho_v}{\varepsilon_0} \qquad \text{... (2)}$$

But in charge free region, Q = 0 hence there can not exist any charge density ρ_v.

$\therefore \qquad \rho_v = 0 \qquad$ ... For charge free region

Substituting in (1) and (2),

$\nabla \cdot \bar{D} = \nabla \cdot \bar{E} = 0 \qquad$ ... For charge free region

Thus divergence of $\bar{D}$ and $\bar{E}$ are zero in a charge free region.

Example 3.14 : *Show that the divergence of flux density due to point charge and uniform line charge is zero.*

Solution : Case I : Point charge

In spherical coordinates, the flux density $\bar{D}$ due to a point charge is given by,

$$\bar{D} = \frac{Q}{4\pi r^2}\,\bar{a}_r$$

Hence for r > 0, $D_r = \frac{Q}{4\pi r^2}$, $D_\theta = D_\phi = 0$

$$\therefore \quad \nabla \cdot \bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) + 0 + 0$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \times \frac{Q}{4\pi r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi}\right)$$

$$= 0 \qquad \ldots \frac{Q}{4\pi} \text{ is constant}$$

Hence the divergence of flux density due to point charge is zero.

Case II : Uniform line charge

For a uniform line charge, the flux density $\bar{D}$ in a cylindrical coordinate system is given by,

$$\bar{D} = \frac{\rho_L}{2\pi r}\,\bar{a}_r$$

Thus for r > 0, $D_r = \frac{\rho_L}{2\pi r}$ and $D_\phi = D_z = 0$

$$\therefore \quad \nabla \cdot \bar{D} = \frac{1}{r}\frac{\partial}{\partial r}(r D_r) + 0 + 0$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r \times \rho_L}{2\pi r}\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\rho_L}{2\pi}\right)$$

$$= 0 \qquad \ldots \frac{\rho_L}{2\pi} \text{ is constant}$$

Thus everywhere except at r = 0, the divergence of flux density due to uniform line charge is zero.

Key Point: *As $\bar{D} = \varepsilon \bar{E}$ and ε is a constant, the divergence of $\bar{E}$ due to point charge and uniform line charge is also zero everywhere except $r = 0$ where it is indeterminate.*

Example 3.15 : *A spherical volume charge density is given by,*

$$\rho_v = \rho_0\left(1 - \frac{r^2}{a^2}\right), \quad r \le a$$

$$= 0 \quad , \quad r > a$$

a) Calculate the total charge Q.

b) Find $\bar{E}$ outside the charge distribution.

c) Find $\bar{E}$ for $r < a$.

d) Show that the maximum value of $\bar{E}$ is at $r = 0.745\ a$. Obtain $|\bar{E}|$ max.

Solution : Note that the ρ_v is dependent on the variable r. Hence though the charge distribution is sphere of radius 'a' we can not obtain Q just by multiplying ρ_v by $\left(\frac{4}{3}\pi a^3\right)$ as ρ_v is not constant. As it depends on r, it is necessary to consider differential volume dv and integrating from r = 0 to a, total Q must be obtained. Thus **if ρ_v depends on r, do not use standard results.**

a) $$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi \qquad \text{... Spherical system}$$

$$\therefore \quad Q = \int_v \rho_v\, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \rho_0\left(1-\frac{r^2}{a^2}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho_0 [-\cos\theta]_0^{\pi}\ [\phi]_0^{2\pi} \int_{r=0}^{a}\left\{r^2 - \frac{r^4}{a^2}\right\} dr$$

$$= \rho_0 [-(-1)-(-1)][2\pi]\left[\frac{r^3}{3}-\frac{r^5}{5a^2}\right]_0^a$$

$$= \rho_0 \times 2 \times 2\pi \times \left[\frac{a^3}{3}-\frac{a^3}{5}\right]$$

$$= \rho_0 \times 4\pi \times \frac{2a^3}{15} = \frac{8\pi}{15}\rho_0 a^3 \text{ C}$$

Outside sphere, $\rho_v = 0$ so Q = 0 for r > a.

b) The total charge enclosed by the sphere can be assumed to be point charge placed at the centre of the sphere as per Gauss's law.

$$\therefore \quad \bar{D} = \frac{Q}{4\pi r^2}\,\bar{a}_r \quad \text{at} \quad r > a$$

$\therefore$ Outside the charge distribution i.e. r > a,

$$|\bar{E}| = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\frac{8\pi}{15}\rho_0 a^3}{4\pi\varepsilon_0 r^2} = \frac{2}{15}\,\frac{\rho_0 a^3}{\varepsilon_0}\,\frac{1}{r^2}$$

$$\therefore \quad \bar{E} = \frac{2}{15}\,\frac{\rho_0 a^3}{\varepsilon_0}\,\frac{1}{r^2}\,\bar{a}_r \text{ V/m}$$

Thus $\bar{E}$ varies with r, outside the charge distribution.

c) For r < a, consider a Gaussian surface as a sphere r having r < a as shown in the Fig. 3.31.

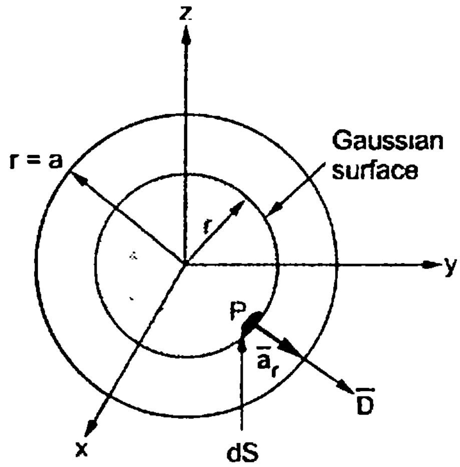


Fig. 3.31

Consider dS at point P normal to $\bar{a}_r$ direction, as $\bar{D}$ and $\bar{E}$ are in $\bar{a}_r$ direction.

$$d\bar{S} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$\bar{D} = D_r \, \bar{a}_r$$

$$\therefore \quad \bar{D} \cdot d\bar{S} = D_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$$\therefore \quad Q_1 = \oint_S \bar{D} \cdot d\bar{S}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, r^2 \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} = 4\pi r^2 \, D_r$$

where Q_1 = Charge enclosed by Gaussian surface

$$\therefore \quad D_r = \frac{Q_1}{4\pi r^2}$$

$$\therefore \quad \bar{D} = \frac{Q_1}{4\pi r^2} \, \bar{a}_r$$

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{Q_1}{4\pi\varepsilon_0 \, r^2} \, \bar{a}_r$$

Let us find Q_1, charge enclosed by Gaussian surface of radius r.

$$Q_1 = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} \rho_0 \left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} \left\{\frac{r^3}{3}-\frac{r^5}{5a^2}\right\}_0^r$$

$$= 4\pi\rho_0\left(\frac{r^3}{3}-\frac{r^5}{5a^2}\right) \text{C}$$

Using in the equation of $\bar{E}$, field intensity for r < a is,

$$\bar{E} = \frac{4\pi\rho_0\left(\frac{r^3}{3}-\frac{r^5}{5a^2}\right)}{4\pi\varepsilon_0 r^2}\bar{a}_r$$

$$= \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3}-\frac{r^3}{5a^2}\right]\bar{a}_r \quad \text{V/m}$$

d) To find $\bar{E}$ to be maximum, inside the sphere i.e. r < a obtain,

$$\frac{d|\bar{E}|}{dr} = 0$$

$$\therefore \quad \frac{d}{dr}\left\{\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{r^3}{5a^2}\right)\right\} = 0$$

$$\therefore \quad \frac{1}{3}-\frac{3r^2}{5a^2} = 0 \qquad \text{as } \rho_v \neq 0, \quad \varepsilon_0 \neq 0$$

$$\therefore \quad r^2 = \frac{5a^2}{9}$$

$$\therefore \quad r = 0.745\text{ a} \qquad \text{... Proved}$$

$$\therefore \quad |\bar{E}|_{max} = \frac{\rho_0}{\varepsilon_0}\left[\frac{0.745\,a}{3}-\frac{(0.745\,a)^3}{5a^2}\right]$$

$$= \frac{0.1656\,a\rho_0}{\varepsilon_0} \quad \text{V/m}$$

Example 3.16 : *If a sphere of radius 'a' has a charge density $\rho_v = k\,r^3$ then find $\bar{D}$ and $\nabla \bullet \bar{D}$ as a function of radius r and sketch the result. Assume k constant.*

Solution : Consider a sphere of radius 'a' as shown in the Fig. 3.32.

Case [1] Consider point P outside sphere such that r > a. The Gaussian surface passes through point P. Now $\bar{D}$ is directed along $\bar{a}_r$ direction hence $\bar{D} = D_r\bar{a}_r$.

$$dS = r^2 \sin\theta\, d\theta\, d\phi \qquad \text{... Spherical system}$$

$$\therefore \quad d\psi = \bar{D} \bullet d\bar{S} = D_r\,\bar{a}_r \bullet r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

$$= D_r\, r^2 \sin\theta\, d\theta\, d\phi$$

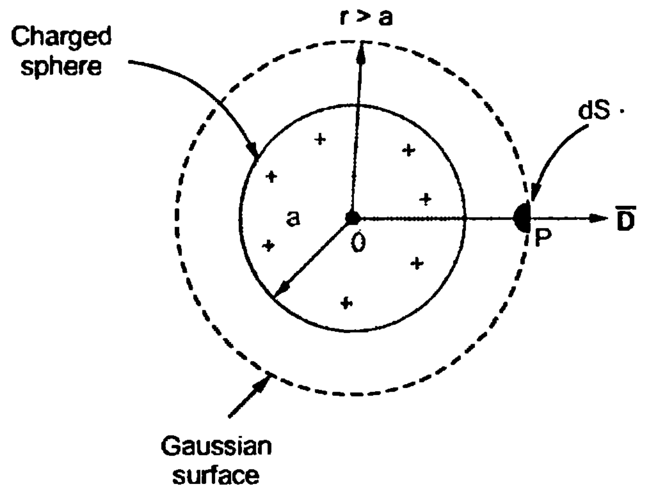


Fig. 3.32

$$\therefore \quad Q = \psi = \oint_S \bar{D} \bullet d\bar{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r\, r^2 \sin\theta\, d\theta\, d\phi = 4\pi r^2 D_r$$

$$\therefore \quad D_r = \frac{Q}{4\pi r^2} \text{ i.e. } \bar{D} = \frac{Q}{4\pi r^2}\, \bar{a}_r$$

Now total charge enclosed is, $Q = \int_v \rho_v\, dv$

$$\therefore \quad Q = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} k\, r^3\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{4\pi k a^6}{6}$$

$$\therefore \quad \bar{D} = D_r\, \bar{a}_r = \frac{k a^6}{6 r^2}\, \bar{a}_r \text{ C/m}^2 \qquad \text{... for } r > a$$

Case [2] Let point P is on the surface of sphere i.e. r = a

$$\therefore \quad \bar{D} = \frac{Q}{4\pi a^2}\, \bar{a}_r \quad \text{and } Q = \frac{4\pi k a^6}{6}$$

$$\therefore \quad \bar{D} = \frac{k a^4}{6}\, \bar{a}_r \text{ C/m}^2 \qquad \text{... for } r = a$$

Case [3] Let point P is inside sphere i.e. r < a. The Gaussian surface passes through point P as shown in the Fig. 3.33.

Again $d\bar{S}$ and $\bar{D}$ are directed radially outwards.

$$\therefore \quad \bar{D} \bullet d\bar{S} = D_r\, r^2 \sin\theta\, d\theta\, d\phi$$

$$\therefore \quad \psi = Q = \oint_S \bar{D} \bullet d\bar{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r\, r^2 \sin\theta\, d\theta\, d\phi = 4\pi r^2 D_r$$

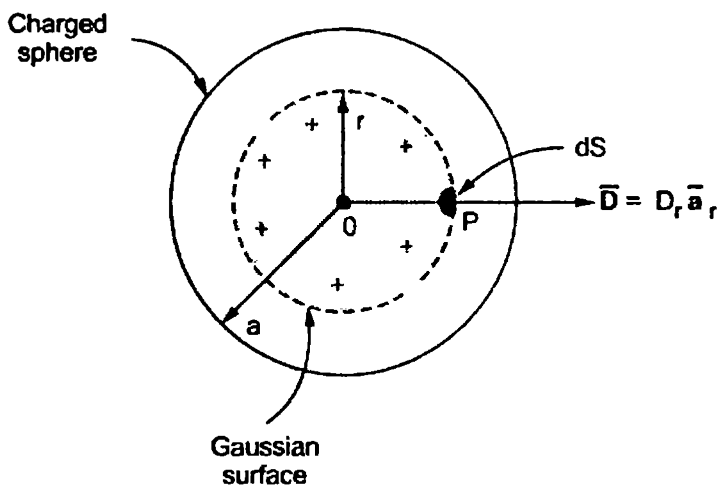


Fig. 3.33

$$\therefore \quad D_r = \frac{Q}{4\pi r^2} \text{ i.e. } \bar{D} = \frac{Q}{4\pi r^2} \bar{a}_r$$

Now charge enclosed by sphere of radius **r only** is to be considered and **not the entire sphere.**

$$\therefore \quad Q = \int_v \rho_v dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} k\, r^3\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{kr^6}{6} 4\pi$$

$$\therefore \quad \bar{D} = \frac{kr^6\, 4\pi}{6\; 4\pi r^2} \bar{a}_r = \frac{k\, r^4}{6} \bar{a}_r \text{ C/m}^2 \qquad \text{... for } 0 < r < a$$

The sketch of $\bar{D}$ against r is,

$$\nabla \bullet \bar{D} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 D_r) \text{ as } \bar{D} \text{ is only in } \bar{a}_r \text{ direction}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \times \frac{ka^6}{6r^2}\right] = 0 \qquad \text{... for } r > a$$

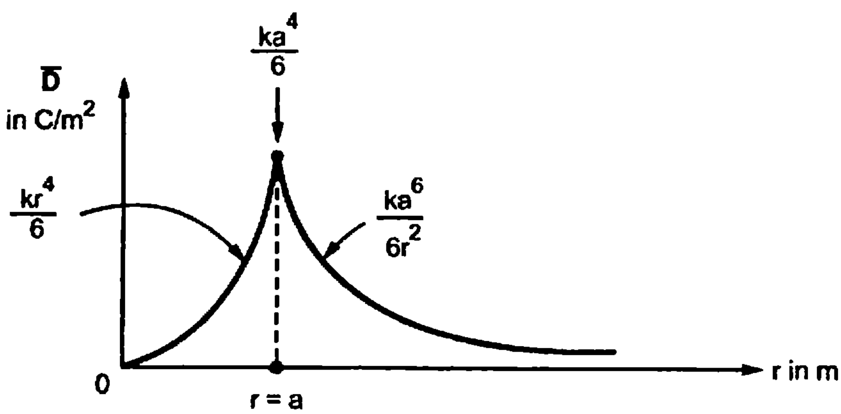


Fig. 3.34

$$\nabla \bullet \overline{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \times \frac{k r^4}{6}\right] = \frac{1}{r^2} \times \frac{k}{6} 6 r^5 = k\, r^3 \qquad \text{... for } r \le a$$

Key Point: *As $\nabla \bullet \overline{D} = \rho_v = k\, r^3$ as given, the results are correct.*

Example 3.17 : *A spherical volume charge density is given by,*

$$\rho_v = \rho_0\left(1 - \frac{r^2}{a^2}\right), \quad r \le a$$

$$= 0 \quad , \quad r > a$$

a) Calculate the total charge Q.

b) Find $\overline{E}$ outside the charge distribution.

c) Find $\overline{E}$ for $r < a$.

d) Show that the maximum value of $\overline{E}$ is at $r = 0.745\ a$. Obtain $|\overline{E}|$ max.

Solution : Note that the ρ_v is dependent on the variable r. Hence though the charge distribution is sphere of radius 'a' we can not obtain Q just by multiplying ρ_v by $\left(\frac{4}{3}\pi a^3\right)$ as ρ_v is not constant. As it depends on r, it is necessary to consider differential volume dv and integrating from r = 0 to a, total Q must be obtained. Thus **if ρ_v depends on r, do not use standard results.**

a) $$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi \qquad \text{... Spherical system}$$

$$\therefore \quad Q = \int_v \rho_v\, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \rho_0\left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho_0 \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} \int_{r=0}^{a} \left\{r^2 - \frac{r^4}{a^2}\right\} dr$$

$$= \rho_0\left[-(-1)-(-1)\right]\left[2\pi\right]\left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]_0^a$$

$$= \rho_0 \times 2 \times 2\pi \times \left[\frac{a^3}{3} - \frac{a^3}{5}\right]$$

$$= \rho_0 \times 4\pi \times \frac{2a^3}{15} = \frac{8\pi}{15}\rho_0\, a^3 \text{ C}$$

Outside sphere, $\rho_v = 0$ so Q = 0 for r > a.

b) The total charge enclosed by the sphere can be assumed to be point charge placed at the centre of the sphere as per Gauss's law.

$$\therefore \quad \overline{D} = \frac{Q}{4\pi r^2}\, \overline{a}_r \quad \text{at} \quad r > a$$

$\therefore$ Outside the charge distribution i.e. r > a,

$$|\bar{E}| = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\frac{8\pi}{15}\rho_0 a^3}{4\pi\varepsilon_0 r^2} = \frac{2}{15}\frac{\rho_0 a^3}{\varepsilon_0}\frac{1}{r^2}$$

$$\therefore \quad \bar{E} = \frac{2}{15}\frac{\rho_0 a^3}{\varepsilon_0}\frac{1}{r^2}\, a_r \text{ V/m}$$

Thus $\bar{E}$ varies with r, outside the charge distribution.

c) For r < a, consider a Gaussian surface as a sphere r having r < a as shown in the Fig. 3.35.

Consider dS at point P normal to $\bar{a}_r$ direction, as $\bar{D}$ and $\bar{E}$ are in $\bar{a}_r$ direction.

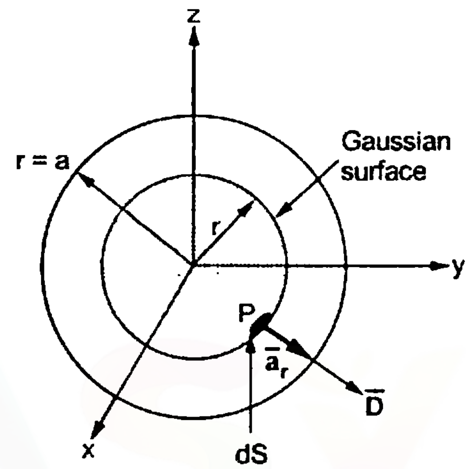


Fig. 3.35

$$d\bar{S} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

$$\bar{D} = D_r \bar{a}_r$$

$$\therefore \quad \bar{D}\cdot d\bar{S} = D_r\, r^2 \sin\theta\, d\theta\, d\phi$$

$$\therefore \quad Q_1 = \oint_S \bar{D}\cdot d\bar{S}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= D_r\, r^2 [-\cos\theta]_0^{\pi}\, [\phi]_0^{2\pi} = 4\pi r^2 D_r$$

where Q_1 = Charge enclosed by Gaussian surface

$$\therefore \quad D_r = \frac{Q_1}{4\pi r^2}$$

$$\therefore \quad \bar{D} = \frac{Q_1}{4\pi r^2}\bar{a}_r$$

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{Q_1}{4\pi\varepsilon_0 r^2}\bar{a}_r$$

Let us find Q_1, charge enclosed by Gaussian surface of radius r.

$$Q_1 = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r} \rho_0\left(1-\frac{r^2}{a^2}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho_0 [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} \left\{\frac{r^3}{3}-\frac{r^5}{5a^2}\right\}_0^r$$

$$= 4\pi\rho_0\left(\frac{r^3}{3}-\frac{r^5}{5a^2}\right) C$$

Using in the equation of $\bar{E}$, field intensity for r < a is,

$$E = \frac{4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right)}{4\pi\varepsilon_0 r^2}\bar{a}_r$$

$$= \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3} - \frac{r^3}{5a^2}\right]\bar{a}_r \quad \text{V/m}$$

d) To find $\bar{E}$ to be maximum, inside the sphere i.e. r < a obtain,

$$\frac{d|\bar{E}|}{dr} = 0$$

$$\therefore \frac{d}{dr}\left\{\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^3}{5a^2}\right)\right\} = 0$$

$$\therefore \frac{1}{3} - \frac{3r^2}{5a^2} = 0 \qquad \text{as } \rho_v \neq 0, \ \varepsilon_0 \neq 0$$

$$\therefore r^2 = \frac{5a^2}{9}$$

$$\therefore r = 0.745\ a$$ ... Proved

$$\therefore |\bar{E}|_{max} = \frac{\rho_0}{\varepsilon_0}\left[\frac{0.745\,a}{3} - \frac{(0.745\,a)^3}{5a^2}\right]$$

$$= \frac{0.1656\,a\rho_0}{\varepsilon_0} \quad \text{V/m}$$

Example 3.18 : *Three point charges are located in air : + 0.008 μC at (0, 0)m, + 0.005 C at (3, 0)m, and at (0, 4) m there is a a charge of – 0.009 μC. Compute total flux over a sphere of 5 m radius with centre (0, 0).* **(UPTU : 2005-06)**

Solution : The arrangement is shown in the Fig. 3.36.

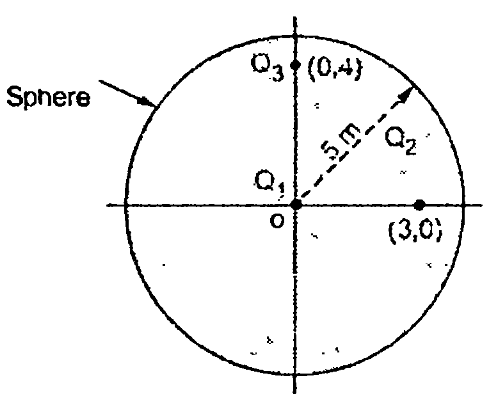


Fig. 3.36

The sphere encloses all the point charges.

$$\therefore Q_{encl} = Q_1 + Q_2 + Q_3$$

$$= 0.008 + 0.005 - 0.009$$

$$= 0.004\ \mu C$$

According to Gauss's law,

$$\psi = Q_{encl} = \mathbf{0.004\ \mu C}$$

This is the flux over a sphere.

Example 3.19 : *Find the divergence of the vector function*

$$\bar{A} = x^2\bar{a}_x + (xy)^2\bar{a}_y + 24(xyz)^2\bar{a}_z$$

Evaluate the volume integral of $\nabla \cdot \bar{A}$ *through the volume of a unit cube centered at the origin.* **(UPTU : 2008-09)**

Solution : $\bar{A} = x^2 a_x + (x^2y^2)\bar{a}_y + 24(x^2y^2z^2)\bar{a}_z$

$$Q = \oint_S \bar{A} \cdot d\bar{S}$$

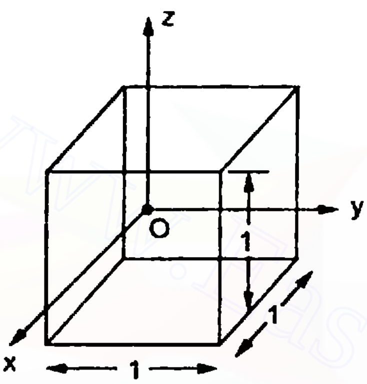


Fig. 3.37

The Fig. 3.37 shows unit cube centered at origin.

For $\bar{A} \cdot d\bar{S}$, consider all six faces of the cube.

Find $d\bar{S}$ for each surface.

1) Front (x = 0.5), $d\bar{S} = dy\, dz\, a_x$

2) Back (x = –0.5), $d\bar{S} = dy\, dz\, (-\bar{a}_x)$

3) Right (y = 0.5), $d\bar{S} = dx\, dz\, (a_y)$

4) Left (y = – 0.5), $d\bar{S} = dx\, dz\, (-\bar{a}_y)$

5) Top (z = 0.5), $d\bar{S} = dx\, dy\, \bar{a}_z$

6) Bottom (z = – 0.5), $d\bar{S} = dx\, dy\, (-\bar{a}_z)$

For front,	$\bar{A} \cdot d\bar{S} = x^2\, dy\, dz$	(x = 0.5)
For back,	$\bar{A} \cdot d\bar{S} = -x^2\, dx\, dz$	(x = – 0.5)
For right,	$\bar{A} \cdot d\bar{S} = x^2y^2\, dy\, dz$	(y = 0.5)
For left,	$\bar{A} \cdot d\bar{S} = -x^2y^2\, dx\, dz$	(y = – 0.5)
For top,	$\bar{A} \cdot d\bar{S} = 24\, x^2y^2z^2\, dx\, dz$	(z = 0.5)
For bottom ,	$\bar{A} \cdot d\bar{S} = -24\, x^2y^2z^2\, dx\, dy$	(z = – 0.5)

$$\therefore \quad \oint_S A \cdot d\bar{S} = \int_{z=-0.5}^{0.5}\int_{\substack{y=-0.5\\(x=0.5)}}^{0.5} x^2 dy\, dz + \int_{z=-0.5}^{0.5}\int_{\substack{y=-0.5\\(x=0.5)}}^{0.5} -x^2 dydz + \int_{z=-0.5}^{0.5}\int_{\substack{y=-0.5\\(y=0.5)}}^{0.5} x^2y^2 dxdz$$

$$+ \int_{z=-0.5}^{0.5}\int_{\substack{x=-0.5\\(y=-0.5)}}^{0.5} -x^2y^2 dxdz + \int_{y=-0.5}^{0.5}\int_{\substack{x=-0.5\\(z=0.5)}}^{0.5} 24x^2y^2z^2 dxdy + \int_{y=-0.5}^{0.5}\int_{\substack{x=-0.5\\(z=-0.5)}}^{0.5} -24x^2y^2z^2 dxdy$$

$$= (0.5)^2 [y]_{-0.5}^{0.5}[z]_{-0.5}^{0.5} - (-0.5)^2[y]_{-0.5}^{0.5}[z]_{-0.5}^{0.5} + (0.5)^2\left[\frac{x^3}{3}\right]_{-0.5}^{0.5}[z]_{-0.5}^{0.5}$$

$$-(-0.5)^2\left[\frac{x^3}{3}\right]_{-0.5}^{0.5}[z]_{-0.5}^{0.5} + 24(0.5)^2\left[\frac{x^3}{3}\right]_{-0.5}^{0.5}\left[\frac{y^3}{3}\right]_{-0.5}^{0.5}$$

$$-24(-0.5)^2\left[\frac{x^3}{3}\right]_{-0.5}^{0.5}\left[\frac{y}{3}\right]_{-0.5}^{0.5}$$

$$= 0$$

Using divergence theorem,

$$\nabla \bullet \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2x + 2x^2y + 48\,x^2y^2z$$

$$\int_v (\nabla \bullet \bar{A})\, dv = \int_{z=0.5}^{0.5}\int_{y=-0.5}^{0.5}\int_{x=-0.5}^{0.5} [2x + 2x^2y + 48x^2y^2z]\, dx\, dy\, dz$$

$$= \int_{z=-0.5}^{0.5}\int_{y=-0.5}^{0.5}\left[x^2 + \frac{2x^3y}{3} + \frac{48x^3y^2z}{3}\right]_{0.5}^{-0.5} dy\, dz$$

$$= \int_{z=-0.5}^{0.5}\int_{y=-0.5}^{0.5}[0.166\,y + 4\,y^2z]\, dy\, dz = \int_{z=-0.5}^{0.5}\left[0.083y^2 + \frac{4y^3}{3}z\right]_{-0.5}^{0.5} dz$$

$$= \int_{z=-0.5}^{0.5} 0.333\, z\, dz = [0.1666\, z^2]_{-0.5}^{0.5} = 0$$

Review Questions

1. *What is electric flux ? Explain the concept of electric flux density.*
2. *Derive the expression for $\bar{D}$ due to a point charge and hence deduce the relationship between $\bar{D}$ and $\bar{E}$.*
3. *State and prove the Gauss's law.*
4. *State the conditions to be satisfied by the special gaussian surfaces.*
5. *Derive the expression for $\bar{D}$ due to a point charge using Gauss's law.*
6. *Explain the Gauss's law applied to the case of infinite line charge and derive the expression for $\bar{D}$ due to the infinite line charge.*
7. *Consider a coaxial cable with inner radius a and outer radius b. Derive the expression for $\bar{D}$ for the region $a < r < b$ using Gauss's law.*

8. *Derive the expression for $\overline{D}$ due to the infinite sheet of charge placed in $z = 0$ plane, using Gauss's law.*
9. *Consider a spherical shell of charge carrying a surface charge density $\rho_S\ C/m^2$. Using Gauss's law derive the expression for $\overline{D}$ in all regions. Sketch the variation of $|\overline{D}|$ against radius r.*
10. *Using Gauss's law, derive $\overline{D}$ in all the regions for a uniformly charged sphere having volume charge density $\rho_v\ C/m^3$. Sketch the variation of $|\overline{D}|$ against the radius r.*
11. *Starting from the Gauss's law as applied to the differential volume element, explain the concept of divergence.*
12. *Define divergence and its physical meaning.*
13. *Derive Maxwell's first equation as applied to the electrostatics, using Gauss's law.*
14. *State the divergence theorem.*
15. *Find $\overline{D}$ at P (6,8, – 10) caused due to*

    a) A point charge of 30 mC at the origin.

    b) A uniform line charge $\rho_l = 40\ \mu C/m$ on the z axis.

    c) A uniform $\rho_S = 57.2\ \mu C/m^2$ on the plane x = 9.

    [Ans. : $5.063\,\bar{a}_x + 6.75\,\bar{a}_y - 8.43\,\bar{a}_z\ \mu C/m^2$, $0.382\,\bar{a}_x + 0.5093\,\bar{a}_y\ \mu C/m^2$, $-28.6\,\bar{a}_x\ \mu C/m^2$]
16. *Find $\overline{D}$ at (4, 0, 3) due to a point charge – 15.734 mC at (4, 0, 0) and a line charge 9.427 mC/m along the y axis.* **[Ans. : $240\,\bar{a}_x + 42\,\bar{a}_z\ \mu C/m^2$]**
17. *Given that $\overline{D} = zr\cos^2\phi\,\bar{a}_z\ C/m^2$, calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1m with $-2 \le z \le 2m$.* **[Ans. : $0.5\ C/m^3$, 4.189 C]**
18. *A spherical symmetrical charge distribution has,*

    $$\rho_v = \frac{\rho_0 r}{\alpha},\quad 0 \le r \le \alpha$$
    $$= 0, \quad r > \alpha$$

    Determine $\overline{D}$ and $\overline{E}$ everywhere. **[Ans. : $\frac{\rho_0 \alpha^3}{4r^2}\,\bar{a}_r$, $\frac{\rho_0 \alpha^3}{4\varepsilon_0 r^2}\,\bar{a}_r$]**
19. *If $\overline{D} = 20xy^2(z+1)\bar{a}_x + 20x^2y(z+1)\bar{a}_y + 10x^2y^2\bar{a}_z\ C/m^3$, calculate charge density at $P(0.3, 0.4, 0.5)$.* **[Ans. : $7.5\ C/m^3$]**
20. *If $\overline{D} = 2xy\,\bar{a}_x + 3yz\,\bar{a}_y + 4zx\,\bar{a}_z$, find the charge enclosed by $-1 \le x \le 2$, $0 \le z \le 4$ and y = 3 plane.* **[Ans. : 216 C]**
21. *If $\overline{D} = 10xy^2\,\bar{a}_x + 15x^2y\,\bar{a}_y + 20x^2y^2z\,\bar{a}_z$, find ρ_v at (1, 1, 1).* **[Ans. : $45\ C/m^3$]**
22. *If $\overline{D} = 5x^2y^2z^2\,\bar{a}_x + 2x^3y^2z\,\bar{a}_y + 6x^4y^3z^2\bar{a}_z$, find ρ_v at (2, 3, 5).* **[Ans. : $30.9\ kC/m^3$]**
23. *Given the flux density in free space $\overline{D} = \frac{r}{4}\,\bar{a}_r$, determine*

    i) Total flux leaving a sphere of r = 0.5 m.

    ii) Total charge enclosed in a sphere of r = 0.4 m.

    iii) Field intensity at r = 0.3 m. **[Ans. : 0.392 nC, 201.062 pC, 8.471 V/m]**

24. *Given that* $\bar{D} = \frac{5x^3}{2}\bar{a}_x C/m^2$, *evaluate both sides of the divergence theorem for the volume of a cube 1m on an edge, centered at the origin and with edges parallel to the axes*

[Ans. : 0.625 C]

25. *Evaluate* $\int \bar{D} \bullet d\bar{S}$ *if* $\bar{D} = yxz\bar{a}_x - y^2\bar{a}_y + yz\bar{a}_z$ *and S is the unit cube bounded by* $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. *Also verify the divergence theorem.* **[Ans. : – 1/4 C]**

University Questions

1. *What is Gauss's Law ? State and prove it.* **[UPTU : 2002-03, 5 Marks]**
2. *State Divergence Theroem and physically interpret the equivalence of the L.H.S. and the R.H.S. terms.* **[UPTU ; 2002-03, 5 Marks]**
3. *Discuss the Gauss's Law and its application.* **[UPTU : 2003-04(B), 5 Marks]**
4. *Relate Electric flux* ψ_e *and electric flux density* $\bar{D}$ *with Electric Field* $\bar{E}$. *Three point charges are located in air : + 0.008 μC at (0, 0)m, + 0.005 μC – 0.009 μC. Compute total flux over a sphere of 5 m radius with centre (0, 0).* **[UPTU : 2005-06, 4 Marks]**
5. *Find the divergence of the vector function*

   $\vec{A} = x^2\bar{a}_x + (xy)^2\bar{a}_y + 24(xyz)^2\bar{a}_z$

   Evaluate the volume integral of $\nabla \cdot \vec{A}$ *through the volume of a unit cube centered at the origin.* **[UPTU : 2008-09, 10 Marks]**

☐☐☐

4 Energy and Potential

4.1 Introduction

Uptill now, we have studied the Coulomb's law and electric field intensity due to various types of charge distributions. Similarly Gauss's law and its applications under various charge distributions are also discussed. In this chapter, another important aspect related to an electrostatic field is discussed, which is electric potential. The electric scalar potential can be conveniently used to obtain electric field intensity $\bar{E}$. This is another method of obtaining vector field $\bar{E}$, from the electric scalar potential. The other parameters such as potential difference, the relation between field intensity and the electric potential, potential gradient are also discussed in this chapter. Before defining an electric potential, let us study the work done in moving a charge in an electric field.

4.2 Work Done

The electric field intensity is defined as the force on a unit test charge at that point at which we want to find the value of $\bar{E}$. Consider an electric field due to a positive charge Q. If a unit test positive charge Q_t is placed at any point in this field, it experiences a repulsive force and tends to move in the direction of the force.

But if a positive test charge Q_t is to be moved towards the positive base charge Q then it is required to be moved against the electric field of the charge Q. i.e. against the repulsive force exerted by charge Q on the test charge Q_t. While doing so, an external source has to do work to move the test charge Q_t against the electric field. This movement of charge requires to expend the energy. This work done becomes the potential energy of the test charge Q_t, at the point at which it is moved.

Consider an earth's gravitational field. An object falls on the earth due to the force exerted by earth's gravitational field. But to move an object away from the earth's gravitational field, the work is required to be done by an external source. The force in opposite direction to that exerted by earth's gravitational field is required to be applied, to move an object against the earth's gravitational field. In such a case, work is said to be done.

Thus, work is said to be done when the test charge is moved against the electric field.

Consider a positive charge Q_1 and its electric field $\bar{E}$. If a positive test charge Q_t is placed in this field, it will move due to the force of repulsion. Let the movement of the charge Q_t is dl. The direction in which the movement has taken place is denoted by unit vector $\bar{a}_L$, in the direction of dl. This is shown in the Fig. 4.1.

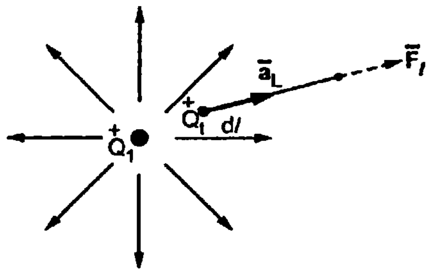


Fig. 4.1

According to Coulomb's law the force exerted by the field $\bar{E}$ is given by,

$$\bar{F} = Q_t \bar{E} \quad N \qquad \text{... (1)}$$

But the component of this force exerted by the field in the direction of dl, is responsible to move the charge Q_t, through the distance dl.

We know that the component of a vector in the direction of the unit vector is the dot product of the vector with that unit vector. Thus the component of $\bar{F}$ in the direction of unit vector $\bar{a}_L$ is given by,

$$\bar{F}_l = \bar{F} \cdot \bar{a}_L = Q_t \bar{E} \cdot \bar{a}_L \quad N \qquad \text{... (2)}$$

This is the force responsible to move the charge Q_t through the distance dl, in the direction of the field.

To keep the charge in equilibrium, it is necessary to apply the force which is equal and opposite to the force exerted by the field in the direction dl.

$$\therefore \quad \bar{F}_{applied} = -\bar{F}_l = Q_t \bar{E} \cdot \bar{a}_L \quad N \qquad \text{... (3)}$$

In this case, the work is said to be done.

Key Point: *Thus keeping the charge in equilibrium means we are moving a charge Q_t, through the distance dl in opposite direction to that of field $\bar{E}$. Hence the work is done.*

Thus there is expenditure of energy which is given by the product of force and the distance.

Hence mathematically the differential work done by an external source in moving the charge Q_t through a distance dl, against the direction of field $\bar{E}$ is given by,

$$dW = \bar{F}_{applied} \times dl = -Q_t \bar{E} \cdot \bar{a}_L \, dl \qquad \text{... (4)}$$

$$\text{But} \quad dl \, \bar{a}_L = d\bar{L} = \text{Distance vector} \qquad \text{... (5)}$$

$$\therefore \quad dW = -Q_t \bar{E} \cdot d\bar{L} \quad J \qquad \text{... (6)}$$

Key Point: *Note that dW is a scalar quantity as $\bar{E} \cdot d\bar{L}$ is the dot product which is a scalar quantity.*

Thus if a charge Q is moved from initial position to the final position, against the direction of electric field $\bar{E}$ then the total work done is obtained by integrating the differential work done over the distance from initial position to the final position.

$$\therefore \quad W = \int_{\text{Initial}}^{\text{Final}} dW = \int_{\text{Initial}}^{\text{Final}} -Q\bar{E} \cdot d\bar{L}$$

$$\therefore \quad \boxed{W = -Q \int_{\text{Initial}}^{\text{Final}} \bar{E} \cdot d\bar{L} \quad J} \quad \ldots (7)$$

The work done is measured in **joules**.

Key Point: *Note that at both the positions initial and final, the charge Q is at rest and not moving, then only the equation (7) is valid.*

4.3 The Line Integral

Consider that the charge is moved from initial position B to the final position A, against the electric field $\bar{E}$ then the work done is given by,

$$W = -Q \int_{B}^{A} \bar{E} \cdot d\bar{L}$$

This is called the **line integral,** where $\bar{E} \cdot d\bar{L}$ gives the component of $\bar{E}$ along the direction $d\bar{L}$.

Mathematical procedure involved in such a line integral, is,

1. Choose any arbitrary path B to A.
2. Break up the path into number of very small segments, which are called differential lengths.
3. Find the component of $\bar{E}$ along each segments.
4. Adding all such components and multiplying by charge, the required work done can be obtained.

Thus line integral is basically a summation and accurate result is obtained when the number of segments becomes infinite.

Let us see an important property of this line integral. Consider an uniform electric field $\bar{E}$. The charge is moved from B to A along the path shown in the Fig. 4.2.

The path B to A is divided into number of small segments.

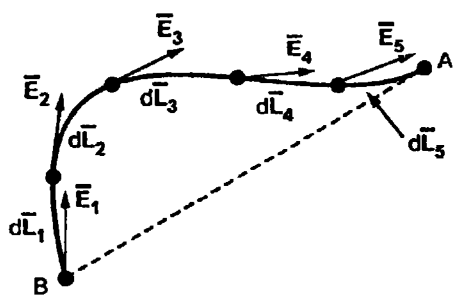


Fig. 4.2

The various distance vectors along the segments choosen are $d\bar{L}_1$, $d\bar{L}_2$, $d\bar{L}_3$, $d\bar{L}_4$ and $d\bar{L}_5$ while the electric field in these directions is $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$, $\bar{E}_4$ and $\bar{E}_5$. Hence the line integral from B to A can be expressed as the summation of dot products.

$$\therefore \quad W = -Q[\bar{E}_1 \cdot d\bar{L}_1 + \bar{E}_2 \cdot d\bar{L}_2 + \ldots + \bar{E}_5 \cdot d\bar{L}_5]$$

But the electric field is uniform and is equal in all directions.

$$\therefore \quad \bar{E}_1 = \bar{E}_2 = \bar{E}_3 = \bar{E}_4 = \bar{E}_5 = \bar{E}$$

$$\therefore \quad W = -Q\bar{E} \cdot [d\bar{L}_1 + d\bar{L}_2 + \ldots + d\bar{L}_5]$$

Now $d\bar{L}_1 + dL_2 + \ldots + d\bar{L}_5$ is the vector addition. So according to method of polygon the sum of all such vectors is the vector joining initial point to final point when all vectors are arranged one after the other in respective directions. This is shown in the Fig. 4.3. Hence the sum of all such vectors is the vector $\bar{L}_{BA}$ joining initial point to final point.

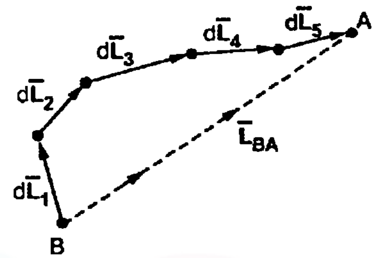


Fig. 4.3

$$\therefore \quad W = -QE \cdot \bar{L}_{BA} \qquad \ldots \text{(Uniform } \bar{E}\text{)}$$

Thus it can be seen that vector sum of small segments choosen along any path, a curve or a straight line remains same as $\bar{L}_{BA}$ and it depends on the initial and final point only.

Key Point: *Hence the work done depends on Q, $\bar{E}$ and $\bar{L}_{BA}$ and does not depend on the path joining B to A. This is true for nonuniform electric field $\bar{E}$ as well.*

Thus, the work done in moving a charge from one location B to another A, in a static, uniform or nonuniform electric field $\bar{E}$ is independent of the path selected. The line integral of $\bar{E}$ is determined completely by the endpoints B and A of the path and not the actual path selected.

Key Point: *This is called conservative property of electric field $\bar{E}$ and field $\bar{E}$ is said to be conservative.*

While solving the problems, it is necessary to select $d\bar{L}$ according to the conditions and co-ordinate system selected. The expressions for $d\bar{L}$ in three co-ordinate systems are given here again for the convenience of the readers.

Differential length vector $d\bar{L}$
Cartesian $\Rightarrow d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$
Cylindrical $\Rightarrow d\bar{L} = dr\,\bar{a}_r + r d\phi\,\bar{a}_\phi + dz\,\bar{a}_z$
Spherical $\Rightarrow d\bar{L} = dr\,\bar{a}_r + r d\theta\,a_\theta + r\sin\theta d\phi\,\bar{a}_\phi$

Table 4.1

4.3.1 Important Comments about Work Done

The work done in moving a point charge in an electric field $\bar{E}$ from position B to A is given by,

$$Q = -Q\int_{B}^{A} \bar{E} \cdot d\bar{L}$$

1. When the movement of the charge Q is **against the direction of E**, then the **work done is positive**, which indicates external source has done the work.

2. When the movement of the charge Q is **in the direction of** $\bar{E}$, then the **work done is negative**, which indicates field itself has done the work, no external source is required.

3. The work done is independent of the path selected from B to A but it depends on end points B and A.

4. When the **path selected** is such that it is **always perpendicular** to $\bar{E}$ i.e. the force on the charge is always exerted at right angles to the direction in which charge is moving, then the **work done is zero**. This indicates θ, the angle between $\bar{E}$ and $d\bar{L}$ is 90°. Due to the dot product, the line integral is zero when $\theta = 90°$.

5. If the path selected is such that it is forming a **closed contour** i.e. starting point is same as the terminating point then the **work done is zero**.

Example 4.1 : *An electrostatic field is given by,*

$$\bar{E} = -8xy\,\bar{a}_x - 4x^2\,\bar{a}_y + \bar{a}_z \;\; V/m$$

The charge of 6 C is to be moved from B (1, 8, 5) to A (2, 18, 6). Find the work done in each of the following cases.

1. *The path selected is $y = 3x^2 + z,\; z = x + 4$*
2. *The straight line from B to A.*

Show that work done remains same and is independent of the path selected.

Solution : The work done is given by,

$$W = -Q\int_{B}^{A} \bar{E} \cdot d\bar{L}$$

Let us differential length $d\bar{L}$ in cartesian co-ordinate system is,

$$d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$\therefore \quad \bar{E} \cdot d\bar{L} = (-8xy\,\bar{a}_x - 4x^2\,\bar{a}_y + \bar{a}_z) \cdot (dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)$$

$$= -8xy\,dx - 4x^2\,dy + dz$$

As $\bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_z = 1$, other dot products are zero.

$$\therefore \quad W = -Q\int_{B}^{A} -8\,xy\,dx - 4x^2 dy + dz$$

$$= -Q\left[\int_{B}^{A} -8\,xy\,dx - \int_{B}^{A} 4\,x^2 dy + \int_{B}^{A} dz\right]$$

Case 1 : The path is $y = 3x^2 + z,\ z = x+4$

$$\therefore \quad y = 3x^2 + x + 4 \quad \text{differentiate}$$

$$\therefore \quad dy = (6x+1)\,dx$$

For $\int_{B}^{A} -8\,xy\,dx \rightarrow$ The limits are x = 1 to x = 2.

For $\int_{B}^{A} -4x^2 y \rightarrow$ The limits are y = 8 to y = 18

For $\int_{B}^{A} dz \rightarrow$ The limits are z = 5 to z = 6.

$$\therefore \quad W = -Q\left[\int_{x=1}^{2} -8\,xy\,dx - \int_{y=8}^{18} 4\,x^2 dy + \int_{z=5}^{6} dz\right]$$

Using $y = 3x^2 + x + 4$ and $dy = (6x + 1)\,dx$ and changing limits of y from 8 to 18 interms of x from 1 to 2 we get

$$\therefore \quad W = -Q\left[\int_{x=1}^{2} -8x\,[3x^2 + x + 4]\,dx - \int_{x=1}^{2} 4\,x^2[6x+1]\,dx + \int_{z=5}^{6} dz\right]$$

$$= -Q\left[\int_{x=1}^{2} [-24x^3 - 8x^2 - 32x]\,dx - \int_{x=1}^{2} (24\,x^3 + 4x^2)\,dx + \int_{z=5}^{6} dz\right]$$

$$= -Q\left[\left(-6x^4 - \frac{8}{3}x^3 - 16x^2 - 6x^4 - \frac{4}{3}x^3\right)_{x=1}^{2} + (z)_5^6\right]$$

$$= -Q\{-256 + 1\} = -6 \times -255 = 1530\ \text{J}$$

Case 2 : Straight line path from B to A.

To obtain the equations of the straight line, any two of the following three equations of planes passing through the line are sufficient,

B (1, 8, 5) and A (2, 18, 6)

$$(y - y_B) = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$

$$(z - z_B) = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$

$$(x - x_B) = \frac{x_A - x_B}{z_A - z_B}(z - z_B)$$

Using the co-ordinates of A and B,

$$y - 8 = \frac{18-8}{2-1}(x-1)$$

$$\therefore \quad y - 8 = 10\,(x-1)$$

$$\therefore \quad y = 10x - 2 \qquad \text{... (1)}$$

$$\therefore \quad dy = 10\,dx$$

And
$$z - 5 = \frac{6-5}{18-8}(y-8)$$

$$\therefore \quad z - 5 = \frac{1}{10}(y-8)$$

$$\therefore \quad 10\,z = y + 42 \qquad \text{... (2)}$$

Now
$$W = -Q\left[\int_{x=1}^{2} -8\,xy\,dx - \int_{y=8}^{18} 4\,x^2 dy + \int_{z=5}^{6} dz\right]$$

$$= -Q\left[\int_{x=1}^{2} -8\,x(10x-2)\,dx - \int_{x=1}^{2} 4\,x^2(10dx) + \int_{z=5}^{6} dz\right]$$

$$= -Q\left\{\left[\frac{-80}{3}x^3 + \frac{16x^2}{2} - \frac{40x^3}{3}\right]_{x=1}^{2} + [z]_5^6\right\}$$

$$= -Q\{-213.33 + 32 - 106.667 + 26.667 - 8 + 13.33 + 1\}$$

$$= -Q[-255] = -6 \times -255 = 1530\ J$$

This shows that **irrespective of path selected, the work done in moving a charge from B to A remains same.**

Example 4.2 : *Consider an infinite line charge along z-axis. Show that the work done is zero if a point charge Q is moving in a circular path of radius r_1, centered at the line charge.*

Solution : The line charge along the z-axis and the circular path along which charge is moving is shown in the Fig. 4.4.

The circular path is in xy plane such that its radius is r_1 and centered at the line charge.

Consider cylindrical co-ordinate system where line charge is along z-axis.

The charge is moving in $\bar{a}_\phi$ direction.

$$\therefore \quad d\bar{L} = r\,d\phi\,\bar{a}_\phi$$

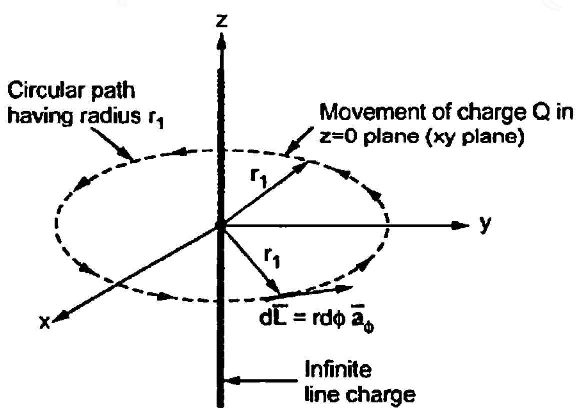


Fig. 4.4

The field $\bar{E}$ due to infinite line charge along z-axis is given in cylindrical co-ordinates as,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \qquad \text{... (Refer Chapter 2)}$$

The circular path indicates that $d\bar{L}$ has no component in $\bar{a}_r$ and $\bar{a}_z$ direction.

$$\therefore \quad d\bar{L} = r\, d\phi\, \bar{a}_\phi$$

$$\therefore \quad W = -Q\int_{\text{initial}}^{\text{final}} \bar{E}\cdot d\bar{L} = -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \cdot r\, d\phi\, \bar{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\varepsilon_0} d\phi\, (\bar{a}_r \cdot \bar{a}_\phi) = 0$$

As $\bar{a}_r \cdot \bar{a}_\phi = 0$ as $\theta = 90°$ between $\bar{a}_r$ and $\bar{a}_\phi$.

This shows that the work done is zero while moving a charge such that path is always perpendicular to the $\bar{E}$ direction.

➡ **Example 4.3 :** *Consider an infinite line charge with density ρ_L C/m, along z-axis. Obtain the work done if a point charge Q is moved from r = a to r = b along a radial path.*

Solution : The line charge and the path of the movement of the point charge Q is shown in the Fig. 4.5.

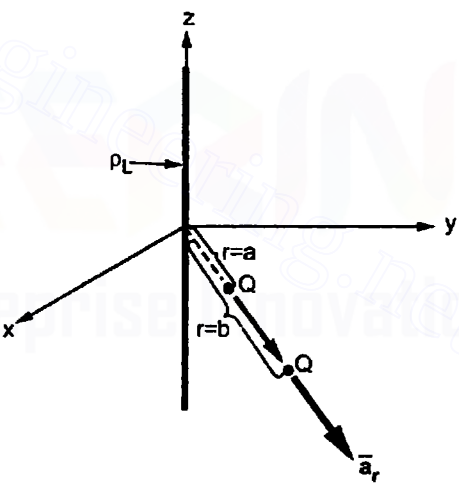


Fig. 4.5

The movement of the point charge Q is along $\bar{a}_r$ direction and hence $d\bar{L}$ has no component in $\bar{a}_\phi$ and $\bar{a}_z$ direction.

$$\therefore \quad d\bar{L} = dr\, \bar{a}_r$$

... In cylindrical system

The field $\bar{E}$ due to infinite line charge along z-axis is given by,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

$$\therefore \quad W = -Q\int_{r=a}^{r=b} \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \cdot dr\, \bar{a}_r \qquad \text{... } (\bar{a}_r \cdot \bar{a}_r = 1)$$

$$\therefore \quad W = -Q\int_{r=a}^{b} \frac{\rho_L}{2\pi\varepsilon_0}\frac{1}{r}\, dr = \frac{-Q\rho_L}{2\pi\varepsilon_0}[\ln r]_a^b$$

$$\therefore \quad \boxed{W = \frac{-Q\rho_L}{2\pi\varepsilon_0}[ln\,b - ln\,a] = \frac{-Q\rho_L}{2\pi\varepsilon_0} ln\frac{b}{a} \quad J}$$

As b > a, *ln* (b/a) is positive and work done is negative. This indicates that the field is doing the work and external source is receiving energy.

4.4 Potential Difference

In the last sections it has been discussed that the work done in moving a point charge Q from point B to A in the electric field $\bar{E}$ is given by,

$$W = -Q\int_B^A \bar{E}\cdot d\bar{L} \qquad \text{... (1)}$$

If the charge Q is selected as **unit test charge** then from the above equation we get the work done in moving unit charge from B to A in the field $\bar{E}$. This **work done in moving unit charge** from point B to A in the field $\bar{E}$ is called **potential difference** between the points B and A. It is denoted by V.

$$\therefore \text{ Potential difference} = V = -\int_B^A \bar{E}\cdot d\bar{L} \qquad \text{... (2)}$$

Thus work done per unit charge in moving unit charge from B to A in the field $\bar{E}$ is called potential difference between the points B and A.

Notation : If B is the initial point and A is the final point then the potential difference is denoted as V_{AB} which indicates the potential difference between the points A and B and unit charge is moved from B to A.

$$\therefore \quad \boxed{V_{AB} = -\int_B^A \bar{E}\cdot d\bar{L}} \qquad \text{... (3)}$$

Key Point: *V_{AB} is positive if the work is done by the external source in moving the unit charge from B to A, against the direction of $\bar{E}$.*

4.4.1 Unit of Potential Difference

The potential difference is work done per unit charge. The work done is measured in joules while the charge in coulombs. Hence unit of potential difference is joules/coulombs (J/C). But practically the unit is called volt (V).

One volt potential difference is one joule of work done in moving unit charge from one point to other in the field $\bar{E}$.

$$\therefore \quad 1 \text{ volt} = \frac{1 \text{ joule}}{1 \text{ coulomb}} \qquad \text{... (4)}$$

4.5 Potential due to Point Charge

Consider a point charge, located at the origin of a spherical co-ordinate system, producing $\bar{E}$ radially in all the directions as shown in the Fig. 4.6.

Assuming free space, the field $\bar{E}$ due to a point charge Q at a point having radial distance r from origin is given by,

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r \qquad \text{... (1)}$$

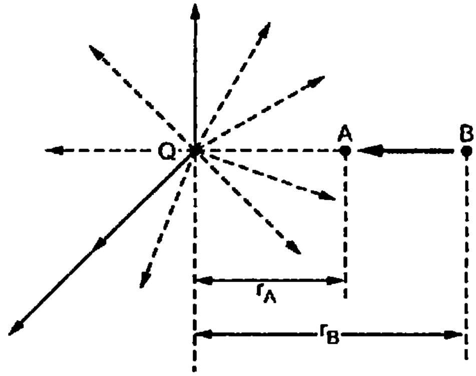


Fig. 4.6 Potential due to a point charge Q

Consider a unit charge which is placed at a point B which is at a radial distance of r_B from the origin. It is moved against the direction of E from point B to point A. The point A is at a radial distance of r_A from the origin.

The differential length in spherical system is,

$$dL = dr\,\bar{a}_r + r d\theta\, a_\theta + r\sin\theta\, d\phi\,\bar{a}_\phi \qquad \text{... (2)}$$

Hence the potential difference V_{AB} between points A and B is given by,

$$V_{AB} = -\int_B^A \bar{E}\cdot d\bar{L} \qquad \text{But } B \Rightarrow r_B \quad \text{and} \quad A \Rightarrow r_A$$

$$\therefore \quad V_{AB} = -\int_{r_B}^{r_A}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r\right)\cdot(dr\,\bar{a}_r + r d\theta\,\bar{a}_\theta + r\sin\theta d\phi\,\bar{a}_\phi)$$

$$\therefore \quad V_{AB} = -\int_{r_B}^{r_A}\frac{Q}{4\pi\varepsilon_0 r^2}dr \qquad \text{... (3)}$$

$$\therefore \quad V_{AB} = -\frac{Q}{4\pi\varepsilon_0}\int_{r_B}^{r_A} r^{-2}dr = \frac{-Q}{4\pi\varepsilon_0}\left[\frac{r^{-1}}{-1}\right]_{r_B}^{r_A}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_B}^{r_A} = \frac{-Q}{4\pi\varepsilon_0}\left[-\frac{1}{r_A}-\left(-\frac{1}{r_B}\right)\right]$$

$$\therefore \quad \boxed{V_{AB} = -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r_A}+\frac{1}{r_B}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A}-\frac{1}{r_B}\right] \text{V}} \qquad \text{... (4)}$$

When $r_B > r_A$, $\frac{1}{r_B} < \frac{1}{r_A}$ and V_{AB} is positive. This indicates the work is done by external source in moving unit charge from B to A.

4.5.1 Concept of Absolute Potential

Instead of potential difference, it is more convenient to express absolute potentials of various points in the field. Such absolute potentials are measured with respect to a specified **reference position**. Such a reference position is assumed to be at zero potential.

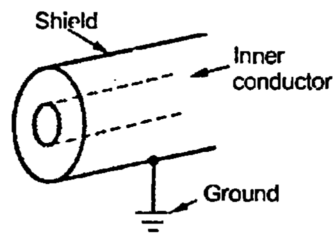


Fig. 4.7 Shielded cable

For practical circuits, zero reference point is selected as **'ground'**. All the potentials of the circuit are measured with respect to ground which is treated to be zero potential reference. The example of such a reference is a shielded cable. The conductor is shielded as shown in the Fig. 4.7. The outer shield is grounded to minimise the interference of the external signals with the signal carried by inner conductor. In such a case outer shield is taken as a reference with respect to which, potentials of various points are defined.

Similalrly in cathode ray tube, the metallic shield around the tube is taken as zero potential reference. With respect to this reference, potentials of various points are measured.

> **Key Point:** *Most widely used reference which is used to develope the concept of absolute potential is infinity. The potential at infinity is treated to be zero and all the potentials at various points in the field are defined with reference to infinity.*

Consider potential difference V_{AB} due to movement of unit charge from B to A in a field of a point charge Q. It is given by equation (4).

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A}-\frac{1}{r_B}\right]$$

Now let the charge is moved from infinity to the point A i.e. $r_B = \infty$. Hence $\frac{1}{r_B} = \frac{1}{\infty} = 0$.

$$\therefore \quad V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A}-\frac{1}{\infty}\right] = \frac{Q}{4\pi\varepsilon_0 r_A}\ \text{V} \qquad \text{... (5)}$$

The quantity represented by equation (5) is called **potential of point A** denoted as V_A.

$$\therefore \quad \boxed{V_A = \frac{Q}{4\pi\varepsilon_0 r_A}\ \text{V}} \qquad \text{... (6)}$$

This is also called **absolute potential** of point A.

Similarly **absolute potential of point B** can be defined as,

$$\therefore \quad \boxed{V_B = \frac{Q}{4\pi\varepsilon_0 r_B}\ \text{V}} \qquad \text{... (7)}$$

This is work done in moving unit charge from infinity at point B.

Hence the potential difference can be expressed as the difference between the absolute potentials of the two points.

$$\therefore \quad \boxed{V_{AB} = V_A - V_B \quad V} \quad \quad ... (8)$$

Thus absolute potential can be defined as,

The absolute potential at any point in an electric field is defined as the work done in moving a unit test charge from the infinity (or reference point at which potential is zero) to the point, against the direction of the field.

Hence absolute potential at any point which is at a distance r from the origin of a spherical system, where point charge Q is located, is given by,

$$\boxed{V = \frac{Q}{4\pi\varepsilon_0 r}} \quad \quad ... (9)$$

The reference point is at infinity.

Key Point: *Note that the potential is a scalar quantity.*

Other way to define potential mathematically is,

$$V = -\int_{\infty}^{r} \bar{E} \cdot d\bar{L} \quad V \quad \quad ... (10)$$

where ∞ is selected as the reference.

Thus,
$$V_A = -\int_{\infty}^{A} \bar{E} \cdot d\bar{L} \quad V \quad \quad ... (11)$$

This is potential of point A with reference at infinity.

4.5.2 Potential due to Point Charge not at Origin

If the point charge Q is not located at the origin of a spherical system then obtain the position vector r′ of the point where Q is located.

Then the absolute potential at a point A located at a distance r from the origin is given by,

$$V(r) = V_A = \frac{Q}{4\pi\varepsilon_0 |r - r'|}$$

$$= \frac{Q}{4\pi\varepsilon_0 R_A} \quad \quad ... (12)$$

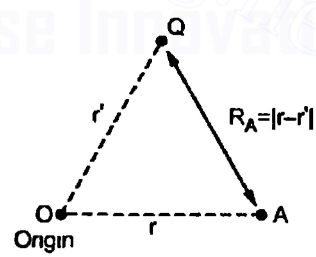


Fig. 4.8

where $R_A = |r - r'|$ = Distance between point at which potential is to be calculated and the location of the charge

Key Point: *R is only the distance and not the vector. The potential is a scalar quantity hence only distance $R = |r - r'|$ is involved in the determination of potential of point A. The reference is still infinity.*

4.5.3 Potential due to Several Point Charges

Consider the various point charges $Q_1, Q_2 \ldots Q_n$ located at the distances $r_1, r_2 \ldots r_n$ from the origin as shown in the Fig. 4.9. The potential due to all these point charges, at point A is to be determined. Use superposition principle.

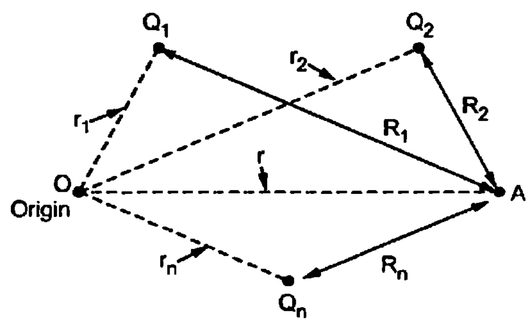


Fig. 4.9 Potential due to several point charges

Consider the point charge Q_1.

The potential V_{A1} due to Q_1 is given by,

$$V_{A1} = \frac{Q}{4\pi\varepsilon_0 |r - r_1|} = \frac{Q}{4\pi\varepsilon_0 R_1} \text{ V}$$

where $R_1 = |r - r_1|$ = Distance between point A and position of Q_1

The potential V_{A2} due to Q_2 is given by,

$$V_{A2} = \frac{Q_2}{4\pi\varepsilon_0 |r - r_2|} = \frac{Q_2}{4\pi\varepsilon_0 R_2} \text{ V}$$

Thus potential V_{An} due to Q_n is given by,

$$V_{An} = \frac{Q_n}{4\pi\varepsilon_0 |r - r_n|} = \frac{Q_n}{4\pi\varepsilon_0 R_n} \text{ V}$$

As the potential is scalar, the net potential at point A is the algebraic sum of the potentials at A due to individual point charges, considered one at a time.

$$\therefore \quad V(r) = V_A = V_{A1} + V_{A2} + \ldots\ldots + V_n$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + \ldots\ldots + \frac{Q_n}{4\pi\varepsilon_0 R_n}$$

$$\therefore \quad \boxed{V_A = V(r) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 |r - r_m|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 R_m} \text{ V}} \quad \ldots (13)$$

Key Point: *Note that the reference point of zero potential is still assumed to be at infinity.*

➠ **Example 4.4 :** *A point charge Q = 0.4 nC is located at the origin. Obtain the absolute potential of A (2, 2, 3).*

Solution : The potential of A due to point charge Q at the origin is given by,

$$V_A = \frac{Q}{4\pi\varepsilon_0 r_A} \quad \text{and} \quad A\ (2, 2, 3),\ Q \text{ at } (0, 0, 0)$$

where $r_A = \sqrt{(2-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{17}$

$$\therefore \quad V_A = \frac{0.4 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times \sqrt{17}} = 0.8719 \text{ V}$$ ... The reference is at infinity.

Example 4.5 : *If same charge Q = 0.4 nC in above example is located at (2, 3, 3) then obtain the absolute potential of point A (2, 2, 3).*

Solution : Now the Q is located at (2, 3, 3).

The potential at A is given by,

$$V_A = \frac{Q}{4\pi\varepsilon_0 R_A} \quad \text{where}$$

$$R_A = |\mathbf{r}-\mathbf{r}'|$$

$$= \sqrt{(2-2)^2+(2-3)^2+(3-3)^2} = 1$$

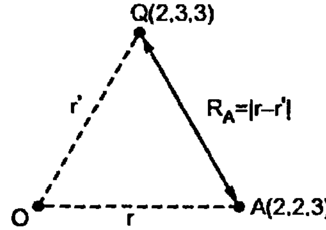


Fig. 4.10

... By distance formula

$$\therefore \quad V_A = \frac{0.4\times10^{-9}}{4\pi\times8.854\times10^{-12}\times1} = 3.595 \text{ V}$$

Example 4.6 : *If the point B is at (– 2, 3, 3) in the above example, obtain the potential difference between the points A and B.*

Solution : $V_{AB} = V_A - V_B$

where V_A and V_B are the absolute potentials of A and B.

Now $V_A = 3.595$ V ... As calculated earlier.

$$V_B = \frac{Q}{4\pi\varepsilon_0 R_B}$$ where R_B is distance between point B and Q (2, 3, 3)

$$\therefore \quad R_B = \sqrt{(-2-2)^2+(3-3)^2+(3-3)^2} = 4$$

$$\therefore \quad V_B = \frac{0.4\times10^{-9}}{4\pi\times8.854\times10^{-12}\times4} = 0.8987 \text{ V}$$

$$\therefore \quad V_{AB} = V_A - V_B = 3.595 - 0.8987 = \mathbf{2.6962\ V}$$

Example 4.7 : *If three charges, 3 μC, 4 μC and 5 μC are located at (0, 0, 0), (2 – 1, 3) and (0, 4, –2) respectively. Find the potential at (1, 0, 1) assuming zero potential at infinity.*

Solution : Let $Q_1 = 3\mu C$, $Q_2 = -4\mu C$

and $Q_3 = 5\mu C$

The potential of A due to Q_1 is,

$$V_{A1} = \frac{Q_1}{4\pi\varepsilon_0 R_1}$$

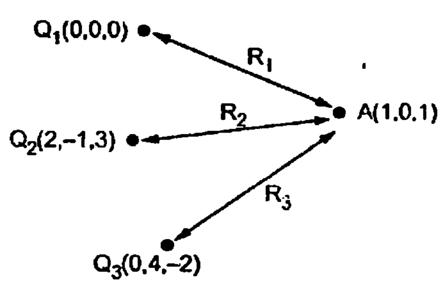


Fig. 4.11

and $$R_1 = \sqrt{(1-0)^2+(0-0)^2+(1-0)^2} = \sqrt{2}$$

$$\therefore \quad V_{A1} = \frac{3\times10^{-6}}{4\pi\varepsilon_0\times\sqrt{2}}$$

$$= 19.0658 \text{ kV}$$

The potential of A due to Q_2 is,

$$V_{A2} = \frac{Q_2}{4\pi\varepsilon_0 R_2}$$

and $$R_2 = \sqrt{(1-2)^2+[0-(-1)]^2+(1-3)^2} = \sqrt{6}$$

$$\therefore \quad V_{A2} = \frac{-4\times10^{-6}}{4\pi\varepsilon_0\times\sqrt{6}} = -14.6769 \text{ kV}$$

The potential of A due to Q_3 is,

$$V_{A3} = \frac{Q_3}{4\pi\varepsilon_0 R_3}$$

and $$R_3 = \sqrt{(1-0)^2+(0-4)^2+[1-(-2)]^2} = \sqrt{26}$$

$$\therefore \quad V_{A3} = \frac{5\times10^{-6}}{4\pi\varepsilon_0\times\sqrt{26}} = 8.8132 \text{ kV}$$

$$\therefore \quad V_A = V_{A1}+V_{A2}+V_{A3} = +13.2021 \text{ kV}$$

4.5.4 Potential Calculation When Reference is other than Infinity

The expressions derived uptill now are under the assumption that the reference position of zero potential is at infinity.

If any other point than infinity is selected as the reference then the potential at a point A due to point charge Q at the origin becomes,

$$\boxed{V_A = \frac{Q}{4\pi\varepsilon_0 R_A} + C}$$

where C = Constant to be determined at choosen reference point where V = 0.

Note that the potential difference between the two points is not the function of C.

Key Point: ***Another important note is that if the potential difference is to be calculated then reference is not needed. The reference is important only when the absolute potential is to be calculated.***

The idea will be clear from the following example.

Example 4.8 : *A point charge of 6 nC is located at origin in free space, find potential of point P if P is located at (0.2, – 0.4, 0.4) and*
a) V = 0 at infinity
b) V = 0 at (1, 0, 0)
c) V = 20 V at (– 0.5, 1, – 1).

Solution : a) The reference is at infinity, hence

$$V_P = \frac{Q}{4\pi\varepsilon_0 R_P}$$

$$R_P = \sqrt{(0.2-0)^2+(-0.4-0)^2+(0.4-0)^2} = 0.6$$

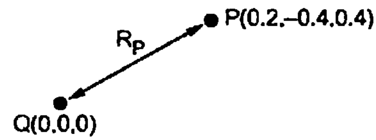


Fig. 4.12

$$\therefore V_P = \frac{6\times10^{-9}}{4\pi\times8.854\times10^{-12}\times0.6} = 89.8774 \text{ V}$$

b) V = 0 at (1, 0, 0). Thus the reference is not at infinity. In such a case potential at P is,

$$V_P = \frac{Q}{4\pi\varepsilon_0 R_P}+C$$

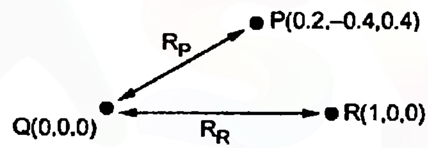


Fig. 4.13

Now V_R at (1, 0, 0) is zero.

$$\therefore \quad V_R = \frac{Q}{4\pi\varepsilon_0 R_R}+C = 0$$

and $$R_R = \sqrt{(1-0)^2+(0)^2+(0)^2} = 1$$

$$\therefore \quad 0 = \frac{6\times10^{-9}}{4\pi\times8.854\times10^{-12}\times1}+C$$

$$\therefore \quad C = -53.9264$$

$$\therefore \quad V_P = \frac{Q}{4\pi\varepsilon_0 R_P}+C = 89.8774-53.9264 = 35.9509 \text{ V}$$

This is with reference to (1, 0, 0) where V = 0 V.

c) Now V = 20 V at (– 0.5, 1, –1). Let this point is M (– 0.5, 1, – 1). The reference is not given as infinity.

$$V_M = \frac{Q}{4\pi\varepsilon_0 R_M}+C$$

and V_M = 20 V

while $$R_M = \sqrt{(-0.5)^2+(1)^2+(-1)^2} = 1.5$$

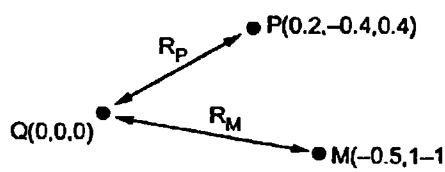


Fig. 4.14

$$\therefore \quad 20 = \frac{6\times10^{-9}}{4\pi\times8.854\times10^{-12}\times1.5}+C$$

$$\therefore \quad C = -15.9509$$

$$\therefore \quad V_P = \frac{Q}{4\pi\varepsilon_0 R_P}+C = 89.8774-15.9509$$

$\therefore \quad V_P = 73.9264 \text{ V}$

Key Point: *Note that distance of P from origin where Q is located is R_P which is same in all the cases. Only 'C' changes as the reference changes hence V_P changes.*

4.6 Potential due to a Line Charge

Consider a line charge having density ρ_L C/m, as shown in the Fig. 4.15.

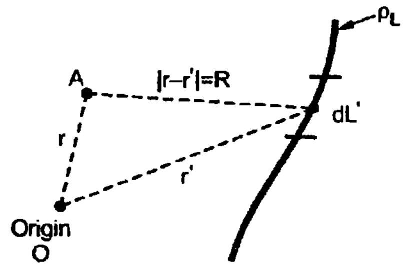


Fig. 4.15

Consider differential length dL′ at a distance r′. Then the differential charge on the length dL′ is given by,

$$dQ = \rho_L(r')\,dL' \qquad \text{... (1)}$$

where $\quad \rho_L(r') =$ Line charge density at r′

Let the potential at A is to be determined. Then,

$$dV_A = \frac{dQ}{4\pi\varepsilon_0 |r-r'|} = \frac{dQ}{4\pi\varepsilon_0 R} \qquad \text{... (2)}$$

The $R = |r - r'|$ indicates the distance of point A from the differential charge.

The dV_A is a differential potential at A. Hence the potential V_A can be obtained by integrating dV_A over the length over which line charge is distributed.

$$\therefore \quad V_A = V(r) = \int_{Line} \frac{dQ}{4\pi\varepsilon_0 R} \quad \text{and using (1),}$$

$$\therefore \quad \boxed{V_A = V(r) = \int_{Line} \frac{\rho_L(r')\,dL'}{4\pi\varepsilon_0 R} \text{ V}} \qquad \text{... (3)}$$

Key Point: *Note that R is the distance and not the vector and for uniform line charge density $\rho_L(r') = \rho_L$.*

Example 4.9 : *Find the potential V on z-axis at a distance z from origin when uniform line charge ρ_L in the form of a ring of radius a is placed in the z = 0 plane.*

Solution : The arrangement is shown in the Fig. 4.16.

The point A (0, 0, z) is on z-axis, at a distance z from the origin while radius of the ring is a.

Consider differential length dL′ at point P on the ring. The ring is in z = 0 plane hence dL′ in cylindrical system is,

$$dL' = r'd\phi = a\,d\phi$$

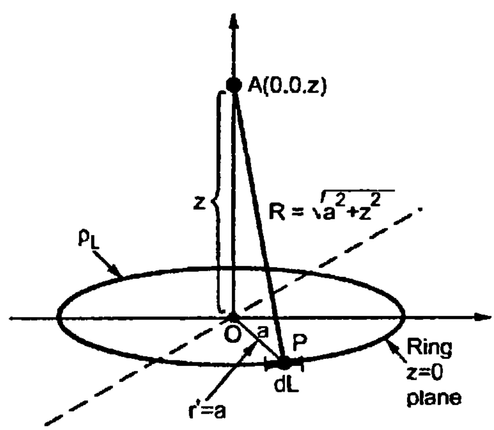


Fig. 4.16

The distance of point A from the differential charge is R = l (PA).

$$\therefore \quad R = \sqrt{a^2+z^2} \qquad \text{... From the Fig. 4.16}$$

$$\text{The charge} \quad dQ = \rho_L(r')dL' = \rho_L\, a\, d\phi \qquad \text{... } \rho_L(r') = \rho_L$$

$$\therefore \quad dV_A = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_L\, a\, d\phi}{4\pi\varepsilon_0\sqrt{a^2+z^2}}$$

Hence the potential of A is to be obtained by integrating dV_A over the circular ring i.e. path with radius $r' = a$ and ϕ varies from 0 to 2π.

$$\therefore \quad V_A = \int_{\phi=0}^{2\pi} \frac{\rho_L\, a\, d\phi}{4\pi\varepsilon_0\sqrt{a^2+z^2}} = \frac{\rho_L\, a}{4\pi\varepsilon_0\sqrt{a^2+z^2}}[\phi]_0^{2\pi}$$

$$= \frac{\rho_L a}{2\varepsilon_0\sqrt{a^2+z^2}} \quad V$$

Important :

Key Point: *Note that the potential at a point can be obtained by two ways.*

1. If $\bar{E}$ is known then use,

$$V = -\int \bar{E}\cdot d\bar{L} + C \quad V$$

C = 0 if reference is infinity.

2. If $\bar{E}$ is not known, then find differential charge dQ considering differential length dL′ and

$$V = \int \frac{dQ}{4\pi\varepsilon_0 R} + C \quad V$$

The integration depends on the charge distribution.

Example 4.10 : *A uniform line charge density ρ_L C/m is existing from – L to + L on y-axis. Find potential at A (a, 0, 0).*

Solution : The arrangement is shown in the xy plane as in the Fig. 4.17.

As $\bar{E}$ is not known in standard form, consider differential length dL′ at a point P, at a distance y from origin, on the charge,

$$\therefore \quad dL' = dy$$

$$\therefore \quad dQ = \rho_L dL' = \rho_L dy$$

The distance of point A from the differential charge is,

$$R_A = \sqrt{a^2+y^2}$$

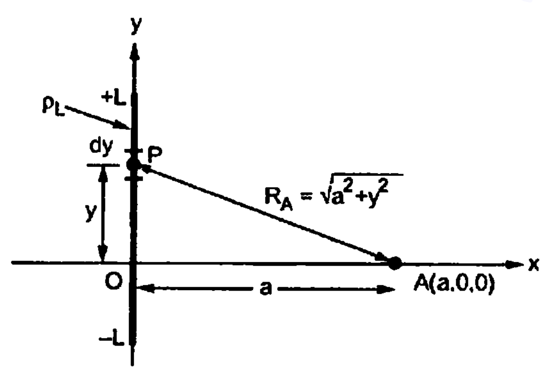


Fig. 4.17

$$\therefore \quad dV_A = \frac{dQ}{4\pi\varepsilon_0 R_A} = \frac{\rho_L dy}{4\pi\varepsilon_0\sqrt{a^2+y^2}}$$

Now integrate over entire length – L to + L.

$$\therefore \quad V_A = \int_{y=-L}^{+L} \frac{\rho_L}{4\pi\varepsilon_0}\frac{dy}{\sqrt{a^2+y^2}}$$

$$= 2\int_{y=0}^{L} \frac{\rho_L}{4\pi\varepsilon_0}\frac{dy}{\sqrt{a^2+y^2}} \quad \text{... Changing limits}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = ln[x+\sqrt{x^2+a^2}] \quad \text{... Standard result}$$

$$\therefore \quad V_A = \frac{2\rho_L}{4\pi\varepsilon_0}\Big[ln[y+\sqrt{y^2+a^2}]\Big]_{y=0}^{y=L}$$

$$= \frac{\rho_L}{2\pi\varepsilon_0}\Big[ln(L+\sqrt{L^2+a^2})-ln(\sqrt{a^2})\Big]$$

$$\therefore \quad \boxed{V_A = \frac{\rho_L}{2\pi\varepsilon_0}ln\left[\frac{L+\sqrt{L^2+a^2}}{a}\right] \text{ V}}$$

4.7 Potential due to Surface Charge

Consider uniform surface charge density ρ_S C/m^2 on a surface, as shown in the Fig. 4.18.

Consider the differential surface area dS′ at point P where ρ_S is indicated as $\rho_S(r')$

The differential charge can be expressed as,

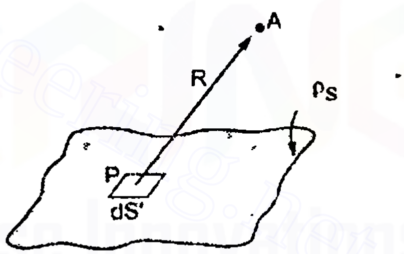


Fig. 4.18 Potential due to surface charge

$$dQ = \rho_S(r')\,dS' \quad \text{... (1)}$$

$$\therefore \quad dV_A = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_S(r')\,dS'}{4\pi\varepsilon_0 R} \quad \text{... (2)}$$

where R = Distance of point A from the differential charge

The total potential at A can be obtained by integrating dV_A over the given surface.

$$\therefore \quad \boxed{V_A = \int_S \frac{\rho_S(r')\,dS'}{4\pi\varepsilon_0 R} \text{ V}} \quad \text{... (3)}$$

Note that for uniform surface charge density $\rho_S(r') = \rho_S$.

4.8 Potential due to Volume Charge

Consider a uniform volume charge density ρ_v C/m^3 over the given volume as shown in the Fig. 4.19.

Consider the differential volume dv′ at point P where the charge density is $\rho_v(r')$.

The differential charge can be expressed as,

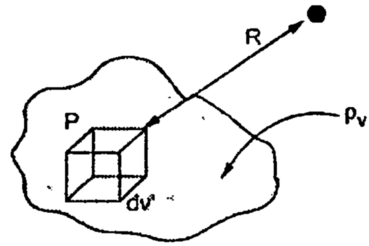


Fig. 4.19 Potential due to volume charge

$$dQ = \rho_v(r')\,dv' \quad \dots (1)$$

$$\therefore \quad dV_A = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_v(r')\,dv'}{4\pi\varepsilon_0 R} \quad \dots (2)$$

where R = Distance of point A from the differential charge

The total potential at A can be obtained by integrating dV_A over the given volume.

$$\therefore \quad \boxed{V_A = \int_v \frac{\rho_v(r')\,dv'}{4\pi\varepsilon_0 R}\ V} \quad \dots (3)$$

Note that for uniform volume charge density $\rho_v(r') = \rho_v$.

Potential difference	$V_{AB} = -\int_B^A \bar{E} \bullet d\bar{L}$
Absolute potential due to point charge	$V_{AB} = \frac{Q}{4\pi\varepsilon_0 R}\ V$
Absolute potential due to line charge	$V_{AB} = \int_L \frac{\rho_l\, dL'}{4\pi\varepsilon_0 R}\ V$
Absolute potential due to surface charge	$V_A = \int_S \frac{\rho_s\, dS'}{4\pi\varepsilon_0 R}\ V$
Absolute potential due to volume charge	$V_A = \int_v \frac{\rho_v\, dv'}{4\pi\varepsilon_0 R}\ V$
If the reference is other than infinity	$V_A = \frac{Q}{4\pi\varepsilon_0 R} + C\ V$
In all the expressions, R is the distance of point A from the charge Q or differential charge dQ.	

Table 4.2

Example 4.11 : *A total charge of 10^{-8} C is distributed uniformly along a ring of radius 5 m. Calculate the potential on the axis of the ring at a point 5 m from the centre of the ring. If the same charge is uniformly distributed on a disc of 5 m radius, what will be the potential on its axis at 5 m from the centre ?* **(UPTU : 2005-06, 5 Marks)**

Solution : The charge is distributed along a ring so it is a line charge. Let r′ = radius of ring = 5 m.

$$\rho_l = \frac{\text{Total charge}}{\text{Circumference}} = \frac{10^{-8}}{2\pi r'}$$

$$= \frac{10^{-8}}{10\pi} = 3.183 \times 10^{-10} \text{ C/m}$$

The ring is shown in the Fig. 4.20. Consider the differential length dL′ on the ring at point P.

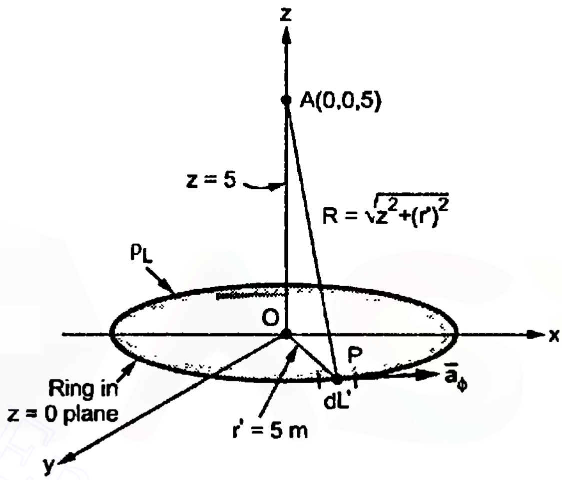


Fig. 4.20

$$dQ = \rho_l \, dL'$$

$$\text{But } dL' = r' d\phi = 5 \, d\phi$$

$$\therefore \quad \overline{dL'} = 5 \, d\phi \, \bar{a}_\phi$$

$$\therefore \quad dQ = 3.183 \times 10^{-10} \times 5 \, d\phi$$

$$dV_A = \frac{dQ}{4\pi\varepsilon_0 R}$$

where R = distance between A and P = $\sqrt{z^2 + (r')^2} = \sqrt{50}$

$$\therefore \quad dV_A = \frac{3.183 \times 10^{-10} \times 5 \, d\phi}{4\pi\varepsilon_0 \times \sqrt{50}} = 2.0228 \, d\phi$$

$$\therefore \quad V_A = \int_{\phi=0}^{2\pi} 2.0228 \, d\phi = 2.0228 \, [\phi]_0^{2\pi} = 2\pi \times 2.0228 = 12.7101 \text{ V}$$

Now the same charge is distributed over a disc of r′ = 5 m

$$\therefore \quad \rho_S = \frac{\text{Total charge}}{\text{Area}} = \frac{10^{-8}}{\pi (r')^2} = \frac{10^{-8}}{\pi \times 25} = 1.2732 \times 10^{-10} \text{ C/m}^2$$

Let the disc is placed in x-y plane as shown in the Fig. 4.21 with z-axis as its axis.

Consider differential surface dS′ at point P having radial distance r′ from the origin.

$$dS' = r' \, dr' \, d\phi$$

$$dQ = \rho_S \, dS' = \rho_S \, r' \, dr' \, d\phi$$

$$dV_A = \frac{dQ}{4\pi\varepsilon_0 R}$$

$$= \frac{\rho_S r' dr' d\phi}{4\pi\varepsilon_0 \sqrt{(r')^2 + z^2}}$$

...R = Distance AP

$$\therefore V_A = \int dV_A = \int_{\phi=0}^{2\pi}\int_{r'=0}^{5} \frac{\rho_S r' dr' d\phi}{4\pi\varepsilon_0\sqrt{(r')^2+25}}$$

...z = 5 m

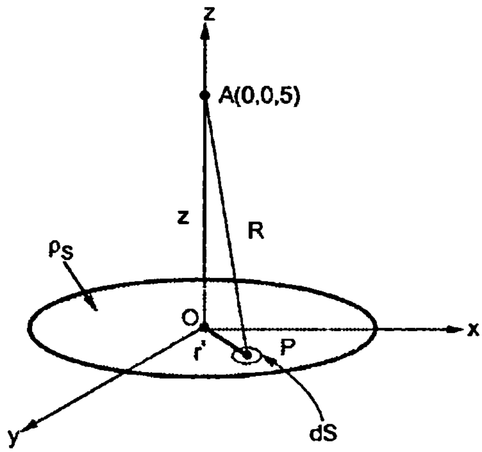


Fig. 4.21

Put $(r')^2 + 25 = u^2$ i.e. $2r'\,dr' = 2u\,du$

For $r' = 0$, $u_1 = 5$ and $r' = 5$, $u_2 = \sqrt{50}$

$$\therefore V_A = \int_{\phi=0}^{2\pi}\int_{u_1}^{u_2} \frac{\rho_S u\,du\,d\phi}{4\pi\varepsilon_0 u} = \frac{\rho_S}{4\pi\varepsilon_0}[u]_{u_1}^{u_2}\,[\phi]_0^{2\pi}$$

$$= \frac{1.2732\times10^{-10}}{4\pi\times8.854\times10^{-12}}\times[\sqrt{50}-5]\,[2\pi-0] = \mathbf{14.8909\ V}$$

4.9 Potential Difference due to Infinite Line Charge

Consider an infinite line charge along z-axis having uniform line charge density ρ_L C/m.

The point B is at a radial distance r_B while point A is at a radial distance r_A from the charge, as shown in the Fig. 4.22.

The $\bar{E}$ due to infinite line charge along z-axis is known and given by,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

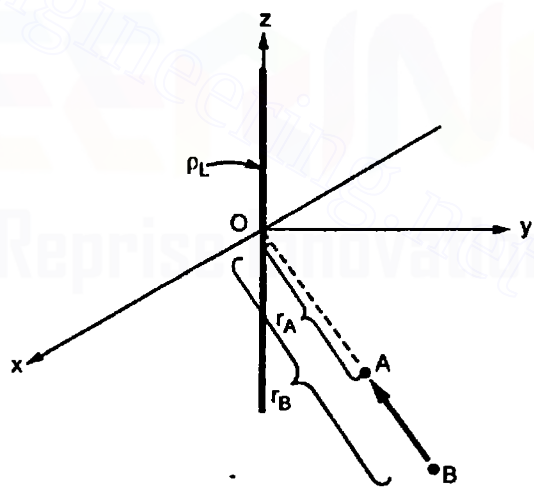


Fig. 4.22

while $d\bar{L} = dr\,\bar{a}_r$

in cylindrical system in radial direction.

$$\therefore \quad V_{AB} = -\int_B^A \bar{E}\cdot d\bar{L} = -\int_{r_B}^{r_A} \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \cdot dr\,\bar{a}_r$$

$$= -\int_{r_B}^{r_A} \frac{\rho_L}{2\pi\varepsilon_0 r} dr = \frac{-\rho_L}{2\pi\varepsilon_0} \int_{r_B}^{r_A} \frac{1}{r} dr = \frac{-\rho_L}{2\pi\varepsilon_0} [ln\ r]_{r_B}^{r_A} = \frac{-\rho_L}{2\pi\varepsilon_0} [ln\ r_A - ln\ r_B]$$

$$\therefore \quad \boxed{V_{AB} = \frac{\rho_L}{2\pi\varepsilon_0} ln \frac{r_B}{r_A}\ V}$$

Important note : This is a standard result and may be used to find potential difference between the points due to infinite line charge. Remember that r_A and r_B are radial distances in cylinderical co-ordinate system i.e. perpendicular distances from charge, thus do not forget to find perpendicular distances r_A and r_B while using this result. **The result can be used for any zero reference as potential difference calculation does not depend on the reference.**

Example 4.12 : *A line y = 1, z = 1 carries a uniform charge of 2 nC/m, find potential at A (5,0,1) if*
i) V = 0 V at 0 (0, 0, 0) ii) V = 100 V at B (1, 2, 1).

Solution : i) The line is shown in the Fig. 4.23, which is parallel to x-axis.

As reference is not at infinity, to find potential at A means potential of A with respect to origin 0.

$$\therefore \quad V_{A0} = \frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{r_0}{r_A}\right]$$

where r_A = Perpendicular distance of A from line

$$= \sqrt{(1-0)^2 + (1-1)^2} = 1\ m$$

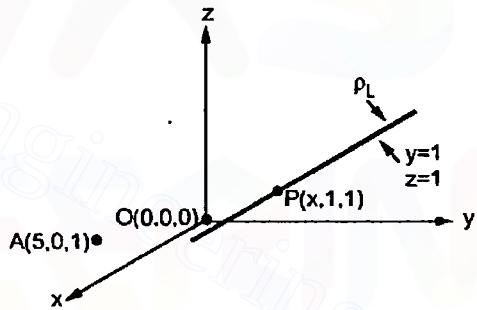


Fig. 4.23

As line is parallel to x-axis, x co-ordinate is not considered.

and $\quad r_0 = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}\ m$

$$\therefore \quad V_{A0} = \frac{2\times 10^{-9}}{2\pi \times 8.854 \times 10^{-12}} ln \frac{\sqrt{2}}{1} = +\ 12.4596\ V$$

$$V_{A0} = V_A - V_0$$

$$\therefore \quad 12.4596 = V_A - 0 \quad \text{... as } V_0 = 0\ V$$

$$\therefore \quad \mathbf{V_A = 12.4596\ V} \quad \text{... This is potential of A}$$

Note : This gives potential difference hence reference is not required which is already being taken care of.

Similarly method avoids the calculation of constant C as reference is other than infinity.

Alternatively : Consider infinite line charge hence,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \qquad \text{and} \qquad d\bar{L} = dr\,\bar{a}_r$$

$$\therefore \quad V = -\int \bar{E} \cdot d\bar{L} = -\int \frac{\rho_L}{2\pi\varepsilon_0 r} dr = -\frac{\rho_L}{2\pi\varepsilon_0} ln[r]$$

As reference is other than infinity,

$$V_A = -\frac{\rho_L}{2\pi\varepsilon_0} ln[r_A] + C$$

To calculate C, use $V_0 = 0$ V at 0 (0, 0, 0).

$$\therefore \quad V_0 = -\frac{\rho_L}{2\pi\varepsilon_0} ln\,[r_0] + C$$

and $\quad r_0 = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$ ... x is not considered.

$$\therefore \quad 0 = -\frac{\rho_L}{2\pi\varepsilon_0} ln[\sqrt{2}] + C$$

$$\therefore \quad C = 12.4596$$

$$\therefore \quad V_A = -\frac{\rho_L}{2\pi\varepsilon_0} ln\,[r_A] + 12.4596$$

and $\quad r_A = \sqrt{(1-0)^2 + (1-1)^2} = 1$

$$\therefore \quad V_A = -\frac{\rho_L}{2\pi\varepsilon_0} ln[1] + 12.4596 = 12.4596 \text{ V}$$

This is potential of A with reference to 0.

The answer remains same.

ii) Now V = 100 V at B (1, 2, 1)

$$\therefore \quad V_{AB} = -\frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{r_B}{r_A}\right]$$

where $\quad r_B = \sqrt{(1-2)^2 + (1-1)^2} = 1$ ... x is not considered

$$\therefore \quad V_{AB} = -\frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{1}{1}\right] = 0 \text{ V}$$

Now $\quad V_{AB} = V_A - V_B$

$$\therefore \quad 0 = V_A - 100$$

$\therefore \quad V_A = 100$ V ... This is potential of A

Alternatively : Using absolute potentials at points,

$$V_A = -\frac{\rho_L}{2\pi\varepsilon_0} ln\,[r_A]+C$$

At B,
$$V_B = -\frac{\rho_L}{2\pi\varepsilon_0} ln\,[r_B]+C$$

$$\therefore \quad 100 = -\frac{\rho_L}{2\pi\varepsilon_0} ln[1]+C \qquad \text{... as } r_B = 1$$

$$\therefore \quad C = 100$$

$$\therefore \quad V_A = -\frac{\rho_L}{2\pi\varepsilon_0} ln[1]+100 = 100\text{ V}$$

This is absolute potential of A, with reference to zero potential point which is same for B as well. So $V_B = 100$ V with respect to same zero potential point hence $V_{AB} = 0$ V.

Example 4.13 : *Two uniform line charges, 8 nC/m are located at x = 1, z = 2 and at x = – 1, y = 2 in free space. If the potential at origin is 100 V, find V at P (4, 1, 3).*

Solution : The two line charges are shown in the Fig. 4.24.

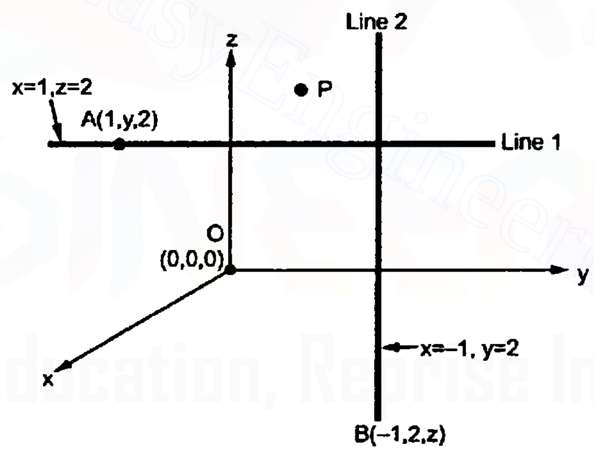


Fig. 4.24

Now V = 100 V at the origin O (0, 0, 0).

Let us obtain potential difference V_{PO} using standard result.

Case 1 : Line charge 1

$$\therefore \quad V_{PO1} = +\frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{r_{O1}}{r_{P1}}\right]$$

where r_{O1} and r_{P1} are perpendicular distances of points O and P from the line 1. The line 1 is parallel to y-axis so do not use y co-ordinates to find r_{O1} and r_{P1}.

$$\therefore \quad r_{O1} = \sqrt{(1-0)^2+(2-0)^2} = \sqrt{5}$$

$$\therefore \quad r_{P1} = \sqrt{(1-4)^2+(2-3)^2} = \sqrt{10}$$

$$\therefore \qquad V_{PO1} = +\frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{\sqrt{5}}{\sqrt{10}}\right] = -49.8386$$

$$\text{But} \qquad V_{PO1} = V_{P1} - V_O \qquad \text{where} \qquad V_O = 100 \text{ V}$$

$$\therefore \qquad -49.8386 = V_{P1} - 100$$

$$\therefore \qquad V_{P1} = 50.16 \text{ V} \qquad \text{... Absolute potential of P due to line charge 1}$$

Case 2 : Line charge 2, which is parallel to z-axis.

Do not consider z co-ordinate to find perpendicular distance.

$$\therefore \qquad r_{O2} = \sqrt{(-1-0)^2 + (2-0)^2} = \sqrt{5}$$

$$\text{and} \qquad r_{P2} = \sqrt{(-1-4)^2 + (2-1)^2} = \sqrt{26}$$

$$\therefore \qquad V_{PO2} = \frac{\rho_l}{2\pi\varepsilon_0} ln\left[\frac{\sqrt{5}}{\sqrt{26}}\right] = -118.5417 \text{ V}$$

$$\text{But} \qquad V_{PO2} = V_{P2} - V_O \qquad \text{where } V_O = 100 \text{ V}$$

$$\therefore \qquad V_{P2} = -118.5417 + 100 = -18.5417 \text{ V}$$

This is absolute potential of P due to line charge 2

$$\therefore \qquad V_P = V_{P1} + V_{P2} = 50.16 - 18.5417 = \mathbf{31.6183 \text{ V}}$$

Note : Students can use the method of using consant C to find absolute potential of P due to line charge 1 and line charge 2. Adding the two, potential of P can be obtained. The answer remains same. For reference, the constant $C_1 = C_2 = 215.721$ for both the line charges.

4.10 Equipotential Surfaces

In an electric field, there are many points at which the electric potential is same. This is because, the potential is a scalar quantity which depends on the distance between the point at which potential is to be obtained and the location of the charge. There can be number of points which can be located at the same distance from the charge. All such points are at the same electric potential. If the surface is imagined, joining all such points which are at the same potential, then such a surface is called **equipotential surface.**

Key Point: *An equipotential surface is an imaginary surface in an electric field of a given charge distribution, in which all the points on the surface are at the same electric potential.*

The potential difference between any two points on the equipotential surface is always zero. Thus the work done in moving a test charge from one point to another in an equipotential surface is always zero. There can be many equipotential surfaces existing in an electric field of a particular charge distribution.

Consider a point charge located at the origin of a sphere. Then potential at a point which is at a radial distance r from the point charge is given by,

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

So at all points which are at a distance r from Q, the potential is same and surface joining all such points is equipotential surface.

Similarly at $r = r_1$, $r = r_2$ ... there exists other equipotential surfaces, in an electric field of point charge, in the form of concentric spheres as shown in the Fig. 4.25.

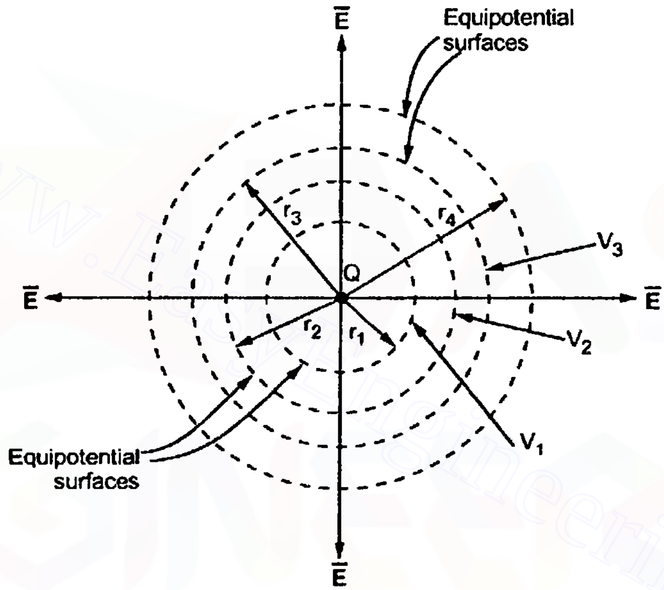


Fig. 4.25 Equipotential surfaces

It can be noted that V is inversely proportional to distance r. Thus V_1 at equipotential surface at $r = r_1$ is highest and it goes on decreasing, as the distance r increases. Thus $V_1 > V_2 > V_3 > .....$ As we move away from the charge, the $\bar{E}$ decreases hence potential of equipotential surfaces goes on decreasing. While potential of equipotential surfaces goes on increasing as we move against the direction of electric field.

For a uniform field $\bar{E}$, the equipotential surfaces are perpendicular to $\bar{E}$ and are equispaced for fixed increment of voltages. Thus if we move a charge along a circular path of radius r_1 as shown in $\bar{a}_\phi$ direction, then work done is zero. This is because $\bar{E}$ and $d\bar{L}$ are perpendicular. Thus $\bar{E}$ and equipotential surface are at right angles to each other.

For a nonuniform field, the field lines tends to diverge in the direction of decreasing $\bar{E}$. Hence equipotential surfaces are still

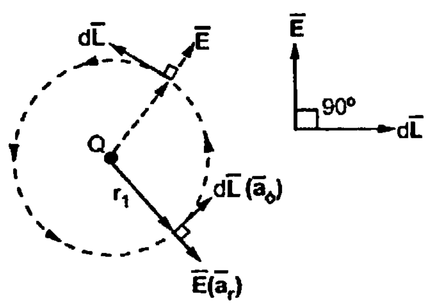
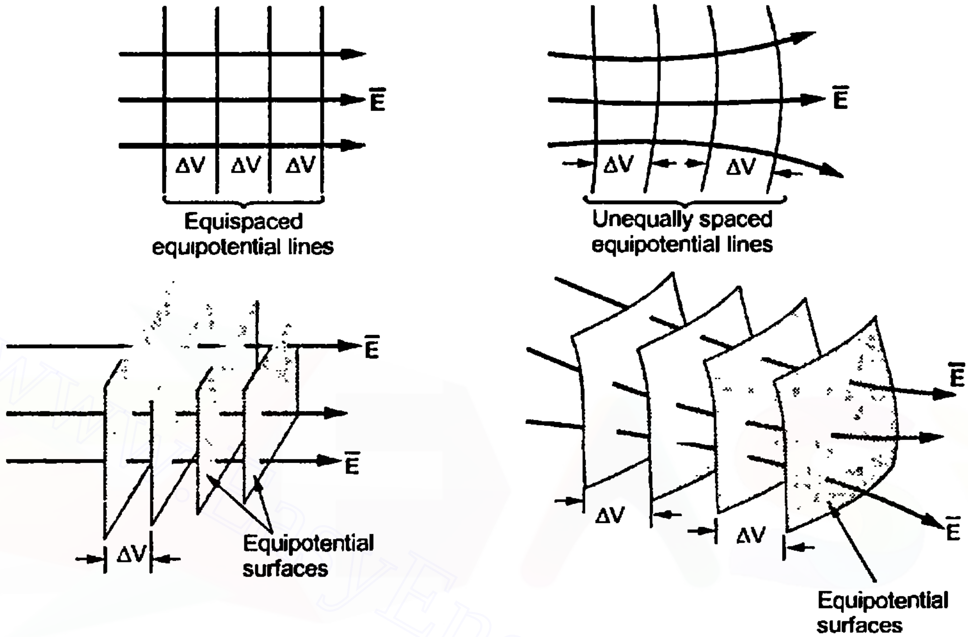


Fig. 4.26

perpendicular to $\bar{E}$ but are not equispaced, for fixed increment of voltages. The equipotential surfaces for uniform and nonuniform field are shown in the Fig. 4.27 (a) and (b).



(a) Uniform field (b) Non-uniform field

Fig. 4.27

4.11 Conservative Field

It is seen that, the work done in moving a test charge around any closed path in a static field $\bar{E}$ is zero. This is because starting and terminating point is same for a closed path. Hence upper and lower limit of integration becomes same hence the work done becomes zero. Such an integral over a closed path is denoted as,

$$\boxed{\oint_{\substack{\text{Closed}\\\text{path}}} \bar{E} \cdot d\bar{L} = 0} \quad \text{... (1)}$$

Key Point: *The $\oint$ sign indicates integral over a closed path. Such a field having property given by equation (1), associated with it, is called* ***conservative field*** *or* ***lamellar field****. This indicates that the work done in $\bar{E}$ and hence potential between two points is independent of the path joining the two points.*

4.12 Potential Gradient

Consider an electric field $\bar{E}$ due to a positive charge placed at the origin of a sphere. Then,

$$V = -\int \bar{E} \cdot d\bar{L} = \frac{Q}{4\pi\varepsilon_0 r}$$

The potential decreases as distance of point from the charge increases. This is shown in the Fig. 4.28.

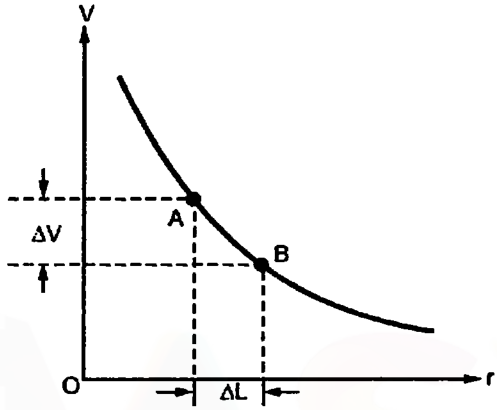


Fig. 4.28 Potential gradient

It is known that the line integral of $\bar{E}$ between the two points gives a potential difference between the two points. For an elementary length ΔL we can write,

$$\therefore \quad V_{AB} = \Delta V = -\bar{E} \cdot \Delta \bar{L}$$

Hence an inverse relation namely the change of potential ΔV, along the elementary length ΔL must be related to $\bar{E}$, as $\Delta L \to 0$.

The rate of change of potential with respect to the distance is called the **potential gradient.**

$$\therefore \quad \boxed{\frac{dV}{dL} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \text{Potential gradient}}$$

Potential gradient is nothing but the slope of the graph of potential against distance at a point where elementary length is considered.

Let us see how this potential gradient is related to the electric field.

4.12.1 Relation between $\bar{E}$ and V

Consider $\bar{E}$ due to a particular charge distribution in space. The electric field $\bar{E}$ and potential V is changing from point to point in space. Consider a vector incremental length $\Delta \bar{L}$ making an angle θ with respect to the direction of $\bar{E}$, as shown in the Fig. 4.29.

To find incremental potential we use,

$$\Delta V = -\bar{E} \cdot \Delta \bar{L} \quad \dots (1)$$

Now $\quad \Delta \bar{L} = \Delta L \, \bar{a}_L \quad \dots (2)$

where $\bar{a}_L$ = Unit vector in the direction of Δ L.

Now the dot product means product of magnitudes of one quantity and

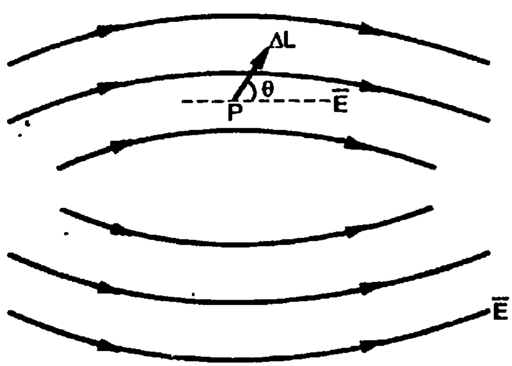


Fig. 4.29 Incremental length at an angle θ

component of other in the direction of first. So $\bar{E} \cdot \Delta\bar{L}$ is the product of component of $\bar{E}$ in the direction of $\bar{a}_L$ and Δ L.

$$\therefore \quad \bar{E} \cdot \Delta\bar{L} = (E_L \, \bar{a}_L) \cdot (\Delta L \bar{a}_L) \qquad \ldots \bar{a}_L \cdot \bar{a}_L = 1$$

$$\therefore \quad \bar{E} \cdot \Delta\bar{L} = E_L \, \Delta L$$

$$\therefore \quad \Delta V = -E_L \, \Delta L \qquad \ldots (3)$$

where E_L = Component of $\bar{E}$ in the direction of $\bar{a}_L$.

In other words, dot product can be expressed in terms of cos θ as.

$$\Delta V = -E \, \Delta L \, \cos\theta \qquad \ldots \text{as } \bar{E} \cdot \Delta\bar{L} = |\bar{E}| \, |\Delta L| \cos\theta$$

$$\therefore \quad \frac{\Delta V}{\Delta L} = -E \cos\theta \qquad \ldots (4)$$

To find ΔV at a point, take lim $\Delta L \to 0$,

$$\therefore \quad \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = -E \cos\theta \qquad \ldots (5)$$

$$\text{But} \quad \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \frac{dV}{dL} = \text{Potential gradient}$$

$$\therefore \quad \frac{dV}{dL} = -E \cos\theta \qquad \ldots (6)$$

At a point P where ΔL is considered, $\bar{E}$ has a fixed value while ΔL is also constant. Hence potential gradient $\frac{dV}{dL}$ can be maximum only when $\cos\theta = -1$ i.e. $\theta = +180°$. This indicates that ΔL **must be in the direction opposite to $\bar{E}$.**

$$\therefore \quad \left.\frac{dV}{dL}\right|_{max} = E \qquad \ldots (7)$$

This equation shows that,

1. Maximum value of the potential gradient gives the magnitude of the electric field intensity $\bar{E}$.

2. The maximum value of rate of change of potential with distance i.e. potential gradient is possible only when the direction of increment in distance is opposite to the direction of $\bar{E}$.

Thus if $\bar{a}_n$ is the unit vector in the direction of **increasing potential** normal to the equipotential surface then $\bar{E}$ can be expressed as,

$$\bar{E} = -\left.\frac{dV}{dL}\right|_{max} \bar{a}_n \qquad \ldots (8)$$

As $\bar{E}$ and potential gradient are in opposite direction, equation (8) has a negative sign.

The equation shows that the magnitude of $\bar{E}$ is given by maximum space rate of change of V while the direction of $\bar{E}$ is normal to the equipotential surface in the direction **of decreasing potential.**

The maximum value of rate of change of potential with distance (dV/dL) is called gradient of V.

The mathematical operation on V by which $-\bar{E}$ is obtained is called **gradient** and denoted as,

$$\text{Gradient of V} = \text{grad V} = \nabla \text{ V} \quad \text{... (9)}$$

$$\therefore \quad \nabla \text{ V} = \text{grad V} = -\bar{E} \text{ V/m} \quad \text{... (10)}$$

$$\text{or} \quad \boxed{\bar{E} = -\nabla \text{ V} = -(\text{grad V})} \quad \text{... (11)}$$

The equation (11) gives the relationship between $\bar{E}$ and V. Now $\bar{E}$ is vector but V is scalar, hence remember that grad V i.e. **gradient of a scalar is a vector.**

4.12.2 The Vector Operator ∇ (Del)

In space the potential V is unique function of x, y and z co-ordinates, in cartensian system denoted as V (x, y, z). Hence its total differential potential dV can be obtained as,

$$dV = \frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz \quad \text{... (12)}$$

In cartesian co-ordinates,

$$\bar{E} = E_x\,\bar{a}_x + E_y\,\bar{a}_y + E_z\,\bar{a}_z \quad \text{... (13)}$$

$$\text{While} \quad d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z \quad \text{... (14)}$$

$$\therefore \quad dV = -\bar{E}\cdot d\bar{L}$$

$$= -[E_x\,dx + E_y\,dy + E_z\,dz] \quad \text{... (15)}$$

Comparing equations (12) and (15) we can write,

$$E_x = -\frac{\partial V}{\partial x},\quad E_y = -\frac{\partial V}{\partial y},\quad E_z = -\frac{\partial V}{\partial z} \quad \text{... (16)}$$

Hence $\bar{E}$ can be expressed interms of (16) as,

$$\bar{E} = -\frac{\partial V}{\partial x}\bar{a}_x - \frac{\partial V}{\partial y}\bar{a}_y - \frac{\partial V}{\partial z}\bar{a}_z$$

$$\therefore \quad \bar{E} = -\left[\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right] V \quad \text{... (17)}$$

The potential V is scalar but the operator on V given in equation is vector and is called del operator denoted as ∇. The operation of del on a scalar V is called grad V.

$$\therefore \quad \boxed{\nabla = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z} \quad \text{... (18)}$$

$$\therefore \quad \boxed{\bar{E} = -\nabla \text{ V}} \quad \text{... (19)}$$

The grad of a scalar is a vector.

The grad V in various co-ordinate systems are,

Sr. No.	Co-ordinate system	Grad V = ∇ V
1.	Cartesian	$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$
2.	Cylindrical	$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z$
3.	Spherical	$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi$

Table 4.3

4.12.3 Properties of Gradient of a Scalar

The various properties of a gradient of a scalar field α are,

1. The gradient $\nabla\alpha$ gives the maximum rate of change of α per unit distance.
2. The gradient $\nabla\alpha$ always indicates the direction of maximum rate of change of α.
3. The gradient $\nabla\alpha$ at any point is perpendicular to the constant α surface which passes through the point.
4. The directional derivative of α along the unit vector $\bar{a}$ is $\nabla\alpha \cdot \bar{a}$ which is projection of $\nabla\alpha$ in the direction of unit vector $\bar{a}$.

   If β is another scalar then,
5. $\nabla(\alpha+\beta) = \nabla\alpha + \nabla\beta$
6. $\nabla(\alpha\beta) = \alpha\nabla\beta + \beta\nabla\alpha$
7. $\nabla\left(\frac{\alpha}{\beta}\right) = \frac{\beta\nabla\alpha - \alpha\nabla\beta}{\beta^2}$

Example 4.14 : *The electric field intensity $\bar{E}$ is negative gradient of the scalar potential. Find $\bar{E}$ at the point $P(0,1,1)$ if,*

a) $V = E_0 e^{-x}\sin\left(\frac{\pi y}{4}\right)$ *... in cartesian*

b) $V = E_0 r\cos\theta$ *... in spherical.*

Solution : a)

$$V = E_0 e^{-x}\sin\left(\frac{\pi y}{4}\right)$$

$$\therefore \quad \bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$\frac{\partial V}{\partial x} = E_0 \sin\left(\frac{\pi y}{4}\right)(-1)e^{-x} \quad \text{... y is constant}$$

$$\frac{\partial V}{\partial y} = E_0\, e^{-x} \cos\left(\frac{\pi y}{4}\right)\frac{\pi}{4} \qquad \text{... x is constant}$$

$$\frac{\partial V}{\partial z} = 0 \qquad \text{... z is absent}$$

$$\therefore \quad \bar{E} = -\left[-E_0\, e^{-x} \sin\frac{\pi y}{4}\bar{a}_x + E_0\, e^{-x}\frac{\pi}{4}\cos\frac{\pi y}{4}\bar{a}_y\right] \text{ V/m}$$

At P (0,1,1) , $\bar{E} = E_0\,[0.7071\,\bar{a}_x - 0.555\,\bar{a}_y]$ V/m

b) $$V = E_0\, r\cos\theta$$

$$\therefore \quad \bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right]$$

$$\frac{\partial V}{\partial r} = E_0\cos\theta,\quad \frac{\partial V}{\partial \theta} = -E_0\, r\sin\theta,\quad \frac{\partial V}{\partial \phi} = 0$$

$$\therefore \quad \bar{E} = -E_0\cos\theta\,\bar{a}_r + E_0\sin\theta\,\bar{a}_\theta \text{ V/m}$$

Convert P (0,1,1) to spherical co-ordinates.

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{2},\quad \phi = \tan^{-1}\frac{y}{x} = \frac{\pi}{2},\quad \theta = \cos^{-1}\frac{z}{r} = 45°$$

$$\therefore \quad \bar{E} = +E_0\,[-0.7071\,\bar{a}_r + 0.7071\,\bar{a}_\theta] \text{ V/m}$$

Example 4.15 : *An electric potential is given by,*

$$V = \frac{60\sin\theta}{r^2} \text{ V}$$

Find V and $\bar{E}$ at P(3, 60°, 25°).

Solution : At P (3, 60°, 25°) , r = 3, θ = 60°, φ = 25°

$$\therefore \quad V = \frac{60\sin 60°}{(3)^2} = 5.7735 \text{ V}$$

$$\bar{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right)$$

$$\therefore \quad \frac{\partial V}{\partial r} = 60\sin\theta(-2)\,r^{-3} = -\frac{120\sin\theta}{r^3} \qquad \text{... θ constant}$$

$$\frac{\partial V}{\partial \theta} = \frac{60}{r^2}\cos\theta \qquad \text{... r constant}$$

$$\frac{\partial V}{\partial \phi} = 0 \qquad \text{... φ absent}$$

$$\therefore \quad \bar{E} = -\left[-\frac{120\sin\theta}{r^3}\,\bar{a}_r + \frac{1}{r}\cdot\frac{60}{r^2}\cos\theta\,\bar{a}_\theta\right]$$

At P,
$$\bar{E} = -\left[\frac{-120\sin 60^\circ}{(3)^3}\,\bar{a}_r + \frac{60}{(3)^3}\cos 60^\circ\,\bar{a}_\theta\right]$$

$$= 3.849\,\bar{a}_r - 1.111\,\bar{a}_\theta \quad \text{V/m}$$

Example 4.16 : *If* $V = 2x^2y + 20z - \dfrac{4}{x^2+y^2}$ *V*

Find E, $\bar{D}$ *and* ρ_v *at P (6, – 2.5, 3).*

So ution :
$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$\frac{\partial V}{\partial x} = 2y(2x) + 0 - 4\left[\frac{-(2x)}{(x^2+y^2)^2}\right] = 4xy + \frac{8x}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial y} = 2x^2 + 0 - 4\left[\frac{-2y}{(x^2+y^2)^2}\right] = 2x^2 + \frac{8y}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial z} = 0 + 20 - 0 = 20$$

$$\therefore \quad \bar{E} = -\left\{\left[4xy + \frac{8x}{(x^2+y^2)^2}\right]\bar{a}_x + \left[2x^2 + \frac{8y}{(x^2+y^2)^2}\right]\bar{a}_y + 20\bar{a}_z\right\}$$

$$\therefore \quad \bar{E} \text{ at P} = -\{[-60 + 0.0268]\bar{a}_x + [72 - 0.0112]\bar{a}_y + 20\bar{a}_z\}$$

$$= +59.9732\,\bar{a}_x - 71.9888\,\bar{a}_y - 20\,\bar{a}_z \quad \text{V/m}$$

$$\therefore \quad \bar{D} \text{ at P} = \bar{E} \text{ at P} \times \varepsilon_0$$

$$= 0.531\,\bar{a}_x - 0.6373\,\bar{a}_y - 0.177\,\bar{a}_z \quad \text{nC/m}^2$$

Now $\rho_v = \nabla \cdot \bar{D}$

and $\bar{D} = \varepsilon_0 E$ hence $\rho_v = (\nabla \cdot \bar{E})\varepsilon_0$

$$\nabla \cdot \bar{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= -\frac{\partial}{\partial x}\left[4xy + \frac{8x}{(x^2+y^2)^2}\right] - \frac{\partial}{\partial y}\left[2x^2 + \frac{8y}{(x^2+y^2)^2}\right] - \frac{\partial}{\partial z}(20)$$

$$= -\left[4y + \left\{\frac{(x^2+y^2)^2 8 - 8x\,2\,(x^2+y^2)\,(2x)}{(x^2+y^2)^4}\right\}\right] - \left[0 + \left\{\frac{(x^2+y^2)^2\,(8) - 8y\,2\,(x^2+y^2)\,(2y)}{(x^2+y^2)^4}\right\}\right] - 0$$

$$= -4y - \frac{8}{(x^2+y^2)^2} + \frac{32\,x^2}{(x^2+y^2)^3} - \frac{8}{(x^2+y^2)^2} + \frac{32\,y^2}{(x^2+y^2)^3}$$

At P, $x = 6$, $y = -2.5$ and $z = 3$.

$$\therefore \quad \nabla \cdot \bar{E} = 10 - 4.4816\times10^{-3} + 0.01527 - 4.4816\times10^{-3} + 2.651\times10^{-3}$$

$$= 10.00895$$

$$\therefore \quad \rho_v \text{ at P} = \varepsilon_0\,[\nabla \cdot \bar{E}] = 8.854\times10^{\,12}\times10.00895$$

$$= 88.6193 \;\; \text{pC/m}^3$$

4.13 Energy Density in the Electrostatic Fields

It is seen that, when a unit positive charge is moved from infinity to a point in a field, the work is done by the external source and energy is expended. If the external source is removed then the unit positive charge will be subjected to a force exerted by the field and will be moved in the direction of force. Thus to hold the charge at a point in an electrostatic field, an external source has to do work. This energy gets stored in the form of potential energy, when the test charge is hold at a point in a field. This is analogous to the water lifted at a height h and stored in a tank. Then it has a potential energy. When external source is removed, the potential energy gets converted to a kinetic energy. In this section, the expression of such a potential energy is derived.

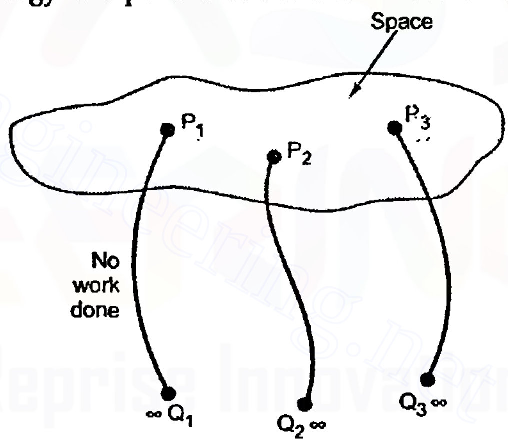


Fig. 4.30

Consider an empty space where there is no electric field at all. The charge Q_1 is moved from infinity to a point in the space say P_1. This requires no work as there is no $\bar{E}$ present. Now the charge Q_2 is to be placed at point P_2 in the space as shown in the Fig. 4.30. But now there is an electric field due to Q_1 and Q_2 is required to be moved against the field of Q_1. Hence the work is required to be done.

Now $\qquad$ Potential = Work done per unit charge $\left(\frac{W}{Q}\right)$

$\therefore$ Work done = Potential V × Charge Q

$\therefore$ Work done to position Q_2 at $P_2 = V_{2,1}\,Q_2$ ... (1)

where $V_{2,1}$ = Potential at P_2 due to P_1

Now let charge Q_3 is to be moved from infinity to P_3. There are electric fields due to Q_1 and Q_2. Hence total work done is due to potential at P_3 due to charge at P_1 and potential at P_3 due to charge at P_2.

$\therefore$ Work done to position Q_3 at $P_3 = V_{3,1}\,Q_3 + V_{3,2}\,Q_3$ ... (2)

Thus for charge Q_n to be placed at P_n, we can write,

$\therefore$ Work done to position Q_n at $P_n = V_{n,1}\,Q_n + V_{n,2}\,Q_n + \ldots$ ... (3)

Hence the total work done in positioning all the charges is,

$$W_E = Q_2\,V_{2,1} + Q_3\,V_{3,1} + Q_3\,V_{3,2} + \ldots \quad \text{... (4)}$$

The total work done is nothing but the potential energy in the system of charges hence denoted as W_E.

If charges are placed in reverse order we can write,

$$W_E = Q_3\,V_{3,4} + Q_2\,V_{2,3} + Q_2\,V_{2,4} + Q_1\,V_{1,2} + Q_1\,V_{1,3} + Q_1\,V_{1,4} + \ldots \quad \text{... (5)}$$

In this expression Q_n is placed first, then Q_{n-1} ... then Q_4, Q_3, Q_2 and finally Q_1.

Adding equation (4) and equation (5),

$$\begin{aligned} 2\,W_E &= Q_1\left(V_{1,2} + V_{1,3} + V_{1,4} + \ldots + V_{1,n}\right) \\ &+ Q_2\left(V_{2,1} + V_{2,3} + V_{2,4} + \ldots + V_{2,n}\right) \\ &+ Q_3\left(V_{3,1} + V_{3,2} + V_{3,4} + \ldots + V_{3,n}\right) + \ldots \end{aligned} \quad \text{... (6)}$$

Each sum of the potentials is the total resultant potential due to all the charges except for the charge at the point at which potential is obtained.

$\therefore$ $V_{1,2} + V_{1,3} + V_{1,4} + \ldots\ldots + V_{1,n} = V_1$

This is potential at P_1 where Q_1 is placed due to all other charges $Q_2, Q_3, \ldots Q_n$.

Similarly, $V_{2,1} + V_{2,3} + V_{2,4} + \ldots + V_{2,n} = V_2$ and so on.

Using in the equation (6),

$$2\,W_E = Q_1\,V_1 + Q_2\,V_2 + Q_3\,V_3 + \ldots$$

$$\therefore \quad \boxed{W_E = \frac{1}{2}\sum_{m=1}^{n} Q_m\,V_m \quad J} \quad \text{... (7)}$$

This is the potential energy stored in the system of n point charges.

It instead of point charges, the region has continuous charge distributions then summation in equation (7) becomes integration.

For line charge ρ_L, $$W_E = \frac{1}{2}\int \rho_l \, dL \, V \quad J \qquad \text{... (8)}$$

For surface charge ρ_S, $$W_E = \frac{1}{2}\int \rho_S \, dS \, V \quad J \qquad \text{... (9)}$$

For volume charge ρ_v, $$W_E = \frac{1}{2}\int \rho_v \, dv \, V \quad J \qquad \text{... (10)}$$

4.13.1 Energy Stored Interms of $\bar{D}$ and $\bar{E}$

Consider the volume charge distribution having uniform charge density $\rho_v \; C/m^3$. Hence the total energy stored is given by the equation (10) as,

$$W_E = \frac{1}{2}\int_{vol} \rho_v \, V \, dv$$

According to Maxwell's first equation,

$$\rho_v = \nabla \cdot \bar{D}$$

$$\therefore \quad W_E = \frac{1}{2}\int_{vol} (\nabla \cdot \bar{D}) \; V \, dv \qquad \text{... (11)}$$

For any vector A and scalar V there is vector identity,

$$\nabla \cdot V\bar{A} = \bar{A} \cdot \nabla V + V(\nabla \cdot \bar{A}) \qquad \text{... (12)}$$

$$\therefore \quad (\nabla \cdot \bar{A}) \, V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V \qquad \text{... (13)}$$

Using equation equation (13) in equation (11) we get,

$$W_E = \frac{1}{2}\int_{vol} (\nabla \cdot V\bar{D} - \bar{D} \cdot \nabla) \, dv$$

$$\therefore \quad W_E = \frac{1}{2}\int_{vol} (\nabla \cdot V\bar{D}) \, dv - \frac{1}{2}\int_{vol} \bar{D} \cdot \nabla V \; dv \qquad \text{... (14)}$$

According to divergence theorem, volume integral can be converted to closed surface integral if closed surface totally surrounds the volume.

$$\therefore \quad \frac{1}{2}\int_{vol} (\nabla \cdot V\bar{D}) \, dv = \frac{1}{2}\oint (V\bar{D}) \cdot d\bar{S} \qquad \text{... (15)}$$

$$\therefore \quad W_E = \frac{1}{2}\oint (V\bar{D}) \cdot d\bar{S} - \frac{1}{2}\int_{vol} \bar{D} \cdot \nabla V \; dv \qquad \text{... (16)}$$

We know that $V \propto \frac{1}{r}$ and $\overline{D} \propto \frac{1}{r^2}$ for point charge, $V \propto \frac{1}{r^2}$, $\overline{D} \propto \frac{1}{r^3}$ for dipoles and so on. So $V\overline{D}$ is proportional to at least $1/r^3$ while dS varies as r^2. Hence total integral varies as $1/r$. As surface becomes very large, $r \to \infty$ and $1/r \to 0$. Hence closed surface integral is zero in the equation (16).

$$\therefore \qquad W_E = -\frac{1}{2} \int_{vol} \overline{D} \cdot \nabla V \; dv \qquad \text{... (17)}$$

But $\qquad \overline{E} = -\nabla V$

$$\therefore \qquad W_E = -\frac{1}{2} \int_{vol} \overline{D} \cdot (-\overline{E}) \, dv \qquad \text{... (18)}$$

$$\therefore \qquad \boxed{W_E = \frac{1}{2} \int_{vol} \overline{D} \cdot \overline{E} \; dv \; J} \qquad \text{... (19)}$$

Now $\qquad \overline{D} = \varepsilon_0 \overline{E}$

$$\therefore \qquad W_E = \frac{1}{2} \int_{vol} \varepsilon_0 \overline{E} \cdot \overline{E} \; dv \; J$$

$$\therefore \qquad \boxed{W_E = \frac{1}{2} \int_{vol} \varepsilon_0 E^2 \; dv \; J} \qquad \text{... as } \overline{E} \cdot \overline{E} = E^2 \qquad \text{... (20)}$$

$$\therefore \qquad \boxed{W_E = \frac{1}{2} \int_{vol} \frac{D^2}{\varepsilon_0} \; dv \; J} \qquad \text{... (21)}$$

In a differential form,

$$d W_E = \frac{1}{2} \overline{D} \cdot \overline{E} \; dv$$

$$\therefore \qquad \boxed{\frac{d W_E}{dv} = \frac{1}{2} \overline{D} \cdot \overline{E} \; J/m^3} \qquad \text{... (22)}$$

This is called **energy density** in the electric field having units J/m^3. If this is integrated over the volume, we get total energy present.

$$\boxed{W_E = \int_{vol} \left(\frac{d W_E}{dv} \right) dv} \qquad \text{... (23)}$$

Example 4.17 : *If $V = x - y + xy + z$ V, find $\overline{E}$ at (1, 2, 4) and the electrostatic energy stored in a cube of side 2 m centered at the origin.* **[UPTU : 2002-03, 2007-08, 5 Marks]**

Solution : $\qquad V = x - y + xy + z$

$$\overline{E} = -\nabla V = -\left[\frac{\partial V}{\partial x} \overline{a}_x + \frac{\partial V}{\partial y} \overline{a}_y + \frac{\partial V}{\partial z} \overline{a}_z \right]$$

$$\frac{\partial V}{\partial x} = 1 + y, \; \frac{\partial V}{\partial y} = -1 + x, \; \frac{\partial V}{\partial z} = 1$$

$$\therefore \quad \bar{E} = -[(1+y)\bar{a}_x + (x-1)\bar{a}_y + \bar{a}_z]$$

At (1, 2, 4); $\quad \bar{E} = -3\bar{a}_x - \bar{a}_z$ V/m

Now $\quad W_E = \frac{1}{2}\int_{vol} \varepsilon_0 |\bar{E}|^2 dv, \quad dv = dx\,dy\,dz$

$$|\bar{E}|^2 = (1+y)^2 + (x-1)^2 + 1^2 = 1 + 2y + y^2 + x^2 - 2x + 1 + 1$$

$$= x^2 + y^2 - 2x + 2y + 3$$

$$\therefore \quad W_E = \frac{\varepsilon_0}{2}\int_{vol}(x^2+y^2-2x+2y+3)\,dx\,dy\,dz$$

The cube is centered at origin hence all the variables x, y and z vary from – 1 to + 1.

$$\therefore \quad W_E = \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\int_{x=-1}^{1}(x^2+y^2-2x+2y+3)\,dx\,dy\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\left[\frac{x^3}{3}+xy^2-\frac{2x^2}{2}+2xy+3x\right]_{x=-1}^{1} dy\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\left[\frac{2}{3}+2y^2+4y+6\right]dy\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\left[\frac{2}{3}y+\frac{2y^3}{3}+\frac{4y^2}{2}+6y\right]_{y=-1}^{1} dz = \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\left(\frac{4}{3}+\frac{4}{3}+12\right)dz$$

$$= \frac{\varepsilon_0}{2}\left[\frac{44z}{3}\right]_{-1}^{1} = \frac{88\varepsilon_0}{6} = 0.12985 \text{ nJ}$$

Example 4.18 : *Point charges $Q_1 = 1$ nC, $Q_2 = -2$ nC, $Q_3 = 3$ nC and $Q_4 = -4$ nC are placed one by one in the same order at (0,0,0), (1,0,0), (0,0,–1) and (0,0,1) respectively. Calculate the energy in the system when all charges are placed.*

Solution : When Q_1 is placed, the work done is zero as $\bar{E} = 0$, hence $W_1 = 0$ J.

When Q_2 is placed, there is field of Q_1 present.

$$\therefore \quad W_2 = Q_2 V_{2.1} = Q_2 \times \frac{Q_1}{4\pi\varepsilon_0 R_{21}}$$

and $R_{21} = 1$.

$$\therefore \quad W_2 = \frac{-2\times10^{-9}\times1\times10^{-9}}{4\pi\times8.854\times10^{-12}}$$

$$= -17.9754 \text{ J}$$

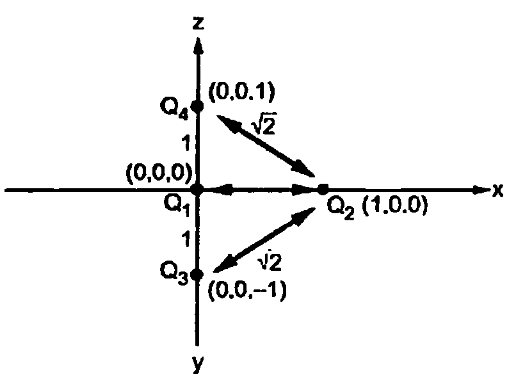


Fig. 4.31

When Q_3 is placed, there is field due to Q_1 and Q_2 both.

$$\therefore \quad W_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$$

$$\therefore \quad W_3 = Q_3\left[\frac{Q_1}{4\pi\varepsilon_0 R_{31}} + \frac{Q_2}{4\pi\varepsilon_0 R_{32}}\right] \quad \text{and } R_{31} = 1,\ R_{32} = \sqrt{2}$$

$$= \frac{3\times10^{-9}}{4\pi\varepsilon_0}\left[\frac{1\times10^{-9}}{1} - \frac{2\times10^{-9}}{\sqrt{2}}\right] = -\ 11.168\ \text{J}$$

When Q_4 is placed, there is field due to Q_1, Q_2 and Q_3.

$$\therefore \quad W_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$$

$$= Q_4\left[\frac{Q_1}{4\pi\varepsilon_0 R_{41}} + \frac{Q_2}{4\pi\varepsilon_0 R_{42}} + \frac{Q_3}{4\pi\varepsilon_0 R_{43}}\right]$$

and $R_{41} = 1,\ R_{42} = \sqrt{2},\ R_{43} = 2$

$$= \frac{-4\times10^{-9}}{4\pi\varepsilon_0}\left[\frac{1\times10^{-9}}{1} - \frac{2\times10^{-9}}{\sqrt{2}} + \frac{3\times10^{-9}}{2}\right] = -\ 39.035\ \text{J}$$

$$\therefore \quad W_E = W_1 + W_2 + W_3 + W_4 = 0 - 17.9754 - 11.168 - 39.035$$

$$= -\ 68.178\ \text{J}$$

Example 4.19 : *The potential field in free space is given by,*

$V = \dfrac{50}{r}, \quad a \le r \le b$ *(spherical)*

i) Show that $\rho_v = 0$ *for* $a < r < b$

ii) Find the energy stored in the region $a < r < b$.

Solution : i) $V = \dfrac{50}{r}$

$$\therefore \quad \bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right]$$

$$\therefore \quad \frac{\partial V}{\partial r} = -\frac{50}{r^2}, \quad \frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \phi} = 0$$

$$\therefore \quad \bar{E} = \frac{50}{r^2}\bar{a}_r$$

$$\therefore \quad \bar{D} = \varepsilon_0 \bar{E} = \frac{50\varepsilon_0}{r^2}\bar{a}_r$$

$$\therefore \quad \rho_v = \nabla\bullet\bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) + 0 + 0$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\times\frac{50\varepsilon_0}{r^2}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}[50\varepsilon_0] = 0\ \text{C/m}^3 \quad \text{... Proved}$$

ii) $$W_E = \frac{1}{2}\int_{vol} \varepsilon_0 |\bar{E}|^2 \, dv$$

$$|\bar{E}|^2 = \frac{(50)^2}{r^4} \quad \text{and} \quad dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\therefore \quad W_E = \frac{\varepsilon_0}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=a}^{b} \frac{50^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{2500\,\varepsilon_0}{2}[-\cos\theta]_0^{\pi}[\phi]_0^{2\pi}\left[-\frac{1}{r}\right]_a^b = 1250\,\varepsilon_0(2)(2\pi)\left[-\frac{1}{b}-\left(-\frac{1}{a}\right)\right]$$

$$= 1.39\times 10^{-7}\left[\frac{1}{a}-\frac{1}{b}\right] \text{J}$$

4.14 An Electric Dipole

The two point charges of equal magnitude but opposite sign, seperated by a very small distance give rise to an electric dipole. The field produced by such a dipole plays an important role in the engineering electromagnetics.

Consider an electric dipole as shown in the Fig. 4.32. The two point charges + Q and – Q are separated by a very small distance d.

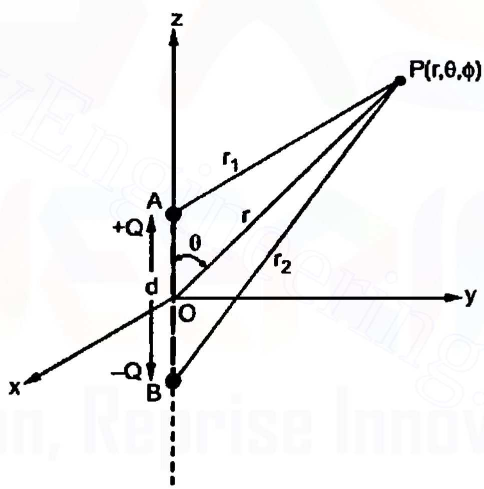


Fig. 4.32 Field due to an electric dipole

Consider a point P (r, θ, ϕ) in spherical co-ordinate system. It is required to find $\bar{E}$ due to an electric dipole at point P. Let O be the midpoint of AB. The distance of point P from A is r_1 while the distance of point P from B is r_2. The distance of point P from point O is r. The distance of separation of charges i.e. d is very small compared to the distances r_1, r_2 and r. The co-ordinates of A are $\left(0, 0, +\frac{d}{2}\right)$ and that of B are $\left(0, 0, -\frac{d}{2}\right)$.

To find $\bar{E}$, we will find out the potential V at point P, due to an electric dipole. Then using $\bar{E} = -\nabla V$, we can find $\bar{E}$ due to an electric dipole.

4.14.1 Expression of $\bar{E}$ due to an Electric Dipole

In spherical co-ordinates, the potential at point P due to the charge + Q is given by,

$$V_1 = \frac{+Q}{4\pi\varepsilon_0 r_1} \quad \text{... (1)}$$

The potential at P due to the charge – Q is given by,

$$V_2 = \frac{-Q}{4\pi\varepsilon_0 r_2} \quad \text{... (2)}$$

The total potential at point P is the algebraic sum of V_1 and V_2.

$$\therefore V = V_1 + V_2$$

$$= \frac{+Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2}$$

$$\therefore V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \quad \text{... (3)}$$

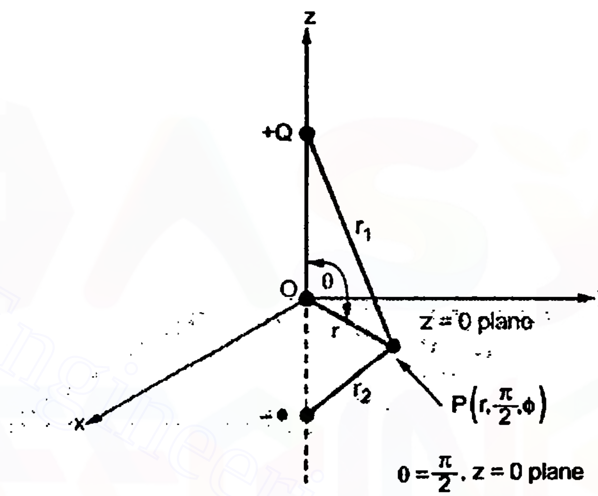


Fig. 4.33 Point P in z = 0 plane

If now point P is located in z = 0 plane as shown in the Fig. 4.33, then $r_2 = r_1$. Hence we get V = 0. Thus the entire **z = 0 plane i.e. xy plane is a zero potential surface.**

All points in z = 0 plane behave similar to the points at infinity as all are at zero potential.

Now consider that P is located far away from the electric dipole. Thus r_1, r_2 and r can be assumed to be parallel to each other as shown in the Fig. 4.34.

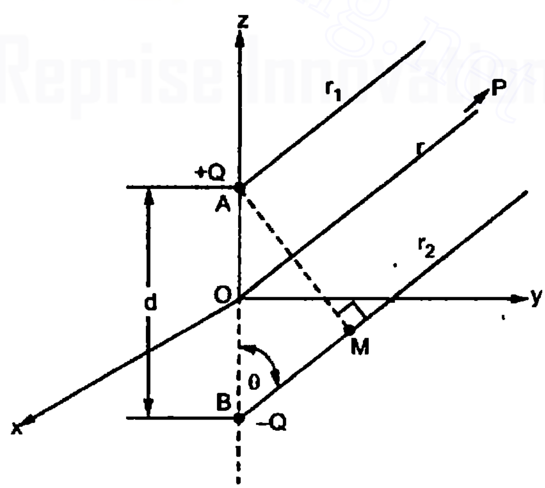


Fig. 4.34 Point P is too far away

AM is drawn perpendicular from A on r_2. The angle made by r_1, r_2 and r with z axis is θ as all are parallel.

$$\therefore \quad BM = AB \cos\theta$$

$$= d\cos\theta \quad \text{... (4)}$$

Now $\quad PB = PM + BM$

and $\quad PA = PM$ as AM is perpendicular.

and $\quad PB = r_2, \quad PA = r_1$

$\therefore \quad BM = PB - PM = r_2 - PM$

While $\quad PM = PA = r_1$

$$\therefore \quad BM = r_2 - r_1 \quad \text{... (5)}$$

$$\therefore \quad r_2 - r_1 = d \cos\theta \quad \text{... (6)}$$

As d is very small, $r_1 \approx r_2 \approx r$ hence $r_1 r_2 = r^2$

$$\therefore \quad V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta}{r^2}\right] V \quad \text{... (7)}$$

Now $$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right]$$

$$\therefore \quad \frac{\partial V}{\partial r} = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right] = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}(r^{-2})\right]$$

$$= \frac{Q\,d\cos\theta}{4\pi\varepsilon_0}\left[-2\,r^{-3}\right] = \frac{-2Q\,d\cos\theta}{4\pi\varepsilon_0\,r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{Q\,d}{4\pi\varepsilon_0\,r^2}[-\sin\theta] \quad \text{and} \quad \frac{\partial V}{\partial \phi} = 0$$

$$\therefore \quad \bar{E} = -\left[\frac{-2Q\,d\cos\theta}{4\pi\varepsilon_0\,r^3}\bar{a}_r - \frac{Q\,d\sin\theta}{4\pi\varepsilon_0\,r^3}\bar{a}_\theta\right]$$

$$\therefore \quad \boxed{\bar{E} = \frac{Q\,d}{4\pi\varepsilon_0\,r^3}\left[2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right]} \quad \text{(Spherical system) ... (8)}$$

This is electric field $\bar{E}$ at point P due to an electric dipole.

4.14.2 Dipole Moment

Let the vector length directed from – Q to + Q i.e. from B to A is $\bar{d}$.

$$\therefore \quad \bar{d} = d\,\bar{a}_z \quad \text{... (9)}$$

Its component along $\bar{a}_r$ direction can be obtained as,

$$d_r = \bar{d}\cdot\bar{a}_r = d\bar{a}_z\cdot\bar{a}_r = d\cos\theta$$

$$\therefore \quad \bar{d} = d\cos\theta\,\bar{a}_r \quad \text{... (10)}$$

Then the product Q $\bar{d}$ is called dipole moment and denoted as $\bar{p}$.

$$\therefore \qquad \boxed{\bar{p} = Q\,\bar{d}} \qquad \text{... (11)}$$

The dipole moment is measured in Cm **(coulomb-metre).**

Now $\quad \bar{p} \cdot \bar{a}_r = Q\,\bar{d} \cdot \bar{a}_r = Q\,d\cos\theta$ ... from (10)

Hence the expression of potential V can be expressed as,

$$\boxed{V = \frac{Q\,d\cos\theta}{4\pi\varepsilon_0\, r^2} = \frac{\bar{p} \cdot \bar{a}_r}{4\pi\varepsilon_0\, r^2}\ \text{V}} \qquad \text{... (12)}$$

Note that,

$\bar{a}_r$ = Unit vector in the direction of distance vector joining the point at which moment exists and point at which V is to be obtained.

$= \dfrac{\bar{r}}{|\bar{r}|}$ and $\bar{r}$ = Vector joining point of dipole moment to P.

It can be noted that the dipole moment and potential will remain same though Q increases and d decreases or viceversa, as long as the product of Q and d remains constant.

Now if $p = |\bar{p}| = Q\,|\bar{d}| = Qd$ then $\bar{E}$ due to a dipole can be expressed interms of **magnitude of dipole moment** as,

$$\boxed{\bar{E} = \frac{p}{4\pi\varepsilon_0\, r^3}\,[2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta]} \qquad \text{... (13)}$$

Observe that,

1. The potential is inversely proportional to the square of the distance from dipole.
2. The electric field is inversely proportional to the cube of the distance from dipole.

A single point charge is called **monopole** in which $V \propto (1/r)$ and $\bar{E} \propto (1/r^2)$.

The arrangement of two point charges is called **dipole** in which $V \propto (1/r^2)$ and $\bar{E} \propto (1/r^3)$.

Similarly symmetrical arrangements of larger number of point charges produce potentials and fields which are inversely proportional to the higher powers of r, such as r^3, r^4 ... etc. Such arrangements are called **multipoles.** The symmetrical arrangement consisting of two dipoles as shown in the Fig. 4.35 is called **quadrupole.** The symmetric arrangement consisting of two quadrupoles is called **octupole** and so on.

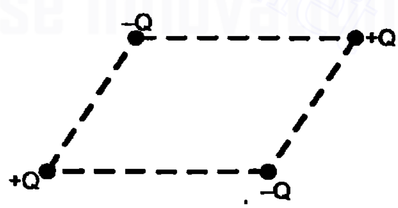


Fig. 4.35 Quadrupole

Example 4.20 : *A dipole having moment $\bar{p} = 3\bar{a}_x - 5\bar{a}_y + 10\bar{a}_z$ nCm is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$.*

Solution : The potential V in terms of dipole moment is,

$$V = \frac{\bar{p} \cdot \bar{a}_r}{4\pi\varepsilon_0\, r^2}$$

Now Q (1, 2, −4) and P (2, 3, 4)

$$\therefore \quad \bar{r} = (2-1)\bar{a}_x + (3-2)\bar{a}_y + [4-(-4)]\,\bar{a}_z$$

$$= \bar{a}_x + \bar{a}_y + 8\bar{a}_z$$

$$\therefore \quad |\bar{r}| = \sqrt{1+1+64} = \sqrt{66}$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{a}_x + \bar{a}_y + 8\bar{a}_z}{\sqrt{66}}$$

$$\therefore \quad \bar{p}\cdot\bar{a}_r = (3\bar{a}_x - 5\bar{a}_y + 10\bar{a}_z)\cdot\frac{(\bar{a}_x + \bar{a}_y + 8\bar{a}_z)}{\sqrt{66}}$$

$$= \frac{3-5+80}{\sqrt{66}} = \frac{78}{\sqrt{66}} \times 10^{-9} \text{ as } \bar{p} \text{ in nCm}$$

$$\therefore \quad V = \frac{\bar{p}\cdot\bar{a}_r}{4\pi\varepsilon_0 r^2} = \frac{78/\sqrt{66}\times 10^{-9}}{4\pi\times 8.854\times 10^{-12}\times(\sqrt{66})^2}$$

$$= \mathbf{1.3074\ V}$$

Examples with Solutions

Example 4.21 : *An electric field is given by,*

$$\bar{E} = 6y^2 z\bar{a}_x + 12xyz\bar{a}_y + 6xy^2\bar{a}_z \ \text{V/m}$$

and $\Delta\bar{L} = -3\bar{a}_x + 5\bar{a}_y - 2\bar{a}_z$ μm.

Find the work done in moving a 2 μC charge along this path if the location of the path is at,
a) P_1 (0, 3, 5) *b)* P_2 (1, 1, 0) *c)* P_3 (− 0.7, − 2, 0.4).

Solution : Note : The paths are located at the points. Hence charge is moved through ΔL rather than from one point to other. It is moved at a point in the direction $\Delta\bar{L}$ through distance ΔL. Hence the length is differential and work done will be also differential. There is no need of integration.

$$\therefore \quad dW = -Q\,\bar{E}\cdot\Delta\bar{L}$$

$$= -Q\,[6y^2 z\,\bar{a}_x + 12\,xyz\bar{a}_y + 6\,xy^2\,\bar{a}_z]\cdot[-3\bar{a}_x + 5\bar{a}_y - 2\bar{a}_z]$$

$$= -Q\,\{-18\,y^2 z + 60\,xyz - 12\,xy^2\}\times 10^{-6} \quad \text{... as μm}$$

$$= -2\times 10^{-6}\{-18\,y^2 z + 60\,xyz - 12\,xy^2\}\,10^{-6}$$

a) At P_1 (0, 3, 5) substitute x = 0, y = 3, z = 5

$$\therefore \quad dW = -2\times 10^{-12}\{-810\} = \mathbf{1620\ pJ}$$

b) At P_2 (1, 1, 0) substitute x = 1, y = 1, z = 0

$$\therefore \quad dW = -2\times 10^{-12}\ \{0+0-12\} = \mathbf{24\ pJ}$$

c) At $P_3(-0.7,-2,0.4)$ substitute $x=-0.7$, $y=-2$, $z=0.4$

$$\therefore \quad dW = -2\times10^{-12}\{-28.8+33.6+33.6\} = -76.8 \text{ pJ}$$

Example 4.22 : *What is the potential at the center of a square with a side $a=2m$ while charges $2\mu C$, $-4\mu C$, $6\mu C$ and $2\mu C$ are located at its four corners ?*

Solution : The arrangement of charges is shown in the Fig. 4.36.

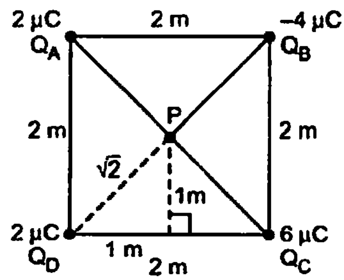


Fig. 4.36

The potential at a point due to a point charge is given by,

$$V = \frac{Q}{4\pi\varepsilon_0 R} \quad \text{where}$$

R = Distance between charge and the point

$$\therefore V_{P1} = \text{Potential of P due to } Q_A = \frac{Q_A}{4\pi\varepsilon_0 R_1}$$

where $R_1 = l\,(AP) = \sqrt{2}$ ... from geometry

Similarly $$V_{P2} = \frac{Q_B}{4\pi\varepsilon_0 R_2}$$

where $R_2 = l\,(BP) = \sqrt{2}$ m

$$V_{P3} = \frac{Q_C}{4\pi\varepsilon_0 R_3}$$

where $R_3 = l\,(CP) = \sqrt{2}$ m

$$V_{P4} = \frac{Q_D}{4\pi\varepsilon_0 R_4} \quad \text{where} \quad R_4 = l\,(DP) = \sqrt{2}\text{ m}$$

$$\therefore \quad V_P = \sum_{m=1}^{4} V_{Pm} = \frac{1}{4\pi\varepsilon_0 R}[Q_A + Q_B + Q_C + Q_D]$$

where $R = R_1 = R_2 = R_3 = R_4 = \sqrt{2}$ m

$$\therefore \quad V_P = \frac{1}{4\pi\times8.854\times10^{-12}\times\sqrt{2}}[2-4+6+2]\times10^{-6}$$

$$= \mathbf{38.131\ kV}$$

Example 4.23 : *Given a point charge of $200\pi\varepsilon_0$ C at $C(3,-1,+2)$, a line charge of $40\pi\varepsilon_0$ C/m on the x-axis and a surface charge of $8\,\varepsilon_0\,C/m^2$ on the plane $x=-3$, all in the free space. Find the potential at P (5, 6, 7) if V = 0 V at Q (0, 0, 1).*

Solution : The various charges are shown in the Fig. 4.37.

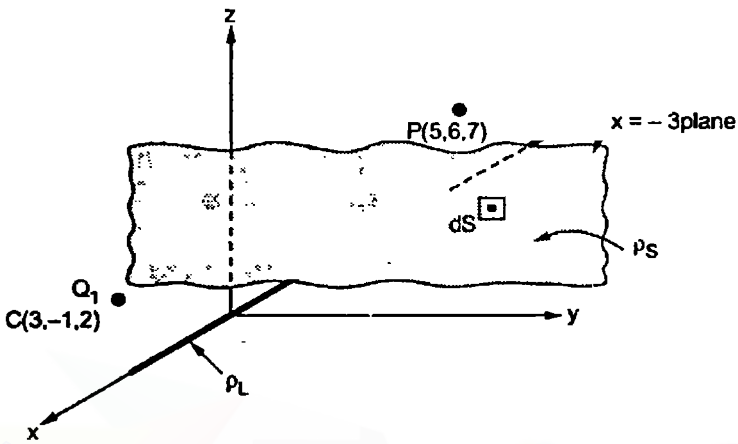


Fig. 4.37

There are three charge configurations.

Case 1 : Point charge $Q_1 = 200\pi\varepsilon_0$ C at C $(3,-1,+2)$.

$$V_P = \frac{Q_1}{4\pi\varepsilon_0 R_1} + C_1$$

where C_1 = constant

$$R_1 = \sqrt{(5-3)^2 + [6-(-1)]^2 + [7-2]^2}$$

$\therefore$ $R_1 = \sqrt{78}$ ... Distance between P and C

To find C_1, V = 0 V at Q (0, 0, 1)

$$\therefore \quad V_Q = \frac{Q_1}{4\pi\varepsilon_0 R_2} + C_1$$

where

$$R_2 = \sqrt{[0-3]^2 + [0-(-1)]^2 + [1-2]^2} = \sqrt{11}$$

$$\therefore \quad 0 = \frac{200\pi\varepsilon_0}{4\pi\varepsilon_0\sqrt{11}} + C_1$$

$$\therefore \quad C_1 = -15.0755$$

$$\therefore \quad V_P = \frac{200\pi\varepsilon_0}{4\pi\varepsilon_0 \times \sqrt{78}} - 15.0755$$

$$= -9.4141 \text{ V}$$

Case 2 : Due to line charge along x-axis.

$$V_{PQ} = \frac{\rho_L}{2\pi\varepsilon_0} \ln \frac{r_Q}{r_P} \quad \text{... Potential difference}$$

As line charge is along x-axis, any point on it (x, 0, 0).

$$\therefore \quad r_Q = \sqrt{(0-0)^2+(1-0)^2} = 1 \quad \text{... } \perp \text{ Distance from line charge}$$

$$\text{and} \quad r_P = \sqrt{(6-0)^2+(7-0)^2} = \sqrt{85} \quad \text{... x not considered}$$

$$\therefore \quad V_{PQ} = \frac{40\pi\varepsilon_0}{2\pi\varepsilon_0} \ln \frac{1}{\sqrt{85}} = -44.4265 \text{ V}$$

$$\text{But} \quad V_{PQ} = V_P - V_Q \quad \text{and} \quad V_Q = 0 \text{ V}$$

$$\therefore \quad V_P = V_{PQ} + V_Q = -44.4265 \text{ V} \quad \text{... Absolute potential of P}$$

Case 3 : Surface charge in the plane x = – 3 i.e. parallel to yz plane.

Note : As $\bar{E}$ due to infinite surface charge is known use $V_{AB} = -\int_B^A \bar{E} \cdot d\bar{L}$.

$$\text{So} \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0} \bar{a}_x \quad \text{... } a_x \text{ is normal to yz plane}$$

Point P is infront of plane as x co-ordinate of P is 5 hence $+\bar{a}_x$.

$$d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$\therefore \quad \bar{E} \cdot d\bar{L} = \frac{\rho_S}{2\varepsilon_0} dx \quad \text{... } \bar{a}_x \cdot \bar{a}_y = \bar{a}_x \cdot \bar{a}_z = 0$$

$$\therefore \quad V_{PQ} = -\int_Q^P \frac{\rho_S}{2\varepsilon_0} dx \quad \text{... Potential between P and Q}$$

$x_Q = 0$ and $x_P = 5$ hence

$$V_{PQ} = -\int_0^5 \frac{\rho_S}{2\varepsilon_0} dx = -\frac{5}{2}\frac{\rho_S}{\varepsilon_0} = -\frac{5}{2} \times \frac{8\varepsilon_0}{\varepsilon_0} = -20 \text{ V}$$

$$\text{But} \quad V_{PQ} = V_P - V_Q \quad \text{and} \quad V_Q = 0 \text{ V}$$

$$\therefore \quad V_P = -20 \text{ V} \quad \text{... Absolute potential of P}$$

$$\therefore \quad \text{Total } V_P = -9.4141 - 44.4265 - 20 = -73.8406 \text{ V}$$

Example 4.24 : *Two concentric cylindrical conductors are arranged to form a coaxial transmission line. Prove that the potential difference between the conductors is given by,*

$$V = \frac{\rho_L}{2\pi\varepsilon} ln \frac{b}{a} \; V \qquad a \le r \le b$$

where a = *radius of inner cylinder*

b = *radius of outer cylinder*

ρ_L = *charge per unit length of the inner conductor.*

Solution : The conductors are shown in the Fig. 4.38. The charge due to inner conductor is a line charge ρ_L over a long distance. $\bar{E}$ due to a very long line charge ρ_L is in $\bar{a}_r$ direction given by,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon r} \bar{a}_r \; V/m$$

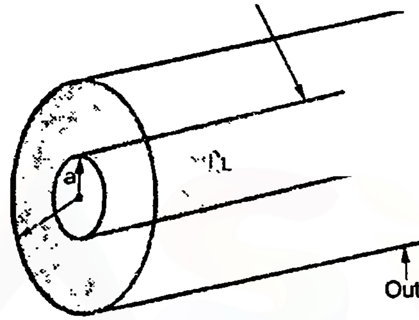


Fig. 4.38

Now the potential difference V_{AB} is to be obtained.

$\overline{dL}$ in cylindrical system is,

$$\overline{dL} = dr\, \bar{a}_r + r\, d\phi \bar{a}_\phi + dz \bar{a}_z$$

$$\therefore \quad \bar{E} \bullet \overline{dL} = \frac{\rho_L}{2\pi\varepsilon r} \bar{a}_r \bullet [dr\, \bar{a}_r + r\, d\phi\, \bar{a}_\phi + dz\, \bar{a}_z]$$

$$= \frac{\rho_L\, dr}{2\pi\varepsilon r}$$

$$\therefore \quad V_{AB} = -\int_B^A \bar{E} \bullet d\bar{L} = -\int_{r=b}^{r=a} \frac{\rho_L\, dr}{2\pi\varepsilon r} = -\frac{\rho_L}{2\pi\varepsilon}[ln\, r]_b^a \qquad \ldots \int \frac{dr}{r} = ln\,[r]$$

$$= -\frac{\rho_L}{2\pi\varepsilon}[ln\, a - ln\, b] = \frac{\rho_L}{2\pi\varepsilon}[ln\, b - ln\, a]$$

$$\therefore \quad \boxed{V_{AB} = \frac{\rho_L}{2\pi\varepsilon} ln\left[\frac{b}{a}\right] V} \quad a \le r \le b \qquad \ldots \text{Proved}$$

Example 4.25 : *If the potential field V is V = 100 (x^2-y^2) find $\bar{E}$, V at a point (2, – 1, 3) and the equation representing the locus of all points having a potential of 300 V.*

Solution : $V = 100\,(x^2 - y^2)$

At (2, – 1, 3), $V = 100\,[(2)^2 - (-1)^2 = 300 \text{ V}$

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$= [-200\, x\, \bar{a}_x - 200\, y\, \bar{a}_y]$$

$$= -200\, x\, \bar{a}_x + 200y\, \bar{a}_y$$

$\therefore$ At (2,– 1, 3), $\bar{E} = -400\,\bar{a}_x - 200\,\bar{a}_y$ V/m

For $V = 300$ V, the equation of locus is,

$$300 = 100\,(x^2 - y^2)$$

$$\therefore \quad x^2 - y^2 = 3$$

Example 4.26 : *Given a field*

$$\bar{E} = \left(\frac{-6y}{x^2}\right)\bar{a}_x + \left(\frac{6}{x}\right)\bar{a}_y + 5\,\bar{a}_z \quad V/m,$$

Find the potential difference V_{AB} given $A(-7,2,1)$ and $B(4,1,2)$.

Solution :

$$\bar{E} = -\frac{6y}{x^2}\,\bar{a}_x + \frac{6}{x}\,\bar{a}_y + 5\,\bar{a}_z$$

$$V_{AB} = -\int_B^A \bar{E}\cdot\overline{dL} \quad \text{where} \quad \overline{dL} = dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z$$

Now $\bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_z = 1$ and all other dot products are zero.

$$\therefore \quad \bar{E}\cdot\overline{dL} = -\frac{6y}{x^2}\,dx + \frac{6}{x}\,dy + 5\,dz$$

$$\therefore \quad V_{AB} = -\int_B^A -\frac{6y}{x^2}\,dx + \frac{6}{x}\,dy + 5\,dz$$

To obtain the integral as it does not depend on the path from B (4, 1, 2) to A(–7, 2, 1) we can divide the path as,

Path 1, B (4, 1, 2) to (–7, 1, 2) → only x varies, y = 1, z = 2.

Path 2, (–7, 1, 2) to (–7, 2, 2) → only y varies, x = – 7, z = 2.

Path 3, (–7, 2, 2) to A (–7, 2, 1) → only z varies, x = – 7, y = 2.

$$\therefore \quad V_{AB} = -\left\{\int_{x=4}^{x=-7} \frac{-6y}{x^2}\,dx + \int_{y=1}^{y=2} \frac{6}{x}\,dy + \int_{z=2}^{z=1} 5\,dz\right\}$$

$$y = 1 \qquad\qquad x = -7$$

$$= -\left\{-6\int_{x=4}^{-7} \frac{1}{x^2}\,dx - \frac{6}{7}\int_{y=1}^{2} dy + 5\int_{z=2}^{1} dz\right\}$$

$$= -\left\{-6\left[-\frac{1}{x}\right]_4^{-7} - \frac{6}{7}\,[y]_1^2 + 5\,[z]_2^1\right\}$$

$$= -\left\{-6\left[+\frac{1}{7}+\frac{1}{4}\right]-\frac{6}{7}[2-1]+5[1-2]\right\}$$

$$= -\{-2.3571-0.85714-5\}$$

$$= +8.2142\text{ V}$$

Alternatively find the equations for straight line path from B to A by using,

$$y-y_B = \frac{y_A-y_B}{x_A-x_B}(x-x_B) \quad \text{and} \quad z-z_B = \frac{z_A-z_B}{y_A-y_B}(y-y_B)$$

and using the relations between x, y and z solve the integrals. From the above equations we get, $x = -11y + 15$ and $z = -y + 3$ so use y interms of x for first integral and x interms of y for the second integral and integrate.

Example 4.27 : *$V = r^2 z \sin\phi$, calculate the energy within the region defined by*
$1 < r < 4$
$-2 < z < 2$
$0 < \phi < \pi/3$

Solution :

$$V = r^2 z \sin\phi \qquad \text{... cylindrical system}$$

$$\therefore \quad \bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{\partial V}{r\partial\phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$= -\left[2rz\sin\phi\bar{a}_r + \frac{1}{r}r^2 z\cos\phi\bar{a}_\phi + r^2\sin\phi\bar{a}_z\right]$$

$$W_E = \frac{1}{2}\int_{vol}\varepsilon_0 |\bar{E}|^2 dv$$

$$|\bar{E}| = \sqrt{4r^2z^2\sin^2\phi + r^2z^2\cos^2\phi + r^4\sin^2\phi}$$

$$\therefore \quad W_E = \frac{\varepsilon_0}{2}\int_{vol}\left[4r^2z^2\sin^2\phi + r^2z^2\cos^2\phi + r^4\sin^2\phi\right]dv$$

$$dv = r\,dr\,d\phi\,dz$$

$$\therefore W_E = \frac{\varepsilon_0}{2}\int_{z=-2}^{2}\int_{\phi=0}^{\pi/3}\int_{r=1}^{4} r^3\left[4z^2\sin^2\phi + z^2\cos^2\phi + r^2\sin^2\phi\right]dr\,d\phi\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-2}^{2}\int_{\phi=0}^{\pi/3}\left\{4z^2\sin^2\phi\left[\frac{r^4}{4}\right]_1^4 + z^2\cos^2\phi\left[\frac{r^4}{4}\right]_1^4 + \left[\frac{r^6}{6}\right]_1^4\sin^2\phi\right\}d\phi\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-2}^{2}\int_{\phi=0}^{\pi/3}\left[255z^2\sin^2\phi + 63.75z^2\cos^2\phi + 682.5\sin^2\phi\right]d\phi\,dz$$

$$= \frac{\varepsilon_0}{2} \int_{\phi=0}^{\pi/3} \left\{ 255 \sin^2\phi \left[\frac{z^3}{3}\right]_{-2}^{2} + 63.75 \left[\frac{z^3}{3}\right]_{2}^{2} \cos^2\phi + 682.5 \sin^2\phi [z]_{-2}^{2} \right\} d\phi$$

$$= \frac{\varepsilon_0}{2} \int_{\phi=0}^{\pi/3} \left[1360 \sin^2\phi + 340 \cos^2\phi + 2730 \sin^2\phi\right] d\phi$$

$$= \frac{\varepsilon_0}{2} \int_{\phi=0}^{\pi/3} \left[4090 \sin^2\phi + 340 \cos^2\phi\right] d\phi$$

$$= \frac{\varepsilon_0}{2} \left\{ 4090 \int_{\phi\ 0}^{\pi/3} \frac{1-\cos 2\phi}{2} d\phi + 340 \int_{\phi=0}^{\pi/3} \frac{1+\cos 2\phi}{2} d\phi \right\}$$

$$= \frac{\varepsilon_0}{2} \left\{ \frac{4090}{2} \left[\phi - \frac{\sin 2\phi}{2}\right]_{\phi=0}^{\pi/3} + \frac{340}{2} \left[\phi + \frac{\sin 2\phi}{2}\right]_{\phi=0}^{\pi/3} \right\}$$

$$= \frac{\varepsilon_0}{2} \left\{ 2045 \left[\frac{\pi}{3} - 0.433\right] + \frac{340}{2} \left[\frac{\pi}{3} + 0.433\right] \right\}$$

$$= \frac{\varepsilon_0}{2} \{(2045 \times 0.6141) + (170 \times 1.48019)\}$$

$$= \mathbf{6.6735\ nJ}$$

Example 4.28 : *Find the rate at which the scalar function*

$V = r^2 \sin 2\phi$ increases in the

i) z direction ii) ϕ direction

Evaluate it at $r = 2$ m and $\phi = 45°$ **[UPTU: 2002-03]**

Solution : The rate means gradient of the scalar.

In cylindrical system, gradient of the scalar in

i) z direction $= \frac{\partial V}{\partial z}\bar{a}_z = \frac{\partial}{\partial z}[r^2 \sin 2\phi]\bar{a}_z = 0\ \bar{a}_z$

ii) ϕ direction $= \frac{1}{r}\frac{\partial V}{\partial \phi}\bar{a}_\phi = \frac{1}{r}\frac{\partial}{\partial \phi}[r^2 \sin 2\phi]\bar{a}_\phi$

$= r\ 2\cos 2\phi\ \bar{a}_\phi$

At r = 2 m and $\phi = 45°$, rate $2 \times 2 \times \cos 90° = 4\ \bar{a}_\phi$

Example 4.29 : *Determine the work done in carrying a charge of –5 C from (2, 1, –1) to (4, 2, –1) in the field $\bar{E} = x\,\bar{a}_x$.* **[UPTU : 2003-04, 5 Marks]**

Solution : The charge is moved from B(2, 1, –1) to A (4, 2, –1).

Now $W = Q\int_B^A \bar{E} \bullet d\bar{L}$ where $d\bar{L} = d_x\bar{a}_x + d_y\bar{a}_y + d_z\bar{a}_z$

$$\therefore \quad \bar{E} \bullet d\bar{L} = x\bar{a}_x \bullet d\bar{L} = x\,dx \quad \ldots \bar{a}_x \bullet \bar{a}_y = \bar{a}_x \bullet \bar{a}_z = 0$$

$$\therefore \quad w = -Q \int_{x=2}^{x=4} x\,dx = -(-5)\left[\frac{x^2}{2}\right]_2^4$$

$$= 5 \times \left[\frac{16}{2} - \frac{4}{2}\right] = 30 \text{ J}$$

Example 4.30 : *Find the potential energy stored in the following free space charge configurations,*

i) A charge Q at each corner of an equilateral triangle of sides d.

ii) A charge Q at each corner of a square of side 'd'.

Solution : i) The arrangement is shown in the Fig. 4.39.

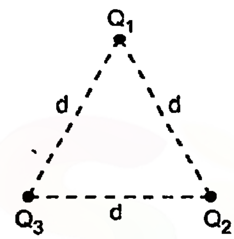


Fig. 4.39

When Q_1 is positioned, no other charge is present. Hence work done $W_1 = 0$ J.

When Q_2 is placed, Q_1 is present hence work done is,

$$W_2 = Q_2 V_{2,1}$$

$$= \frac{Q_2 Q_1}{4\pi\varepsilon_0 R_{21}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 d}$$

When Q_3 is placed, Q_1 and Q_2 are present hence work done is,

$$W_3 = Q_3 V_{3,1} + Q_3 V_{3,2} = Q_3\left[\frac{Q_1}{4\pi\varepsilon_0 R_{31}} + \frac{Q_2}{4\pi\varepsilon_0 R_{23}}\right]$$

Now $\quad R_{31} = R_{23} = d$

$$\therefore \quad W_3 = \frac{Q_3}{4\pi\varepsilon_0 d}[Q_1 + Q_2]$$

$$\therefore \quad W_E = W_1 + W_2 + W_3 = \frac{1}{4\pi\varepsilon_0 d}[Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3]$$

but $\quad Q_1 = Q_2 = Q_3 = Q$

$$\therefore \quad W_E = \frac{3Q^2}{4\pi\varepsilon_0 d} \text{ J}$$

ii) The arrangement is shown in the Fig. 4.40.

$R_{12} = d, \quad R_{23} = d, \quad R_{34} = d, \quad R_{41} = d,$

$$R_{31} = 2 \times \frac{\sqrt{2}\,d}{2} = \sqrt{2}\,d = R_{24}$$

For Q_1 which is placed first, $W_1 = 0$.

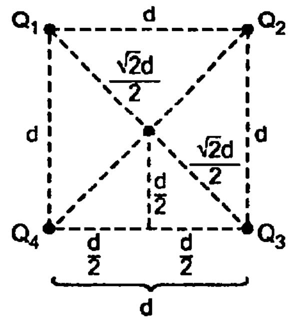


Fig. 4.40

For Q_2, $\quad W_2 = Q_2 V_{2,1}$

$$= \frac{Q_2 Q_1}{4\pi\varepsilon_0 R_{21}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 d}$$

For Q_3, $W_3 = Q_3 V_{3,1} + Q_3 V_{3,2} = Q_3\left[\frac{Q_1}{4\pi\varepsilon_0 R_{31}} + \frac{Q_2}{4\pi\varepsilon_0 R_{32}}\right]$

$$= \frac{Q_1 Q_3}{4\pi\varepsilon_0 \sqrt{2}\, d} + \frac{Q_2 Q_3}{4\pi\varepsilon_0 d}$$

For Q_4, $\quad W_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$

$$= Q_4\left[\frac{Q_1}{4\pi\varepsilon_0 R_{41}} + \frac{Q_2}{4\pi\varepsilon_0 R_{42}} + \frac{Q_3}{4\pi\varepsilon_0 R_{43}}\right]$$

$$= \frac{Q_1 Q_4}{4\pi\varepsilon_0 d} + \frac{Q_2 Q_4}{4\pi\varepsilon_0 d\sqrt{2}} + \frac{Q_3 Q_4}{4\pi\varepsilon_0 d}$$

And $\quad Q_1 = Q_2 = Q_3 = Q_4 = Q$

$$\therefore \quad W_E = W_1 + W_2 + W_3 + W_4 = \frac{Q^2}{4\pi\varepsilon_0 d}\left[1 + \frac{1}{\sqrt{2}} + 1 + 1 + \frac{1}{\sqrt{2}} + 1\right]$$

$$\therefore \quad W_E = \frac{5.414\, Q^2}{4\pi\varepsilon_0 d} \text{ J}$$

Example 4.31 : *Two charges of opposite sign and magnitude 1 μμC are located 1 m apart. Find the potential at a point located midway between the two charges and 50 cm from the line connecting the charges. What shall be potential if the charges have similar sign ?*

[UPTU : 2005-06, 5 Marks]

Solution : The charges are shown in the Fig. 4.41 (a).

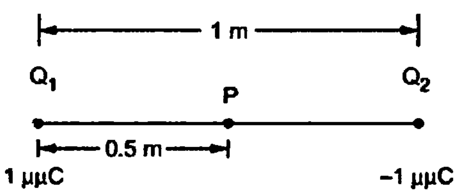


Fig. 4.41 (a)

$$V_{P1} = \frac{Q_1}{4\pi\varepsilon_0 R_{P1}} \quad \text{where } R_{P1} = 0.5$$

$$= \frac{1\times10^{-12}}{4\pi\varepsilon_0 \times 0.5} = 0.01797 \text{ V}$$

$$V_{P2} = \frac{Q_2}{4\pi\varepsilon_0 R_{P2}} \quad \text{where } R_{P2} = 0.5$$

$$= \frac{-1\times10^{-12}}{4\pi\varepsilon_0 \times 0.5} = -0.01797 \text{ V}$$

$$\therefore \quad V_P = V_{P1} + V_{P2} = 0 \text{ V}$$

If the charges are similar then,

$$V_P = 2 \times 0.01797 = 0.03595 \text{ V}$$

Now point P is at 50 cm from the line containing the charges.

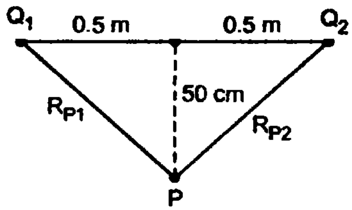


Fig. 4.41 (b)

$$R_{P1} = \sqrt{(0.5)^2 + (0.5)^2} = 0.7071 \text{ m}$$

$$R_{P2} = R_{P1} = 0.7071 \text{ m}$$

$$\therefore\ V_{P1} = \frac{Q_1}{4\pi\varepsilon_0 R_{P1}} = \frac{1\times10^{-12}}{4\pi\varepsilon_0 \times 0.7071}$$

$$= 0.01271 \text{ V}$$

$$\therefore\ V_{P2} = \frac{Q_2}{4\pi\varepsilon_0 R_{P2}} = \frac{-1\times10^{-12}}{4\pi\varepsilon_0 \times 0.7071}$$

$$= -0.01271 \text{ V}$$

$$\therefore \quad V_P = V_{P1} + V_{P2} = 0 \text{ V}$$

If the charges are similar then,

$$V_P = 2 \times 0.01271 = 0.01797 \text{ V}$$

Review Questions

1. *Define a work done and obtain the line integral to calculate the work done in moving a point charge Q in an electric field $\bar{E}$.*
2. *Show that the line integral $-Q\int_B^A \bar{E} \cdot d\bar{L}$ is not dependent on the path selected between B to A but only depends on the end points B and A.*
3. *Prove that if the path selected is such that it is always perpendicular to $\bar{E}$, the work done is zero.*
4. *If a field is given by $\bar{E} = y\bar{a}_x + x\bar{a}_y + 2\bar{a}_z$ V/m then determine the work done in moving a point charge of 2 C from B(1,0,1) to A (0.8, 0.6, 1) along the following paths :*
   1. Circle $x^2 + y^2 = 1$ and $z = 1$
   2. Straight line from B to A.
   Show that in both cases work done remains unchanged. **[Ans. : − 0.96 J]**
5. *If $\bar{E} = -8xy\,\bar{a}_x - 4x^2\bar{a}_y + \bar{a}_z$ V/m, then find the work done in carrying a 6 C charge from (1,8,5) to (2,18,6) along the path $y = 3x^2 + z$, $z = x + 4$.* **[Ans. : 1530 J]**
6. *Find the work done in moving a point charge $Q = 10\ \mu C$ from the origin to $P\left(3\ m, \frac{\pi}{4}, \frac{\pi}{2}\right)$ in the spherical system. The electrostatic field is, $\bar{E} = 10r\,\bar{a}_r + \frac{5}{r\sin\theta}\,\bar{a}_\phi$ V/m.* **[Ans. : − 475 µJ]**

7. *Find the work done in moving 4 μC charge from the origin to (2, −1, 4) through the field given by, $\bar{E} = 2xyz\,\bar{a}_x + x^2z\,\bar{a}_y + x^2y\,\bar{a}_z$ V/m via the path.*

   a) *Straight line segment z = 2x and x = − 2y*

   b) *Curve $= -2y^3$, $z = 4y^2$.* **[Ans. : 64 μJ]**

8. *If $\bar{E} = -50y\,\bar{a}_x - 50x\,\bar{a}_y + 30\bar{a}_z$ V/m then find the differential amount of work done in moving 2 μC charge through a distance of 5 μm from*

   a) *P(1, 2, 3) towards Q(2, 4, 1)* b) *Q(2, 4, 1) towards P(1, 2, 3).*

   [Hint : $d\bar{L} = dl\,\bar{a}_L = 5\times10^{-6}\bar{a}_L$. Obtain $\bar{a}_L$ in both cases and then $dW = -Q\bar{E}\cdot d\bar{L}$. Use $\bar{E}$ at starting points.] **[Ans. : 866.66 pJ, – 1533.33 pJ]**

9. *A point charge Q_1 is located at the origin in the free space. Find the work done in carrying a charge Q_2 from $B(r_B, \theta_B, \phi_B)$ to $C(r_A, \theta_B, \phi_B)$ with θ and ϕ held constant.*

   [Hint : $\bar{E}$ due to Q_1 at origin in spherical system is $\frac{Q_1}{4\pi\varepsilon_0 r^2}\bar{a}_r$ and $d\bar{L}$ in spherical system. Then

   $$W = -Q_2\int_{r_B}^{r_A}\bar{E}\cdot d\bar{L}$$

   [Ans. : $\frac{Q_1 Q_2}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$ J]

10. *If three point charges, 3μC, −4μC and 5μC are located at (0,0,0), (2,−1,3) and (0,4,−2) respectively, then find the potential at (−1,5,2) assuming V(∞) = 0.* **[Ans. : 10.23 kV]**

11. *A total charge of 40/3 nC is uniformly distributed over a circular disc of radius 2 m. Find the potential on the axis of the disc 2 m from the plane of the disc.* **[Ans. : 49.63 V]**

12. *A point charge of 15 nC is situated at the origin and another point charge of – 12 nC is located at the point (3, 3, 3) m. Find the potential at the point (0, – 3, – 3).* **[Ans. : 19.82 V]**

13. *A uniform line charge of 0.8 nC/m lies along the z-axis in free space. Find the potential at P(3, 4, 5) if the potential at Q(2, 8, 3) is zero.* **[Ans. : 23.75 V]**

14. *A point charge of 16 nC is located at Q(2, 3, 5) in free space and a uniform line charge of 5 nC/m is at the intersection of the planes x = 2 and y = 4. If the potential at the origin is 100 V, find V at (4, 1, 3).* **[Ans. : 137.544 V]**

15. *A line charge of 20/3 nC/m is uniformly distributed along a circular ring of radius r = 2m. Find the potential at a point on the axis of a ring 5 m from the plane of the ring.* **[Ans. : 140 V]**

16. *A negative point charge of magnitude 2 μC is located in air at the origin and two positive point charges of 1 μC each are at points y = ± 3 m, calculate the electric potential at a point 4 m from the origin on the x-axis.* **[Ans. : – 899.18 V]**

17. *Calculate the potential V_{AB} for r_A = 6 m with respect to r_B = 18 m due to a point charge Q = 500 pC at the origin.* **[Ans. : 0.5 V]**

18. *A positive point charge of magnitude 10 μC is situated at point x = 0, y = + 2 m and negative point charge of – 10 μC is situated at point x = 0, y = – 2 m. Calculate V at x = 0, y = –1 m.* **[Ans. : – 60 kV]**

19. *A scalar potential is given by,*

    $V = 7y^2 + 12x \;\; V$

    Find $\bar{E}$ *and its value at* $(0,0,0), (4,0,0)$ *and* $(0,4,0)$.

    [Ans. : $-12\bar{a}_x - 14y\bar{a}_y$ V/m, $-12\bar{a}_x, -12\bar{a}_x, -12\bar{a}_x - 56\bar{a}_y$]

20. *A scalar potential is given by,*

    $V = 5x + 4y^2 + 2z^3 \;\; V$

    Find $\bar{E}$ *at* $(2, 3, 4)$. **[Ans. : $-\left(5\bar{a}_x + 24\bar{a}_y + 96\bar{a}_z\right)$ V/m]**

21. *The potential in a certain region is given as,*

    $V = x^2 + 3y^2 + 9z \;\; V$

    Find the electric field intensity at $P(1, -2, 3)$. **[Ans. : $-2\bar{a}_x + 12a_y - 9\bar{a}_z$ V/m]**

22. *If* $V = 3x^2 - y + 3z$ *then find*

    a) V *b)* $\bar{E}$ *and c)* $|\bar{D}|$ *at* $(3, -2, 4)$. **[Ans. : 41 V, $-18\bar{a}_x + \bar{a}_y - 12\bar{a}_z$, 191.74 pC / m^2]**

23. *If an electric potential is given by*

    $V = \dfrac{10}{r^2}\sin\theta\cos\phi$ *V, find* $\bar{D}$ *at* $P\left(2, \dfrac{\pi}{2}, 0\right)$.

    [Ans. : $22.125 \times 10^{-12}\,\bar{a}_r\,C / m^2$]

24. *Point charges* $+3\mu C$ *and* $-3\mu C$ *are located at* $(0, 0, 1\,mm)$ *and* $(0, 0, -1\,mm)$ *respectively in free space.*

    a) Find the dipole moment $\bar{p}$.

    b) Find $\bar{E}$ *at* $P(r = 2, \theta = 40°, \phi = 50°)$ **[Ans. : $6\bar{a}_z$ nCm, $10.33\,\bar{a}_r + 4.33\,\bar{a}_\theta$ V/m]**

25. *Three point charges − 1 nC, 4 nC and 3 nC are located at* $(0,0,0)$, $(0,0,1)$ *and* $(1,0,0)$ *respectively. Find the energy in the system.* **[Ans. : 13.37 nJ]**

University Questions

1. *Explain the electric flux density and electrostatic energy.* **[UPTU: 2003-04(B), 5 Marks]**

2. *Two charges of opposite sign and magnitude 1 μμC are located 1 m apart. Find the potential at a point located midway between the two charges and 50 cm from the line connecting the charges. What shall be potential if the charges have similar sign ?* **[UPTU : 2005-06, 5 Marks]**

3. *A total charges of* 10^{-8} *is distributed uniformly along a ring of radius 5 m. Calculate the potential on the axis of the ring at a point 5 m from the centre of the ring. If the same charge is uniformly distributed on a disc of 5 m radius, what will be the potential on its axis at 5 m from the centre ?*

   [UPTU : 2005-06, 5 Marks]

4. *Determine the energy stored in the electric field in concentric spherical shell.*

   [UPTU : 2005-06, 10 Marks]

5. *If* $V = x - y + xy + z$ *V, find* $\bar{E}$ *at (1, 2, 4) and the electrostatic energy stored in cube of side 2 m centered at the origin.* **[UPTU : 2007-08, 5 Marks]**

6. *State and explain Gauss's law. Derive an expression for the potential at a point outside a hollow sphere having a uniform charge density.* **[UPTU : 2008-09, 10 Marks]**

❑❑❑

5 Conductors, Dielectrics and Capacitance

5.1 Introduction

It is known that the flow of charges constitutes an electric current. The current can be measured by measuring how many charges are passing through a specified surface or a point in a material per second. The flow of charge per unit time i.e. rate of flow of charge at a specified point or across a specified surface is called **an electric current**. It is measured in the unit **Ampere**, which is coulombs/sec (C/s). Thus mathematically the electric current can be expressed as,

$$\boxed{I = \frac{dQ}{dt} \text{ C/s i.e. A}}$$

The Ohm's law relates the applied voltage, an electric current and a resistance. The relation is simple and straight forward for simple d.c. circuits. But in electromagnetic engineering, motion of charges in various media such as liquid, gas, dielectrics etc. is considered. In such media, both positive and negative charges are present with different characteristics. The basic Ohm's law is not sufficient to find current through such media. Hence in electromagnetic engineering instead of current, the current density plays an important role. This chapter explains the current density, continuity equation and the properties of conductors and dielectrics. Then it explains the boundary conditions, the concept of capacitance and calculation of capacitance under various conditions.

5.2 Current and Current Density

The current is defined as the rate of flow of charge and is measured in amperes.

Key Point : *A current of 1 ampere is said to be flowing across the surface when a charge of one coulomb is passing across the surface in one second.*

The current is considered to be the motion of the positive charges. The conventional current is due to the flow of electrons, which are negatively charged. Hence the direction of conventional current is assumed to be opposite to the direction of flow of the electrons.

The current which exists in the conductors, due to the drifting of electrons, under the influence of the applied voltage is called **drift current**.

While in dielectrics, there can be flow of charges, under the influence of the electric field intensity. Such a current is called the **displacement current or convection current.** The current flowing across the capacitor, through the dielectric separating its plates is an example of the convection current.

The analysis of such currents, in the field theory is based on defining a current density at a point in the field.

The current density is a vector quantity associated with the current and denoted as $\bar{J}$.

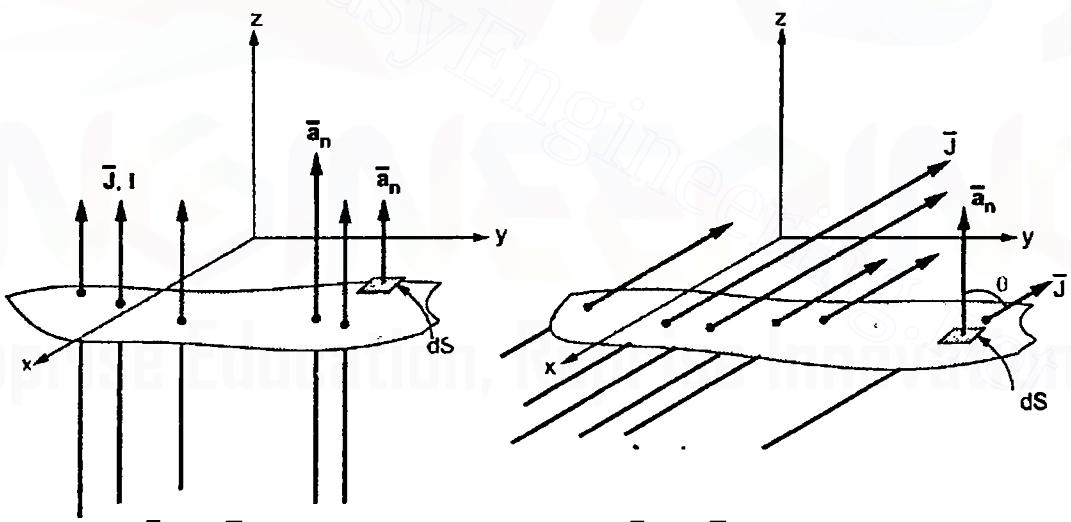
Key Point : *The current density is defined as the current passing through the unit surface area, when the surface is held normal to the direction of the current.*

The current density is measured in amperes per square metres (A/m^2).

5.2.1 Relation between I and $\bar{J}$

Consider a surface S and I is the current passing through the surface. The direction of current is normal to the surface S and hence direction of $\bar{J}$ is also normal to the surface S.

Consider an incremental surface area dS as shown in the Fig. 5.1 (a) and $\bar{a}_n$ is the unit vector normal to the incremental surface dS.



(a) $\bar{J}$ and $d\bar{S}$ are normal

(b) $\bar{J}$ and $d\bar{S}$ are not at right angles

Fig. 5.1

$$\therefore \quad d\bar{S} = dS\,\bar{a}_n \quad \text{while} \quad \bar{J} = J\,\bar{a}_n \qquad \text{... (1)}$$

Then the differential current dI passing through the differential surface dS is given by the dot product of the current density vector $\bar{J}$ and $d\bar{S}$.

$$\therefore \quad dI = \bar{J} \cdot d\bar{S} \quad \text{(dot product)} \qquad \text{... (2)}$$

When $\bar{J}$ and $d\bar{S}$ are at right angles $(\theta = 90°)$ then

$$dI = J\,\bar{a}_n \cdot dS\,\bar{a}_n = J\,dS \qquad \text{... (3)}$$

and $$I = \int_S J\, dS \qquad \dots (4)$$

where J = Current density in A/m^2.

But if $\bar{J}$ is **not normal** to the differential area $d\bar{S}$ then the total current is obtained by integrating the incremental current which is dot product of $\bar{J}$ and $d\bar{S}$, over the surface S. This is shown in the Fig. 5.1 (b). Thus in general,

$$\boxed{I = \int_S \bar{J} \cdot d\bar{S} \quad \text{(Dot product)}} \qquad \dots (5)$$

Thus if $\bar{J}$ is in A/m^2 and $d\bar{S}$ is in m^2 then the current obtained is in amperes (A). It may be noted that $\bar{J}$ need not be uniform over S and S need not be a plane surface.

5.2.2 Relation between $\bar{J}$ and ρ_v

The set of charged particles give rise to a charge density ρ_v in a volume v. The current density $\bar{J}$ can be related to the velocity with which the volume charge density i.e. charged particles in volume v crosses the surface S at a point. This is shown in the Fig. 5.2. The velocity with which the charge is getting transferred is $\bar{v}$ m/s. It is a vector quantity.

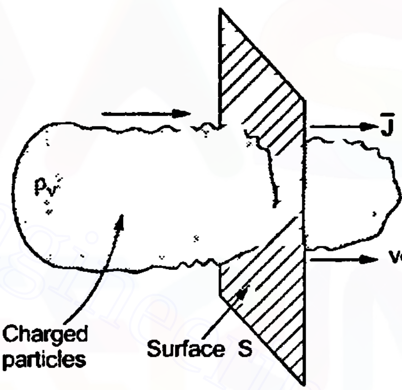


Fig. 5.2

To derive the relation between $\bar{J}$ and ρ_v, consider differential volume Δv having charge density ρ_v as shown in the Fig. 5.3. The elementary charge that volume carries is,

$$\Delta Q = \rho_v \Delta v \qquad \dots (6)$$

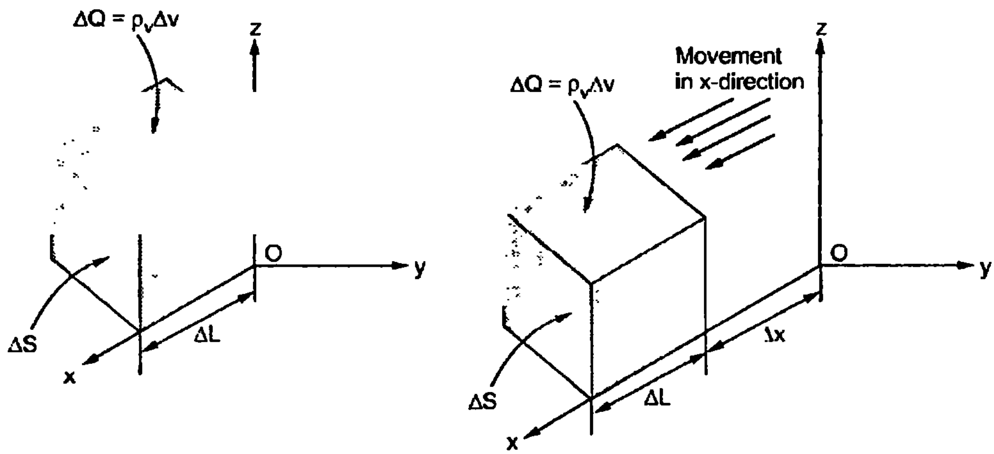


Fig. 5.3 Incremental charge moving in x-direction

Let ΔL is the incremental length while ΔS is the incremental surface area hence incremental volume is,

$$\Delta v = \Delta S \, \Delta L \qquad \ldots (7)$$

$$\therefore \quad \Delta Q = \rho_v \, \Delta S \, \Delta L \qquad \ldots (8)$$

Let the charge is moving in x-direction with velocity $\bar{v}$ and thus velocity has only x component v_x.

[Note : Velocity is denoted by small italic letter while the volume is denoted by small normal letter.]

In the time interval Δt the element of charge has moved through distance Δx, in x-direction as shown in the Fig. 5.3. The direction is normal to the surface ΔS and hence resultant current can be expressed as,

$$\Delta I = \frac{\Delta Q}{\Delta t} \qquad \ldots (9)$$

But now, $\Delta Q = \rho_v \, \Delta S \Delta x$ as the charge corresponding the length Δx is moved and responsible for the current.

$$\therefore \quad \Delta I = \rho_v \, \Delta S \frac{\Delta x}{\Delta t} \qquad \ldots (10)$$

But $\quad \dfrac{\Delta x}{\Delta t}$ = Velocity in x-direction i.e. v_x

$$\therefore \quad \Delta I = \rho_v \, \Delta S v_x \qquad \ldots (11)$$

Note that $\quad v_x$ = x component of velocity $\bar{v}$

But $\quad \Delta I = \bar{J} \, \Delta S$ when $\bar{J}$ and ΔS are normal

Here $\bar{J}$ and ΔS are normal to each other hence comparing the two equations,

$$J_x = \rho_v \, v_x = \text{x component of } \bar{J} \qquad \ldots (12)$$

In this case $\bar{J}$ has only x component.

In general, the relation between J and ρ_v can be expressed as,

$$\boxed{\bar{J} = \rho_v \, \bar{v}} \qquad \ldots (13)$$

where $\quad \bar{v}$ = Velocity vector

Such a current is called convection current and the current density is called convection current density.

Key Point : *The convection current density is linearly proportional to the charge density and the velocity with which the charge is transferred.*

5.3 Continuity Equation

The continuity equation of the current is based on the **principle of conservation of charge**. The principle states that,

The charges can neither be created nor be destroyed.

Consider a closed surface S with a current density $\bar{J}$, then the total current I crossing the surface S is given by,

$$\boxed{I = \oint_S \bar{J} \cdot d\bar{S}} \quad \ldots (1)$$

The current flows outwards from the closed surface. It has been mentioned earlier that **the current means the flow of positive charges.** Hence the current I is constituted due to outward flow of positive charges from the closed surface S. According to principle of conservation of charge, there must be decrease of an equal amount of positive charge inside the closed surface. Hence the outward rate of flow of positive charge gets balanced by the rate of decrease of charge inside the closed surface.

Let Q_i = Charge within the closed surface

$-\dfrac{dQ_i}{dt}$ = Rate of decrease of charge inside the closed surface

The negative sign indicates decrease in charge.

Due to principle of conservation of charge, this rate of decrease is same as rate of outward flow of charge, which is a current.

$$\therefore \quad I = \oint_S \bar{J} \cdot d\bar{S} = -\frac{dQ_i}{dt} \quad \ldots (2)$$

This is the **integral form of the continuity equation of the current.**

The negative sign in the equation indicates outward flow of current from the closed surface. So the equation (2) is indicating **outward flowing current I.**

If the current is entering the volume then

$$\oint_S \bar{J} \cdot d\bar{S} = -I = +\frac{dQ_i}{dt}$$

The point form of the continuity equation can be obtained from the integral form. Using the **divergence theorem,** convert the surface integral in integral form to the volume integral.

$$\therefore \quad \boxed{\oint_S \bar{J} \cdot d\bar{S} = \int_{vol} (\nabla \cdot \bar{J})\, dv} \quad \ldots (3)$$

$$\therefore \quad -\frac{d\,Q_i}{dt} = \int_{vol} (\nabla \cdot \bar{J})\, dv \quad \ldots (4)$$

But $$Q_i = \int_{vol} \rho_v \, dv \quad \ldots (5)$$

where ρ_v = Volume charge density

$$\therefore \quad \int_{vol} (\nabla \cdot \bar{J})\, dv = -\frac{d}{dt}\left[\int_{vol} \rho_v\, dv\right] = -\int_{vol} \frac{\partial \rho_v}{\partial t}\, dv \quad \text{... (6)}$$

For a constant surface, the derivative becomes the partial derivative.

$$\therefore \quad \int_{vol} (\nabla \cdot \bar{J})\, dv = \int_{vol} -\frac{\partial \rho_v}{\partial t}\, dv \quad \text{... (7)}$$

If the relation is true for any volume, it must be true even for incremental volume Δv.

$$\therefore \quad (\nabla \cdot \bar{J})\, \Delta v = -\frac{\partial \rho_v}{\partial t}\, \Delta v \quad \text{... (8)}$$

$$\therefore \quad \boxed{\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}} \quad \text{... (9)}$$

This is the **point form or differential form of the continuity equation of the current.**

The equation states that the current or the charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

5.3.1 Steady Current

For steady currents which are not the functions of time, $\partial \rho_v / \partial t = 0$ hence,

$$\boxed{\nabla \cdot \bar{J} = 0 \qquad \text{(Steady current)}} \quad \text{... (10)}$$

For such currents, the rate of flow of charge remains constant with time. The steady currents have no sources or sinks, as it is constant.

Example 5.1 : *In cylindrical co-ordinates, $\bar{J} = 10 e^{-100r}\, \bar{a}_\phi$ A/m^2. Find the current crossing through the region $0.01 \le r \le 0.02$ m, $0 < z \le 1$ m and intersection of this region with the $\phi =$ constant plane.*

Solution : The current is given by integral form of the continuity equation as,

$$I = \int_S \bar{J} \cdot d\bar{S}$$

Now $\quad d\bar{S} = dr\, dz\, \bar{a}_\phi \quad$ ... normal to $\bar{a}_\phi$ direction as $\bar{J}$ is in $\bar{a}_\phi$ direction

$$\therefore \quad \bar{J} \cdot d\bar{S} = [10 e^{-100r}\, \bar{a}_\phi] \cdot [dr\, dz\, \bar{a}_\phi]$$

$$= 10 e^{100r}\, dr\, dz \quad \text{... } (\bar{a}_\phi \cdot \bar{a}_\phi) = 1$$

$$\therefore \quad I = \int_S 10 e^{-100r}\, dr\, dz = \int_{z=0}^{1} \int_{r=0.01}^{0.02} 10 e^{-100r}\, dr\, dz$$

$$= 10\left[\frac{e^{-100r}}{-100}\right]_{0.01}^{0.02} [z]_0^1 = 10 \times \left[\frac{e^{-2}}{-100} - \frac{e^{-1}}{-100}\right] [1]$$

$$= 10 \times [-1.353 \times 10^{-3} + 3.678 \times 10^{-3}] = 2.326 \times 10^{-2} \text{ A}$$

Example 5.2 : *Find the total current in outward direction from a cube of 1 m, with one corner at the origin and edges parallel to the co-ordinate axes if,*
$J = 2x^2\,\bar{a}_x\, 2xy^3\,\bar{a}_y + 2xy\,\bar{a}_z$ *A/m²*.

Solution : The cube is shown in the Fig. 5.4.

According to continuity equation,

$$I = \oint_S \bar{J} \cdot d\bar{S} = \int_{vol} (\nabla \cdot \bar{J})\, dv$$

The cube is a volume hence use volume integral.

$$dv = dx\,dy\,dz \quad \text{and}$$

$$\nabla \cdot \bar{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

$$= \frac{\partial[2x^2]}{\partial x} + \frac{\partial[2xy^3]}{\partial y} + \frac{\partial[2xy]}{\partial z}$$

$$= 4x + 6xy^2 + 0 = 4x + 6xy^2$$

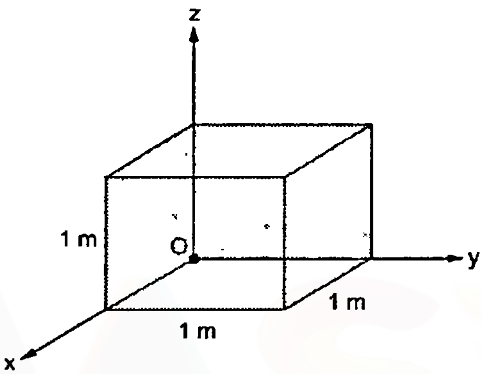


Fig. 5.4

$$\therefore I = \int_{vol} (4x + 6xy^2)\, dx\,dy\,dz = \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (4x + 6xy^2)\; dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{4x^2}{2} + \frac{6x^2y^2}{2}\right]_0^1 dy\,dz = \int_{z=0}^{1}\int_{y=0}^{1} (2 + 3y^2)\,dy\,dz$$

$$= \int_{z=0}^{1}\left[2y + \frac{3y^3}{3}\right]_0^1 dz = \int_{z=0}^{1} (2+1)\,dz$$

$$= 3[z]_0^1 = 3\ \text{A}$$

Example 5.3 : *A current density* $\bar{J} = \dfrac{100\cos\theta}{r^2+1}\,\bar{a}_r$ *A/m² in the spherical co-ordinate system.*

a) How much current flows through the spherical cap r = 3 m, $0 < \theta < \frac{\pi}{6}$, $0 < \phi < 2\pi$

b) The same total current as found in (a) flows through the spherical cap r = 10 m, $0 < \theta < \alpha$, $0 < \phi < 2\pi$. *What should be the value of* α *?*

Solution : a) From the continuity equation of current,

$$I = \oint_S \bar{J} \cdot d\bar{S} = \int_{vol} (\nabla \cdot \bar{J})\, dv$$

As r = 3 m is constant, use surface integral.

$$d\bar{S} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r \qquad \text{... As } \bar{J} \text{ is in } \bar{a}_r \text{ direction}$$

$$\therefore \quad \bar{J}\cdot d\bar{S} = \frac{100\cos\theta}{(r^2+1)} r^2 \sin\theta\, d\theta\, d\phi$$

$$\therefore \quad \oint_S \bar{J}\cdot d\bar{S} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \frac{100\cos\theta}{(r^2+1)} r^2 \sin\theta\, d\theta\, d\phi$$

$$\therefore \quad I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \frac{100\, r^2}{r^2+1} \times \frac{2\cos\theta\sin\theta}{2}\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \frac{100\, r^2}{r^2+1} \times \frac{\sin 2\theta}{2}\, d\theta\, d\phi \qquad \text{... } 2\sin\theta\cos\theta = \sin 2\theta$$

$$= \frac{100\, r^2}{2(r^2+1)}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/6} [\phi]_0^{2\pi} \qquad \text{and } r = 3 \text{ m}$$

$$= \frac{50\times 9}{10}\times\left[\frac{-\cos 2\times\frac{\pi}{6}}{2} - \frac{-\cos 0}{2}\right][2\pi] = \mathbf{70.6858\ A}$$

... Use radian mode to calculate cos

b) Now r = 10 m and limits for θ are 0 to α,

$$\therefore \quad I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha} \frac{100\, r^2}{(r^2+1)} \frac{\sin 2\theta}{2}\, d\theta\, d\phi$$

$$70.6858 = \frac{50\, r^2}{r^2+1}\left[-\frac{\cos 2\theta}{2}\right]_0^{\alpha} [\phi]_0^{2\pi} \qquad \text{... Same I as before}$$

$$\therefore \quad 70.6858 = \frac{50\times(10)^2}{(101)}\times\left[\frac{-\cos 2\alpha}{2} - \frac{-\cos 0}{2}\right][2\pi]$$

$$\therefore \quad 0.4545 = -\cos 2\alpha + 1$$

$$\therefore \quad \cos 2\alpha = 0.5455$$

$$\therefore \quad 2\alpha = 56.9411^\circ \text{ or } 0.9938 \text{ rad}$$

$$\therefore \quad \alpha = \mathbf{28.47^\circ} \text{ or } \mathbf{0.4969 \text{ rad}}$$

5.4 Conductors

Let us study the behaviour and properties of the conductors. Under the effect of applied electric field, the available free electrons start moving. The moving electrons strike the adjacent atoms and rebound in the random directions. This is called drifting of the electrons. After some time, the electrons attain the constant average velocity called **drift**

velocity (v_d). The current constituted due to the drifting of such electrons in metallic conductors is called **drift current**. The drift velocity is directly proportional to the applied electric field.

$$\therefore \qquad \bar{v}_d \propto \bar{E} \qquad \text{... (1)}$$

The constant of proportionality is called **mobility** of the electrons in a given material and denoted as μ_e. It is positive for the electrons.

$$\therefore \qquad \boxed{\bar{v}_d = -\mu_e \bar{E}} \qquad \text{... (2)}$$

The **negative sign** indicates that the velocity of the electrons is against the direction of field $\bar{E}$.

$$\text{Now} \quad \mu \text{ (Mobility)} = \frac{\text{Velocity}}{\text{Field}} = \frac{\text{m/s}}{\text{V/m}} = \frac{\text{m}^2}{\text{V-s}}$$

Thus mobility is measured in square metres per volt-second (m^2/V-s). The typical values of mobility are 0.0012 for aluminium, 0.0032 for copper etc.

According to relation between $\bar{J}$ and $\bar{v}$ we can write,

$$\bar{J} = \rho_v \bar{v} \qquad \text{... (3)}$$

But in the material, the number of protons and electrons is same and it is always electrically neutral. Hence $\rho_v = 0$ for the neutral materials. The drift velocity is the velocity of free electrons hence the above relation can be expressed as,

$$\boxed{\bar{J} = \rho_e \bar{v}_d} \qquad \text{... (4)}$$

where ρ_e = Charge density due to free electrons

The charge density ρ_e can be obtained as the product of number of free electrons/m^3 and the charge 'e' on one electron. Thus ρ_e = ne where n is number of free electrons per m^3.

Substituting equation (2) in equation (4) we get,

$$\boxed{\bar{J} = -\rho_e \mu_e \bar{E}} \qquad \text{... (5)}$$

5.4.1 Point Form of Ohm's Law

The relationship between $\bar{J}$ and $\bar{E}$ can also be expressed in terms of conductivity of the material.

Thus for a metallic conductor,

$$\boxed{\bar{J} = \sigma \bar{E}} \qquad \text{... (6)}$$

where σ = Conductivity of the material

The conductivity is measured in mhos per metre (℧/m). The equation (6) is called **point form of Ohm's law**. The unit of conductivity is also called Siemens per metre (S/m).

The typical values of conductivity are 3.82×10^7 for aluminium, 5.8×10^7 for copper etc. expressed in mho/m. For the metallic conductors the conductivity is constant over wide ranges of current density and electric field intensity. In all directions, metallic conductors have same properties hence called **isotropic** in nature. Such materials obey the Ohm's law very faithfully.

Comparing the equation (5) and equation (6) we can write,

$$\sigma = -\rho_e \mu_e \qquad \text{... (7)}$$

This is conductivity interms of mobility of the charge density of the electrons.

The resistivity is the reciprocal of the conductivity. The conductivity depends on the temperature. As the temperature increases, the vibrations of crystalline structure of atoms increases. Due to increased vibrations of electrons, drift velocity decreases, hence the mobility and conductivity decreases. So **as temperature increases, the conductivity decreases and resistivity increases.**

5.4.2 Resistance of a Conductor

Consider that the voltage V is applied to a conductor of length L having uniform cross-section S, as shown in the Fig. 5.5.

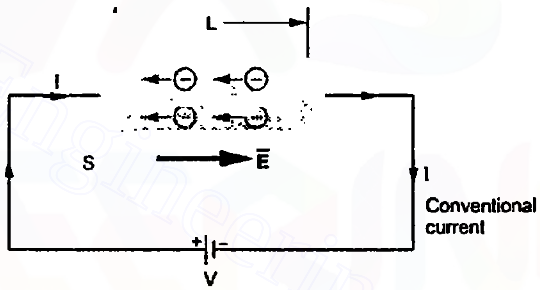


Fig. 5.5 Conductor subjected to voltage V

The direction of $\bar{E}$ is same as the direction of conventional current, which is opposite to the flow of electrons. The electric field applied is uniform and its magnitude is given by,

$$E = \frac{V}{L} \qquad \text{... (8)}$$

The conductor has uniform cross-section S and hence we can write,

$$I = \int_S \bar{J} \cdot d\bar{S} = JS \qquad \text{... (9)}$$

The current direction is normal to the surface S.

Thus, $$J = \frac{I}{S} = \sigma E \qquad \text{... (10)}$$

And using equation (8) in equation (10) we get,

$$J = \frac{\sigma V}{L} \quad \text{where } \sigma = \text{Conductivity of the material} \qquad \text{... (11)}$$

$$\therefore \qquad V = \frac{JL}{\sigma} = \frac{IL}{\sigma S} = \left(\frac{L}{\sigma S}\right) I \qquad \text{... (12)}$$

$$\therefore \qquad \boxed{R = \frac{V}{I} = \frac{L}{\sigma S}} \qquad \text{... (13)}$$

Thus the ratio of potential difference between the two ends of the conductors to the current flowing through it is **resistance of the conductor.**

The equation (12) is nothing but the Ohm's law in its normal form given by V = IR. The equation is true for the uniform fields and resistance is measured is ohms (Ω).

For **nonuniform fields**, the resistance R is defined as the ratio V to I where V is the potential difference betweem two specified equipotential surfaces in the material and I is the current crossing the more positive surface of the two, into the material. Mathematically the resistance for nonuniform fields is given by,

$$\boxed{R = \frac{V_{ab}}{I} = \frac{-\int_b^a \bar{E} \cdot d\bar{L}}{\int_S \bar{J} \cdot d\bar{S}} = \frac{-\int_b^a \bar{E} \cdot d\bar{L}}{\int_S \sigma \bar{E} \cdot d\bar{S}}} \qquad \text{... (14)}$$

The numerator is a line integration giving potential difference across two ends while the denominator is a surface integration giving current flowing through the material.

The resistance can also be expressed as,

$$\boxed{R = \frac{L}{\sigma S} = \frac{\rho_c L}{S}\ \Omega} \qquad \text{... (15)}$$

where $\rho_c = \dfrac{1}{\sigma}$ = Resistivity of the conductor in Ω-m

5.4.3 Properties of Conductor

Consider that the charge distribution is suddenly unbalanced inside the conductor. There are number of electrons trying to reside inside the conductor. All the electrons are negatively charged and they start repelling each other due to their own electric fields. Such electrons get accelerated away from each other, till all the electrons causing interior imbalance, reach at the surface of the conductor. The conductor is surrounded by the insulating medium and hence electrons just driven from the interior of the conductor, reside over the surface. Thus,

1. Under static conditions, **no charge** and **no electric field** can exist at any point **within the conducting material.**
2. The charge can exist on the surface of the conductor giving rise to surface charge density.
3. Within a conductor, the charge density is always zero.
4. The charge distribution on the surface depends on the shape of the surface.
5. The conductivity of an ideal conductor is infinite.
6. The conductor surface is an equipotential surface.

Example 5.4 : *A wire of diameter 2 mm and the conductivity 5×10^7 ℧/m has 10^{29} free electrons per m^3. It is subjected to an electric field of 10 mV/m. Determine, a) The free electron charge density b) The current density c) The current in the wire d) The drift velocity of the electrons.*
Given : The charge of one electron = -1.6×10^{-19} C.

Solution : a) $n = 10^{29}$ electrons/m^3 and $e = -1.6 \times 10^{-19}$ C

$\therefore \quad \rho_e$ = Free electron charge density = ne

$$= 10^{29} \times (-1.6 \times 10^{-19}) = -1.6 \times 10^{10} \text{ C/m}^3$$

b) $$J = \sigma E = 5 \times 10^7 \times 10 \times 10^{-3} = \textbf{500 kA/m}^2$$

c) $$I = JS = J \times \frac{\pi}{4} d^2 = 500 \times 10^3 \times \frac{\pi}{4} \times (2 \times 10^{-3})^2 = \textbf{1.5707 A}$$

d) $$J = \rho_e v_d$$

$$\therefore \quad 500 \times 10^3 = -1.6 \times 10^{10} v_d$$

$$\therefore \quad v_d = -3.125 \times 10^{-5} \text{ m/s} \quad \text{... Drift velocity}$$

The negative sign indicates it is opposite to the direction of the applied electric field $\bar{E}$.

Example 5.5 : *An aluminium conductor is 2000 ft long and has a circular cross-section with a diameter of 20 mm. If there is a d.c. voltage of 1.2 V between the ends find :*
a) The current density b) The current c) Power dissipated from the knowledge of circuit theory. Assume $\sigma = 3.82 \times 10^7$ mho/m for aluminium.

Solution : $L = 2000 \text{ ft} = 2000 \times (30 \times 10^{-2}) \text{ m} = 600 \text{ m}$

$$E = \frac{V}{L} = \frac{1.2}{600} = 2 \times 10^{-3} \text{ V/m}$$

a) $$J = \sigma E = 3.82 \times 10^7 \times 2 \times 10^{-3} = \textbf{76.4 kA/m}^2$$

b) $$I = JS = J \times \frac{\pi}{4} d^2 = 76.4 \times 10^3 \times \frac{\pi}{4} \times \left(20 \times 10^{-3}\right)^2 = \textbf{24 A}$$

c) $$P = \text{Power dissipated} = VI = \frac{V^2}{R} = I^2 R \text{ W}$$

$$= 1.2 \times 24 = \textbf{28.802 W}$$

5.5 Relaxation Time

The medium is called **homogeneous** when the physical characteristics of the medium do not vary from point to point but remain same everywhere throughout the medium. If the characteristics vary from point to point, the medium is called **nonhomogeneous** or **heterogeneous.** While the medium is called **linear** with respect to the electric field if the flux density $\bar{D}$ is directly proportional to the electric field $\bar{E}$. The relationship is through the permittivity of the medium. If $\bar{D}$ is not directly proportional to $\bar{E}$, the material is called **nonlinear.**

Consider a conducting material which is linear and homogeneous. The current density for such a material is,

$$\bar{J} = \sigma \bar{E} \quad \text{where} \quad \sigma = \text{Conductivity}$$

But $\bar{D} = \varepsilon \bar{E}$ ... Linear material

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\varepsilon}$$

$$\therefore \quad \bar{J} = \sigma \frac{\bar{D}}{\varepsilon} = \frac{\sigma}{\varepsilon} \bar{D} \quad \text{... (1)}$$

The point form of the continuity equation states that,

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{... (2)}$$

$$\therefore \quad \nabla \cdot \left(\frac{\sigma}{\varepsilon} \bar{D} \right) = -\frac{\partial \rho_v}{\partial t} \quad \text{... (3)}$$

$$\therefore \quad \frac{\sigma}{\varepsilon} \nabla \cdot \bar{D} = -\frac{\partial \rho_v}{\partial t} \quad \text{... (4)}$$

But $\nabla \cdot \bar{D} = \rho_v$ ... (5)

$$\therefore \quad \frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t} \quad \text{... (6)}$$

$$\therefore \quad \frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0 \quad \text{... (7)}$$

This is a differential equation in ρ_v whose solution is given by,

$$\boxed{\rho_v = \rho_0 e^{-(\sigma/\varepsilon)t} = \rho_0 e^{-t/\tau}} \quad \text{... (8)}$$

where ρ_0 = Charge density at (t = 0)

This shows that if there is a temporary imbalance of electrons inside the given material, the charge density decays exponentially with a time constant $\tau = \varepsilon / \sigma$ sec. This time is called **relaxation time.**

The **relaxation time** (τ) is defined as the time required by the charge density to decay to 36.8 % of its initial value.

$$\therefore \quad \boxed{\tau = \text{Relaxation time} = \frac{\varepsilon}{\sigma} \text{ sec}} \quad \text{... (9)}$$

For a pure conductor, the τ is very very small, of the order of 10^{-19} sec and thus for any imbalance inside the conductor, the charge density reduces to zero very quickly, forcing the electrons causing imbalance, to the surface of the conductor.

Key Point : ***This shows that under static conditions no free charge can remain within the conductor and it gets evenly distributed over the surface of the conductor.***

Example 5.6 : *Determine the relaxation time for silver, having $\sigma = 6.17 \times 10^7$ mho/m. If charge of density ρ_0 is placed within a silver block, find the charge density after one time constant and five time constants. Assume $\varepsilon = \varepsilon_0$.*

Solution : For silver $\sigma = 6.17 \times 10^7$ mho/m and $\varepsilon = \varepsilon_0$

$$\therefore \quad \tau = \text{Relaxation time} = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_0}{\sigma}$$

$$= \frac{8.854 \times 10^{-12}}{6.17 \times 10^7} = 1.435 \times 10^{-19} \text{ sec}$$

The charge density ρ_0 decays exponentially.

$$\therefore \quad \rho = \rho_0 e^{-t/\tau}$$

$$\text{At } t = 1\tau, \quad \rho = \rho_0 e^{-1} = 0.3768 \rho_0$$

$$\text{At } t = 5\tau, \quad \rho = \rho_0 e^{-5} = 6.73 \times 10^{-3} \rho_0$$

5.6 Dielectric Materials

It is seen that the conductors have large number of free electrons while insulators and dielectric materials do not have free charges. The charges in dielectrics are bound by the finite forces and hence called **bound charges.** As they are bound and not free, they cannot contribute to the conduction process. But if subjected to an electric field $\bar{E}$, they shift their relative positions, against the normal molecular and atomic forces. This shift in the relative positions of bound charges, allows the dielectric to store the energy.

The shifts in positive and negative charges are in opposite directions and under the influence of an applied electric field $\bar{E}$ such charges act like small electric dipoles.

Key Point : ***When the dipole results from the displacement of the bound charges, the dielectric is said to be polarized.***

And these electric dipoles produce an electric field which opposes the externally applied electric field. This process, due to which separation of bound charges results to produce electric dipoles, under the influence of electric field $\bar{E}$, is called ***polarization.***

5.6.1 Polarization

To understand the polarization, consider an atom of a dielectric. This consists of a nucleus with positive charge and negative charges in the form of revolving electrons in the orbits. The negative charge is thus considered to be in the form of cloud of electrons. This is shown in the Fig. 5.6.

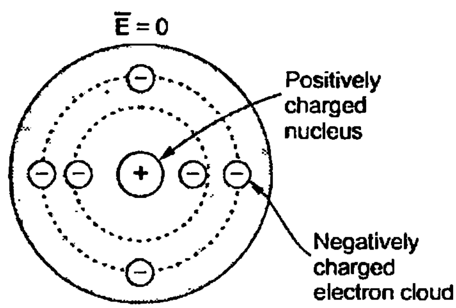
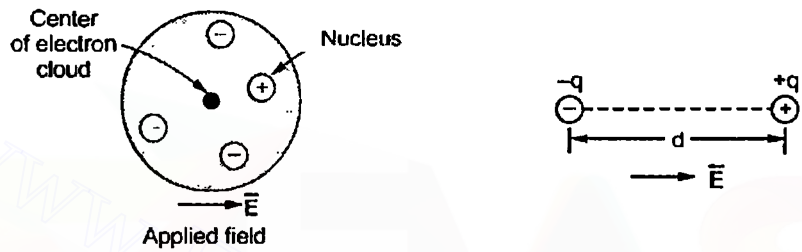


Fig. 5.6 Unpolarized atom of a dielectric

Note that $\bar{E}$ applied is zero. The number of positive charges is same as negative charges and hence atom is electrically neutral. Due to symmetry, both positive and negative charges can be assumed to be point charges of equal amount, coinciding at the centre. Hence there cannot exist an electric dipole. This is called **unpolarized atom.**

When electric field $\bar{E}$ is applied, the symmetrical distribution of charges gets disturbed. The positive charges experience a force $\bar{F} = Q\bar{E}$ while the negative charges experience a force $\bar{F} = -Q\bar{E}$ in the opposite direction.

Now there is separation between the nucleus and the centre of the electron cloud as shown in the Fig. 5.7 (a). Such an atom is called **polarized atom.**



(a) Polarized atom **(b) Equivalent dipole**

Fig. 5.7

It can be seen that an electron cloud has a centre separated from the nucleus. This forms an electric dipole. The equivalent dipole formed is shown in the Fig. 5.7 (b). The dipole gets aligned with the applied field. This process is called **polarization** of dielectrics.

There are two types of dielectrics,

1. Nonpolar and 2. Polar.

In **nonpolar** molecules, the dipole arrangement is totally absent, in absence of electric field E. It results only when an externally field $\bar{E}$ is applied to it. In **polar** molecules, the permanent displacements between centres of positive and negative charges exist. Thus dipole arrangements exist without application of $\bar{E}$. But such dipoles are randomly oriented. Under the application of $\bar{E}$, the dipoles experience torque and they align with the direction of the applied field $\bar{E}$. This is called polarization of polar molecules.

The examples of nonpolar molecules are hydrogen, oxygen and the rare gases. The examples of polar molecules are water, sulphur dioxide, hydrochloric acid etc.

5.6.2 Mathematical Expression for Polarization

When the dipole is formed due to polarization, there exists an electric dipole moment $\bar{p}$.

$$\boxed{\bar{p} = Q\,\bar{d}} \quad \text{... (1)}$$

where Q = Magnitude of one of the two charges

$\bar{d}$ = Distance vector from negative to positive charge

Let n = Number of dipoles per unit volume

Δv = Total volume of the dielectric

N = Total dipoles = $n \Delta v$

Then the total dipole moment is to be obtained using superposition principle as,

$$\bar{P}_{total} = Q_1 \bar{d}_1 + Q_2 \bar{d}_2 + ... + Q_n \bar{d}_n = \sum_{i=1}^{n\Delta v} Q_i \bar{d}_i \quad ... (2)$$

If dipoles are randomly oriented, $\bar{p}_{total}$ is zero but if dipoles are aligned in the direction of applied $\bar{E}$ then $\bar{p}_{total}$ has a significant value.

The polarization $\bar{P}$ is defined as the total dipole moment per unit volume.

$$\therefore \quad \boxed{\bar{P} = \lim_{\Delta v \to 0} \frac{\sum_{i=1}^{n\Delta v} Q_i \bar{d}_i}{\Delta v}} \quad ... (3)$$

It is measured in coulombs per square metre (C/m^2).

It can be seen that the units of polarization are same as that of flux density $\bar{D}$. Thus polarization increases the electric flux density in a dielectric medium. Hence we can write, flux density in a dielectric is,

$$\boxed{\bar{D} = \varepsilon_0 \bar{E} + \bar{P}} \quad ... (4)$$

For isotropic and linear medium, the $\bar{P}$ and $\bar{E}$ are parallel to each other at every point and related to each other as,

$$\boxed{\bar{P} = \chi_e \varepsilon_0 \bar{E}} \quad ... (5)$$

where χ_e = Dimensionless quantity called **electric susceptibility** of the material.

The susceptibility tells us how sensitive is a given dielectric to the applied electric field $\bar{E}$.

Substituting (5) in (4),

$$\bar{D} = \varepsilon_0 \bar{E} + \chi_e \varepsilon_0 \bar{E}$$

$$\therefore \quad \bar{D} = (\chi_e + 1)\varepsilon_0 \bar{E} \quad ... (6)$$

$$\therefore \quad \bar{D} = \varepsilon \bar{E} \quad ... (7)$$

where $\varepsilon = \varepsilon_R \varepsilon_0 \quad ... (8)$

The quantity $\chi_e + 1$ is defined as **relative permittivity** or **dielectric constant** of the dielectric material.

$$\therefore \quad \boxed{\varepsilon_R = \chi_e + 1} \quad ... (9)$$

While the ε is called **permittivity** of the dielectric. Note that in **anisotropic** or **nonisotropic** materials the $\bar{D}$, $\bar{E}$ and $\bar{P}$ are not parallel to each other and ε and χ_e vary in all directions and have nine different components. The discussion of anisotropic materials is beyond the scope of this book.

5.6.3 Properties of Dielectric Materials

The various properties of dielectric materials are,

1. The dielectrics do not contain any free charges but contain **bound charges.**
2. Bound charges are under the internal molecular and atomic forces and cannot contribute to the conduction.
3. When subjected to an external field $\bar{E}$, the bound charges shift their relative positions. Due to this, small electric dipoles get induced inside the dielectric. This is called **polarization.**
4. Due to the polarization, the dielectrics can store the energy.
5. Due to the polarization, the flux density of the dielectric increases by amount equal to the polarization.
6. The induced dipoles produce their own electric field and align in the direction of the applied electric field.
7. When polarization occurs, the volume charge density is formed inside the dielectric while the surface charge density is formed over the surface of the dielectric.
8. The electric field outside and inside the dielectric gets modified due to the induced electric dipoles.

5.6.4 Dielectric Strength

The ideal dielectric is nonconducting but practically no dielectric can be ideal. As the electric field applied to dielectric increases sufficiently, due to the force exerted on the molecules, the electrons in the dielectric become free. Under such large electric field, the dielectric becomes conducting due to presence of large number of free electrons. This condition of dielectric is called **dielectric breakdown.** All kinds of dielectrics such as solids, liquids and gases show the tendency of breakdown under large electric field. The breakdown depends on the nature of material, the time and magnitude of applied electric field and atmospheric conditions such as temperature, moisture, humidity etc.

Key Point : *The minimum value of the applied electric field at which the dielectric breaks down is called* ***dielectric strength*** *of that dielectric.*

The dielectric strength is measured in V/m or kV/cm. It also can be stated as the maximum value of electric field under which a dielectric can sustain without breakdown. Once breakdown occurs, dielectric starts conducting and no longer behaves as dielectric. Hence all the dielectrics are assumed to be either ideal or are not in a breakdown condition.

Example 5.7 : *Find the magnitude of $\bar{D}$ and $\bar{P}$ for a dielectric material in which $|\bar{E}| = 0.15$ mV/m and $\chi_e = 4.25$.*

Solution : For a dielectric medium,

$$\bar{D} = \varepsilon_0 \varepsilon_R \bar{E}$$

where $\varepsilon_R = \chi_e + 1 = 4.25 + 1 = 5.25$

$$\therefore \quad |\bar{D}| = 8.854\times10^{-12}\times5.25\times0.15\times10^{-3}$$

$$= 6.9725\times10^{-15} \text{ C/m}^2$$

and $\bar{P} = \chi_e \varepsilon_0 \bar{E}$

$$\therefore \quad |\bar{P}| = 4.25\times8.854\times10^{-12}\times0.15\times10^{-3}$$

$$= 5.644\times10^{-15} \text{ C/m}^2$$

Example 5.8 : *Find the polarization in dielectric material with $\varepsilon_R = 2.8$ if $\bar{D} = 3\times10^{-7}$ C/m^2.*

Solution : For the dielectric,

$$\bar{P} = \chi_e \varepsilon_0 \bar{E}$$

Now $\varepsilon_R = \chi_e + 1$

$$\therefore \quad \chi_e = \varepsilon_{R-1} = 2.8 - 1 = 1.8$$

And $\bar{D} = \varepsilon_0 \varepsilon_R \bar{E}$

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0 \varepsilon_R} = \frac{3\times10^{-7}}{8.854\times10^{-12}\times2.8}$$

$$= 12.101\times10^3 \text{ V/m}$$

$$\therefore \quad \bar{P} = 1.8\times8.854\times10^{-12}\times12.101\times10^3$$

$$= 1.9285\times10^{-7} \text{ C/m}^2$$

5.7 Boundary Conditions

When an electric field passes from one medium to other medium, it is important to study the conditions at the boundary between the two media. The conditions existing at the boundary of the two media when field passes from one medium to other are called **boundary conditions.** Depending upon the nature of the media, there are two situations of the boundary conditions,

1. Boundary between conductor and free space.
2. Boundary between two dielectrics with different properties.

The free space is nothing but a dielectric hence first case is nothing but the boundary between conductor and a dielectric. For studying the boundary conditions, the Maxwell's equations for electrostatics are required.

$$\oint \bar{E} \cdot \overline{dL} = 0 \quad \text{and} \quad \oint \bar{D} \cdot d\bar{S} = Q$$

Similarly the field intensity $\bar{E}$ is required to be decomposed into two components namely tangential to the boundary (E_{tan}) and normal to the boundary $(\bar{E}_N)$.

$$\therefore \qquad \bar{E} = \bar{E}_{tan} + \bar{E}_N$$

Similar decomposition is required for flux density $\bar{D}$ as well.

Let us study the various cases of boundary conditions in detail.

5.8 Boundary Conditions between Conductor and Free Space

Consider a boundary between conductor and free space. The conductor is ideal having infinite conductivity. Such conductors are copper, silver etc. having conductivity of the order of 10^6 S/m and can be treated ideal. For ideal conductors it is known that,

1. The field intensity inside a conductor is zero and the flux density inside a conductor is zero.

2. No charge can exist within a conductor. The charge appears on the surface in the form of surface charge density.

3. The charge density within the conductor is zero.

Thus $\bar{E}$, $\bar{D}$ and ρ_v within the conductor are zero. While ρ_S is the surface charge density on the surface of the conductor.

To determine the boundary conditions let us use the closed path and the Gaussian surface.

Consider the conductor free space boundary as shown in the Fig. 5.8.

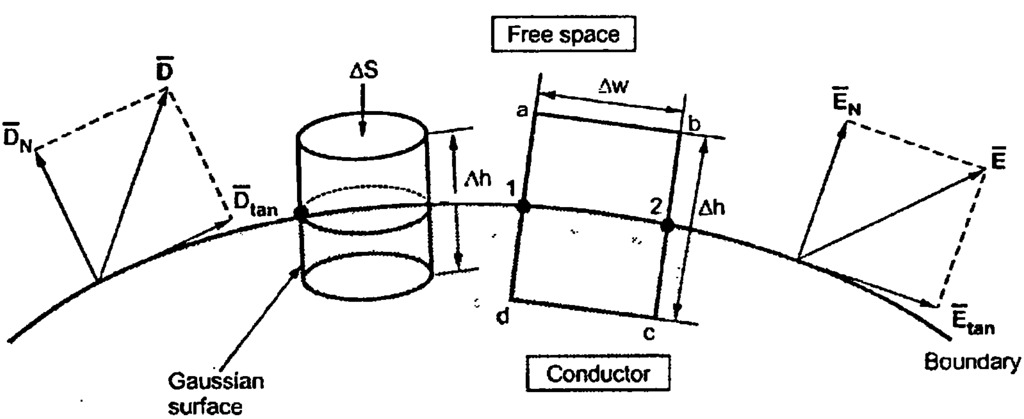


Fig. 5.8 Boundary between conductor and free space

5.8.1 $\bar{E}$ at the Boundary

Let $\bar{E}$ be the electric field intensity, in the direction shown in the Fig. 5.11, making some angle with the boundary. This $\bar{E}$ can be resolved into two components :

1. The component tangential to the surface ($\bar{E}_{tan}$).
2. The component normal to the surface ($\bar{E}_N$).

It is known that,

$$\oint \bar{E} \cdot d\bar{L} = 0 \quad \text{... (1)}$$

The integral of $\bar{E} \cdot d\bar{L}$ carried over a closed contour is zero i.e. work done in carrying unit positive charge along a closed path is zero.

Consider a rectangular closed path abcda as shown in the Fig. 5.8. It is traced in clockwise direction as a-b-c-d-a and hence $\oint \bar{E} \cdot d\bar{L}$ can be divided into four parts.

$$\oint \bar{E} \cdot d\bar{L} = \int_a^b \bar{E} \cdot d\bar{L} + \int_b^c \bar{E} \cdot d\bar{L} + \int_c^d \bar{E} \cdot d\bar{L} + \int_d^a \bar{E} \cdot d\bar{L} = 0 \quad \text{... (2)}$$

The closed contour is placed in such a way that its two sides a-b and c-d are parallel to tangential direction to the surface while the other two are normal to the surface, at the boundary.

The rectangle is an elementary rectangle with elementary height Δh and elementary width Δw. The rectangle is placed in such a way that half of it is in the conductor and remaining half is in the free space. Thus $\Delta h/2$ is in the conductor and $\Delta h/2$ is in the free space.

Now the portion c-d is in the conductor where $\bar{E} = 0$ hence the corresponding integral is zero.

$$\therefore \quad \int_a^b \bar{E} \cdot d\bar{L} + \int_b^c \bar{E} \cdot d\bar{L} + \int_d^a \bar{E} \cdot d\bar{L} = 0 \quad \text{... (3)}$$

As the width Δw is very small, E over it can be assumed constant and hence can be taken out of integration.

$$\therefore \quad \int_a^b \bar{E} \cdot d\bar{L} = \bar{E} \int_a^b d\bar{L} = \bar{E}(\Delta w) \quad \text{... (4)}$$

But Δw is along tangential direction to the boundary in which direction $\bar{E} = \bar{E}_{tan}$.

$$\therefore \quad \int_a^b \bar{E} \cdot d\bar{L} = E_{tan}(\Delta w) \quad \text{where} \quad E_{tan} = |\bar{E}_{tan}| \quad \text{... (5)}$$

Now b-c is parallel to the normal component so we have $\bar{E} = \bar{E}_N$ along this direction. Let $E_N = |\bar{E}_N|$.

Over the small height Δh, E_N can be assumed constant and can be taken out of integration.

$$\therefore \quad \int_b^c \bar{E} \cdot d\bar{L} = \bar{E}\int_b^c d\bar{L} = E_N \int_b^c d\bar{L} \quad \text{... (6)}$$

But out of b-c, b-2 is in free space and 2-c is in the conductor where $\bar{E} = 0$.

$$\therefore \quad \int_b^c d\bar{L} = \int_b^2 d\bar{L} + \int_2^c d\bar{L} = \frac{\Delta h}{2} + 0 = \frac{\Delta h}{2} \quad \text{... (7)}$$

$$\therefore \quad \int_b^c \bar{E} \cdot d\bar{L} = E_N\left(\frac{\Delta h}{2}\right) \quad \text{... (8)}$$

Similarly for path d-a, the condition is same as for the path b-c, only direction is opposite.

$$\therefore \quad \int_d^a \bar{E} \cdot d\bar{L} = -E_N\left(\frac{\Delta h}{2}\right) \quad \text{... (9)}$$

Substituting equations (4), (8) and equation (9) in (3) we get,

$$\therefore \quad E_{tan}\,\Delta w + E_N\left(\frac{\Delta h}{2}\right) - E_N\left(\frac{\Delta h}{2}\right) = 0 \quad \text{... (10)}$$

$$\therefore \quad E_{tan}\,\Delta w = 0 \qquad \text{But } \Delta w \neq 0 \text{ as finite}$$

$$\therefore \quad \boxed{E_{tan} = 0} \quad \text{... (11)}$$

Thus the tangential component of the electric field intensity is zero at the boundary between conductor and free space.

Key Point : ***Thus the $\bar{E}$ at the boundary between conductor and free space is always in the direction perpendicular to the boundary.***

$$\text{Now} \quad \bar{D} = \varepsilon_0 \bar{E} \quad \text{for free space}$$

$$\therefore \quad \boxed{D_{tan} = \varepsilon_0 E_{tan} = 0} \quad \text{... (12)}$$

Thus the tangential component of electric flux density is zero at the boundary between conductor and free space.

Key Point : *Hence electric flux density $\bar{D}$ is also only in the normal direction at the boundary between the conductor and the free space.*

5.8.2 D_N at the Boundary

To find normal component of $\bar{D}$, select a closed Gaussian surface in the form of right circular cylinder as shown in the Fig. 5.8. Its height is Δh and is placed in such a way that $\Delta h/2$ is in the conductor and remaining $\Delta h/2$ is in the free space. Its axis is in the normal direction to the surface.

According to Gauss's law, $\oint_S \bar{D} \cdot d\bar{S} = Q$

The surface integral must be evaluated over three surfaces,

i) Top, ii) Bottom and iii) Lateral.

Let the area of top and bottom is same equal to ΔS.

$$\therefore \quad \int_{top} \bar{D} \cdot d\bar{S} + \int_{bottom} \bar{D} \cdot d\bar{S} + \int_{lateral} \bar{D} \cdot d\bar{S} = Q \qquad \text{... (13)}$$

The bottom surface is in the conductor where $\bar{D} = 0$ hence corresponding integral is zero.

The top surface is in the free space and we are interested in the boundary condition hence top surface can be shifted at the boundary with $\Delta h \to 0$.

$$\therefore \quad \int_{top} \bar{D} \cdot d\bar{S} + \int_{lateral} \bar{D} \cdot d\bar{S} = Q \qquad \text{... (14)}$$

The lateral surface area is $2\pi r \Delta h$ where r is the radius of the cylinder. But as $\Delta h \to 0$, this area reduces to zero and corresponding integral is zero.

While only component of $\bar{D}$ present is the normal component having magnitude D_N. The top surface is very small over which D_N can be assumed constant and can be taken out of integration.

$$\therefore \quad \int_{top} \bar{D} \cdot d\bar{S} = D_N \int_{top} d\bar{S} = D_N \Delta S \qquad \text{... (15)}$$

From Gauss's law,

$$\therefore \quad D_N \Delta S = Q \qquad \text{... (16)}$$

But at the boundary, the charge exists in the form of surface charge density $\rho_S \; C/m^2$.

$$\therefore \quad Q = \rho_S \Delta S \qquad \text{... (17)}$$

Equating equation (16) and (17),

$$\therefore \quad D_N \Delta S = \rho_S \Delta S$$

$$\therefore \quad \boxed{D_N = \rho_S} \qquad \text{... (18)}$$

Thus the flux leaves the surface normally and the normal component of flux density is equal to the surface charge density.

$$\therefore \quad D_N = \varepsilon_0 E_N = \rho_S \qquad \text{... (19)}$$

$$\therefore \quad \boxed{E_N = \frac{\rho_S}{\varepsilon_0}} \qquad \text{... (20)}$$

Key Point : *Note that as the tangential component of $\bar{E}$ i.e. $E_{tan} = 0$, the surface of the conductor is an equipotential surface. The potential difference along any path on the surface of the conductor is $-\int \bar{E} \cdot d\bar{L}$ and as $E = \bar{E}_{tan} = 0$, the potential difference is zero. Thus all points on the conductor surface are at the same potential.*

5.8.3 Boundary Conditions between Conductor and Dielectric

The free space is a dielectric with $\varepsilon = \varepsilon_0$. Thus if the boundary is between conductor and dielectric with $\varepsilon = \varepsilon_0 \varepsilon_r$.

$$\therefore \quad E_{tan} = D_{tan} = 0 \qquad \text{... (21)}$$

$$D_N = \rho_S \qquad \text{... (22)}$$

$$\text{and} \quad E_N = \frac{\rho_S}{\varepsilon} = \frac{\rho_S}{\varepsilon_0 \varepsilon_r} \qquad \text{... (23)}$$

Example 5.9 : *A potential field is given as $V = 100\, e^{-5x} \sin 3y \cos 4z$ V. Let point P $(0.1, \pi/12, \pi/24)$ be located at a conductor free space boundary. At point P, find the magnitudes of,*

a) V b) $\bar{E}$ c) E_t d) E_N e) $\bar{D}$ f) D_N g) ρ_S.

Solution : a) At P, $x = 0.1$, $y = \frac{\pi}{12}$, $z = \frac{\pi}{24}$

$$\therefore \quad V = 100\, e^{-0.5} \sin \frac{3\pi}{12} \cos \frac{4\pi}{24} = 37.1422 \text{ V} \qquad \text{... Use radian mode}$$

b) $$\bar{E} = -\nabla V = -\left(\frac{\partial V_x}{\partial x} \bar{a}_x + \frac{\partial V_y}{\partial y} \bar{a}_y + \frac{\partial V_z}{\partial z} \bar{a}_z \right)$$

$$= -100[-5 e^{5x} \sin 3y \cos 4z \bar{a}_x + e^{-5x} (3)(\cos 3y)(\cos 4z) \bar{a}_y$$
$$+ e^{-5x} (\sin 3y)(4)(-\sin 4z) \bar{a}_z]$$

At P, $$\bar{E} = [-100[-1.857 \bar{a}_x + 1.114 \bar{a}_y - 0.85776 \bar{a}_z]$$

$$= \mathbf{+\ 185.7\ \bar{a}_x - 111.4\ \bar{a}_y + 85.776\ \bar{a}_z \ \ V/m}$$

$$\therefore \quad |\bar{E}| = \mathbf{232.9206\ V/m}$$

c) $$E_t = \mathbf{0\ \ V/m} \text{ as P is on the boundary}$$

d) $$E_N = |\bar{E}| = \mathbf{232.9206\ V/m}$$

e) $$\bar{D} = \varepsilon_0 \bar{E} = 8.854 \times 10^{-12} [185.7 \bar{a}_x - 111.4 \bar{a}_y + 85.776 \bar{a}_z]$$

$$= \mathbf{1.588\ \bar{a}_x - 0.9529\ \bar{a}_y + 0.7337\ \bar{a}_z \ \ nC/m^2}$$

$$\therefore \quad |\bar{D}| = \mathbf{1.992\ \ nC/m^2}$$

f) $$D_N = |\bar{D}| = \mathbf{1.992\ \ nC/m^2}$$

g) $$D_N = \rho_S = \mathbf{1.992\ \ nC/m^2}$$

5.9 Boundary Conditions between Two Perfect Dielectrics

Let us consider the boundary between two perfect dielectrics. One dielectric has permittivity ε_1 while the other has permittivity ε_2. The interface is shown in the Fig. 5.9.

The $\bar{E}$ and $\bar{D}$ are to be obtained again by resolving each into two components, tangential to the boundary and normal to the surface.

Consider a closed path abcda rectangular in shape having elementary height Δh and elementary width Δw, as shown in the Fig. 5.9. It is placed in such a way that $\Delta h/2$ is in dielectric 1 while the remaining is dielectric 2. Let us evaluate the integral of $\bar{E} \cdot d\bar{L}$ along this path, tracing it in clockwise direction as a-b-c-d-a.

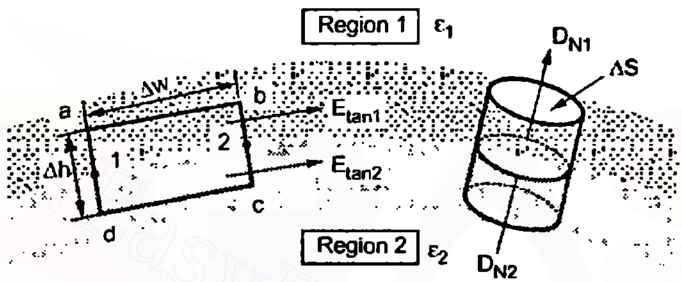


Fig. 5.9 Boundary between two perfect dielectrics

$$\oint \bar{E} \cdot d\bar{L} = 0 \qquad \text{... (1)}$$

$$\therefore \int_a^b \bar{E} \cdot d\bar{L} + \int_b^c \bar{E} \cdot d\bar{L} + \int_c^d \bar{E} \cdot d\bar{L} + \int_d^a \bar{E} \cdot d\bar{L} = 0 \qquad \text{... (2)}$$

Now $\quad \bar{E}_1 = \bar{E}_{1t} + \bar{E}_{1N}$ ... (3)

and $\quad \bar{E}_2 = \bar{E}_{2t} + \bar{E}_{2N}$ ... (4)

Both $\bar{E}_1$ and $\bar{E}_2$ in the respective dielectrics have both the components, normal and tangential.

Let $\quad |\bar{E}_{1t}| = E_{tan1}, \quad |\bar{E}_{2t}| = E_{tan2}$

$\quad |\bar{E}_{1N}| = E_{1N}, \quad |\bar{E}_{2N}| = E_{2N}$

Now for the rectangle to be reduced at the surface to analyse boundary conditions, $\Delta h \to 0$. As $\Delta h \to 0$, $\int_b^c$ and $\int_d^a$ become zero as these are line integrals along Δh and $\Delta h \to 0$.

Hence equation (2) reduces to,

$$\int_a^b \bar{E} \cdot d\bar{L} + \int_c^d \bar{E} \cdot d\bar{L} = 0 \qquad \text{... (5)}$$

Now a-b is in dielectric 1 hence the corresponding component of $\bar{E}$ is E_{tan1} as a-b direction is tangential to the surface.

$$\therefore \quad \int_a^b \bar{E}\cdot d\bar{L} = E_{tan\,1}\int_a^b d\bar{L} = E_{tan\,1}\,\Delta w \quad \text{... (6)}$$

While c-d is in dielectric 2 hence the corresponding component of $\bar{E}$ is E_{tan2} as c-d direction is also tangential to the surface. But direction c-d is opposite to a-b hence corresponding integral is negative of the integral obtained for path a-b.

$$\therefore \quad \int_c^d \bar{E}\cdot d\bar{L} = -E_{tan\,2}\,\Delta w \quad \text{... (7)}$$

Substituting equation (6) and equation (7) in equation (5) we get,

$$E_{tan\,1}\,\Delta w - E_{tan\,2}\,\Delta w = 0 \quad \text{... (8)}$$

$$\therefore \quad \boxed{E_{tan\,1} = E_{tan\,2}} \quad \text{... (9)}$$

Thus the tangential components of field intensity at the boundary in both the dielectrics remain same i.e. electric field intensity is continuous across the boundary.

The relation between $\bar{D}$ and $\bar{E}$ is known as,

$$\bar{D} = \varepsilon\bar{E} \quad \text{... (10)}$$

Hence if $D_{tan\,1}$ and $D_{tan\,2}$ are magnitudes of the tangential components of $\bar{D}$ in dielectric 1 and 2 respectively then,

$$D_{tan\,1} = \varepsilon_1 E_{tan\,1} \text{ and } D_{tan\,2} = \varepsilon_2 E_{tan\,2} \quad \text{... (11)}$$

$$\therefore \quad \frac{D_{tan\,1}}{\varepsilon_1} = \frac{D_{tan\,2}}{\varepsilon_2} \quad \text{... (12)}$$

$$\therefore \quad \boxed{\frac{D_{tan\,1}}{D_{tan\,2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}} \quad \text{... (13)}$$

Thus tangential components of $\bar{D}$ undergoes some change across the interface hence tangential $\bar{D}$ is said to be discontinuous across the boundary.

To find the **normal components**, let us use Gauss's law. Consider a Gaussian surface in the form of right circular cylinder, placed in such a way that half of it lies in dielectric 1 while the remaining half in dielectric 2. The height $\Delta h \to 0$ hence flux leaving from its lateral surface is zero. The surface area of its top and bottom is ΔS.

$$\therefore \quad \oint \bar{D}\cdot d\bar{S} = Q \quad \text{... (14)}$$

$$\therefore \left[\int_{top} + \int_{bottom} + \int_{lateral}\right]\bar{D}\cdot d\bar{S} = Q \quad \text{... (15)}$$

But $\int_{\text{lateral}} \overline{D} \cdot d\overline{S} = 0 \quad \text{as} \quad \Delta h \to 0$ ... (16)

$$\therefore \int_{\text{top}} \overline{D} \cdot d\overline{S} + \int_{\text{bottom}} \overline{D} \cdot d\overline{S} = Q \qquad \text{... (17)}$$

The flux leaving normal to the boundary is normal to the top and bottom surfaces.

$\therefore |\overline{D}| = D_{N1}$ for dielectric 1 and D_{N2} for dielectric 2.

And as top and bottom surfaces are elementary, flux density can be assumed constant and can be taken out of integration.

$$\therefore \quad \int_{\text{top}} \overline{D} \cdot d\overline{S} = D_{N1} \int_{\text{top}} d\overline{S} = D_{N1} \Delta S \qquad \text{... (18)}$$

For top surface, the direction of D_N is entering the boundary while for bottom surface, the direction of D_N is leaving the boundary. **Both are opposite in direction, at the boundary.**

$$\therefore \quad \int_{\text{bottom}} \overline{D} \cdot d\overline{S} = -D_{N2} \int_{\text{bottom}} d\overline{S} = -D_{N2} \Delta S \qquad \text{... (19)}$$

$$\therefore D_{N1} \Delta S - D_{N2} \Delta S = Q \qquad \text{... (20)}$$

But $\quad Q = \rho_S \Delta S$ ... (21)

$$\therefore \quad D_{N1} - D_{N2} = \rho_S \qquad \text{... (22)}$$

There is no free charge available in perfect dielectric hence no free charge can exist on the surface. All charges in dielectric are bound charges and are not free. Hence **at the ideal dielectric media boundary the surface charge density ρ_S can be assumed zero.**

$$\therefore \quad \rho_S = 0$$

$$\therefore \quad D_{N1} - D_{N2} = 0$$

$$\therefore \quad \boxed{D_{N1} = D_{N2}} \qquad \text{... (23)}$$

Hence **the normal component of flux density $\overline{D}$ is continuous at the boundary between the two perfect dielectrics.**

Now $D_{N1} = \varepsilon_1 E_{N1}$ and $D_{N2} = \varepsilon_2 E_{N2}$

$$\therefore \quad \frac{D_{N1}}{D_{N2}} = \frac{\varepsilon_1 E_{N1}}{\varepsilon_2 E_{N2}} = 1 \qquad \text{... As } D_{N1} = D_{N2}$$

$$\therefore \quad \boxed{\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}} \qquad \text{... (24)}$$

The normal components of the electric field intensity $\overline{E}$ are inversely proportional to the relative permittivities of the two media.

$$\therefore \quad \boxed{D_2 = D_1\sqrt{\cos^2\theta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \sin^2\theta_1}} \qquad \text{... (32)}$$

Similarly magnitude of E_2 can be obtained as,

$$\boxed{E_2 = E_1\sqrt{\sin^2\theta_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \cos^2\theta_1}} \qquad \text{... (33)}$$

The equation shows that

1. $\bar{D}$ is larger in the region of larger permittivity.
2. $\bar{E}$ is larger in the region of smaller permittivity.
3. $|\bar{D}_1| = |\bar{D}_2|$ if $\theta_1 = \theta_2 = 0°$.
4. $|\bar{E}_1| = |\bar{E}_2|$ if $\theta_1 = \theta_2 = 90°$.

To find the angles θ_1 and θ_2, with respect to normal **use the dot product** if normal direction to the boundary is known.

Example 5.10 : *The region with $z<0$ is characterised by ε_{r2} = 2 and $z>0$ by $\varepsilon_{r1} = 5$. If $\bar{D}_1 = 2\bar{a}_x + 5\bar{a}_y - 3\bar{a}_z$ nC / m², find : a) $\bar{D}_2$ b) $\bar{D}_{N2}$ c) $\bar{D}_{tan\,2}$ d) Energy density in each region e) The angle that $\bar{D}_2$ makes with z axis f) $\frac{|D_2|}{|D_1|}$ g) $\frac{|P_2|}{|P_1|}$.*

Solution : The two media are separated by z = 0 plane and $\pm\bar{a}_z$ are the directions of normal to the surface.

$$\bar{D}_1 = 2\bar{a}_x + 5\bar{a}_y - 3\bar{a}_z \quad \text{nC/m}^2$$

$$\bar{D}_1 = \bar{D}_{N1} + \bar{D}_{tan\,1}$$

Normal direction to the surface is $\pm\bar{a}_z$ hence the part of $\bar{D}_1$ in the direction of $\pm\bar{a}_z$ is $\bar{D}_{N1}$.

$$\therefore \quad \bar{D}_{N1} = -3\,\bar{a}_z \quad \text{nC/m}^2$$

$$\therefore \quad \bar{D}_{tan\,1} = \bar{D} - \bar{D}_{N1} = 2\bar{a}_x + 5\bar{a}_y \quad \text{nC/m}^2$$

According to boundary conditions,

$$\bar{D}_{N1} = \bar{D}_{N2} = -3\,a_z$$

while
$$\frac{\bar{D}_{tan\,1}}{\bar{D}_{tan\,2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

$$\therefore \quad \frac{2\bar{a}_x + 5\bar{a}_y}{\bar{D}_{tan\,2}} = \frac{5}{2}$$

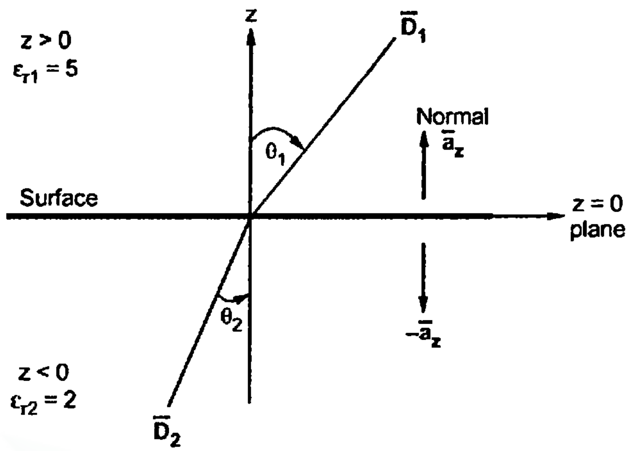


Fig. 5.11

$$\therefore \quad \bar{D}_{tan\,2} = \frac{2}{5}(2\bar{a}_x + 5\bar{a}_y) = 0.8\,\bar{a}_x + 2\bar{a}_y \quad nC/m^2$$

$$\therefore \quad \bar{D}_2 = \bar{D}_{N2} + \bar{D}_{tan\,2} = 0.8\,\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z \quad nC/m^2$$

$$\text{Energy density } W_{E1} = \frac{1}{2}\frac{|D_1|^2}{\varepsilon_1} = \frac{1}{2}\frac{|\bar{D}_1|^2}{\varepsilon_0\varepsilon_{r1}}$$

$$= \frac{1}{2} \times \frac{\left(\sqrt{(2)^2 + (5)^2 + (-3)^2}\right)^2 \times (10^{-9})^2}{8.854 \times 10^{-12} \times 5}$$

$$= 0.4291 \quad \mu J/m^3$$

$$\text{and} \quad W_{E2} = \frac{1}{2}\frac{|D_2|^2}{\varepsilon_2} = \frac{1}{2}\frac{\left[\sqrt{(0.8)^2 + (2)^2 + (-3)^2} \times 10^{-9}\right]^2}{8.854 \times 10^{-12} \times 2}$$

$$= 0.3851 \quad \mu J/m^3$$

To find angle of $\bar{D}_2$ with z axis i.e. $-\bar{a}_z$ is to be obtained by dot product.

$$\therefore \quad \bar{D}_2 \cdot (-\bar{a}_z) = |\bar{D}_2|\,|\bar{a}_z| \cos\theta_2$$

$$\therefore \quad [0.8\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z] \cdot (-\bar{a}_z) = \sqrt{(0.8)^2 + (2)^2 + (-3)^2} \cos\theta_2 \qquad \ldots |\bar{a}_z| = 1$$

$$\therefore \quad +3 = 3.6932 \cos\theta_2$$

$$\therefore \quad \theta_2 = 35.678^\circ$$

$$\text{Alternatively,} \quad \tan\theta_2 = \frac{D_{tan\,2}}{D_{N2}} = \frac{\sqrt{(0.8)^2 + (2)^2}}{3}$$

$$\therefore \quad \theta_2 = 35.678^\circ$$

Now $$D_2 = \sqrt{(0.8)^2+(2)^2+(-3)^2} = 3.6932$$

$$D_1 = \sqrt{(2)^2+(5)^2+(-3)^2} = 6.1644$$

$$\therefore \quad \frac{|D_2|}{|D_1|} = \frac{3.6932}{6.1644} = 0.599$$

$$|\bar{P}| = \chi_e \varepsilon_0 |\bar{E}| = \chi_e \varepsilon_0 \frac{|\bar{D}|}{(\chi_e+1)\varepsilon_0}$$

as $$|\bar{D}| = (\chi_e+1)\varepsilon_0 |\bar{E}|$$

$$\therefore \quad |\bar{P}| = \frac{(\chi_e)}{(\chi_e+1)}|\bar{D}| \quad \text{But} \quad \varepsilon_R = \chi_e+1$$

$$\therefore \quad |\bar{P}| = \frac{(\varepsilon_R-1)}{\varepsilon_R}|\bar{D}|$$

$$\therefore \quad \frac{|P_2|}{|P_1|} = \frac{\varepsilon_{R2}-1}{\varepsilon_{R2}} \times |\bar{D}_2| \times \frac{\varepsilon_{R1}}{\varepsilon_{R1}-1} \times \frac{1}{|\bar{D}_1|}$$

$$= 0.599 \times \frac{(2-1)}{2} \times \frac{5}{(5-1)} = 0.3743$$

Example 5.11 : *Given that $\bar{E}_1 = 2\bar{a}_x - 3\bar{a}_y + 5\bar{a}_z$ V/m at the charge free dielectric interface as shown in the Fig. 5.12. Find $\bar{D}_2$ and the angle θ_1, θ_2.*

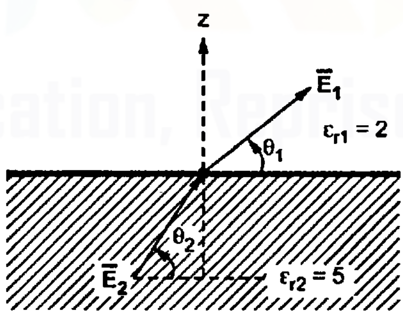


Fig. 5.12

Solution : As shown, z axis is normal to the surface. So part of $\bar{E}_1$ which is in the direction of $\bar{a}_z$ is normal component of $\bar{E}_1$.

$$\therefore \quad \bar{E}_{N1} = 5\bar{a}_z \quad \text{V/m}$$

And $$\bar{E}_1 = \bar{E}_{N1} + E_{\tan 1}$$

$$\therefore \quad \bar{E}_{tan\,1} = \bar{E}_1 - \bar{E}_{N1} = 2\bar{a}_x - 3\bar{a}_y \quad V/m \qquad \text{... (1)}$$

At the boundary of perfect dielectrics,

$$\bar{E}_{tan\,1} = \bar{E}_{tan\,2} = 2\bar{a}_x - 3\bar{a}_y \quad V/m \qquad \text{... (2)}$$

Now $\quad \bar{D}_{tan\,2} = \varepsilon_2\,\bar{E}_{tan\,2} = \varepsilon_0 \varepsilon_{r2}\,\bar{E}_{tan\,2}$ ... (3)

And $\quad \bar{D}_{N1} = \varepsilon_1\,\bar{E}_{N1} = \varepsilon_0\,\varepsilon_{r1}\,\bar{E}_{N1}$ ... (4)

But $\quad \bar{D}_{N2} = \bar{D}_{N1} = \varepsilon_0 \varepsilon_{r1}\,\bar{E}_{N1}$ ... (5)

And $\quad \bar{D}_2 = \bar{D}_{N2} + D_{tan\,2} = \varepsilon_0 \varepsilon_{r1}\,E_{N1} + \varepsilon_0 \varepsilon_{r2}\,\bar{E}_{tan2}$

$$= \varepsilon_0\,[5(2\bar{a}_x - 3\bar{a}_y) + 2(5\bar{a}_z)]$$

$$= 8.854 \times 10^{-12}\left[10\bar{a}_x - 15\bar{a}_y + 10\bar{a}_z\right]$$

$$\therefore \quad \bar{D}_2 = \mathbf{88.54\,\bar{a}_x - 132.81\,\bar{a}_y + 88.54\,\bar{a}_z \quad pC/m^3}$$

As D_{N1}, E_{N1} are in same direction and D_1, E_1 are in same direction,

$$D_{N1} = D_1 \cos\theta_1' \quad \text{i.e.} \quad E_{N1} = E_1 \cos\theta_1'$$

where θ_1' is angle measured w.r.t. normal.

$$|E_{N1}| = 5 \quad \text{and} \quad |E_1| = \sqrt{(2)^2 + (-3)^2 + (5)^2} = 6.1644$$

$$\therefore \quad \cos\theta_1' = \frac{E_{N1}}{E_1} = \frac{5}{6.1644}$$

$$\therefore \quad \theta_1' = 35.795°$$

This θ_1' is angle made by $\bar{E}_1$ with the normal while θ_1 is shown with respect to horizontal.

$$\therefore \quad \theta_1 = 90 - \theta_1' = 90 - 35.795 = \mathbf{54.205°}$$

Similarly if θ_2' is angle made by $\bar{E}_2$ with the normal then,

$$\cos\theta_2' = \frac{E_{N2}}{E_2} = \frac{D_{N2}}{D_2} = \frac{D_{N1}}{D_2} \qquad \text{... } D_{N2} = D_{N1}$$

$$= \frac{\varepsilon_0 \varepsilon_{r1}\left|\bar{E}_{N1}\right|}{\left|\bar{D}_2\right|}$$

$$= \frac{\varepsilon_0 \times 2 \times 5}{\varepsilon_0 \times \sqrt{10^2 + (-15)^2 + (10)^2}} = \frac{10}{20.6155} = 0.485$$

$$\therefore \quad \theta_2' = 60.982°$$

$$\therefore \quad \theta_2 = 90 - \theta_2' = \mathbf{29.017°}$$

5.10 Concept of Capacitance

Consider two conducting materials M_1 and M_2 which are placed in a dielectric medium having permittivity ε. The material M_1 carries a positive charge Q while the material M_2 carries a negative charge, equal in magnitude as Q. There are no other charges present and total charge of the system is zero. In conductors, charge cannot reside within the conductor and it resides only on the surface. Thus for M_1 and M_2, charges + Q and – Q reside on the surfaces of M_1 and M_2 respectively. This is shown in the Fig. 5.13.

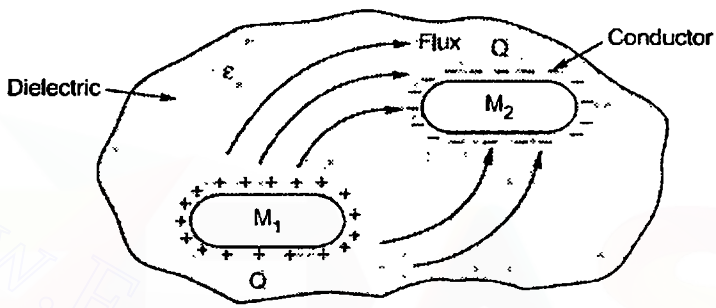


Fig. 5.13 Concept of capacitance

Such a system which has two conducting surfaces carrying equal and opposite charges, separated by a dielectric is called **capacitive system** giving rise to a **capacitance.**

The electric field is normal to the conductor surface and the electric flux is directed from M_1 towards M_2 in such a system. There exists a potential difference between the two surfaces of M_1 and M_2. Let this potential is V_{12}. The ratio of the magnitudes of the total charge on any one of the two conductors and potential difference between the conductors is called the **capacitance** of the two conductor system denoted as C.

$$\therefore \quad C = \frac{Q}{V_{12}} \quad \text{... (1)}$$

In general, $$C = \frac{Q}{V} \quad \text{... (2)}$$

where Q = Charge in coulombs

V = Potential difference in volts

The capacitance is measured in **farads** (F) and

$$1 \text{ Farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}}$$

As charge Q resides only on the surface of the conductor, it can be obtained from the Gauss's law as,

$$Q = \oint_S \bar{D} \cdot d\bar{S} = \oint_S \varepsilon_0 \varepsilon_r \bar{E} \cdot d\bar{S} = \oint_S \varepsilon \bar{E} \cdot d\bar{S}$$

While V is the work done in moving unit positive charge from negative to the positive surface and can be obtained as,

$$V = -\int_{l} \bar{E} \cdot \overline{dL} = -\int^{+} \bar{E} \cdot \overline{dL}$$

Hence capacitance can be expressed as,

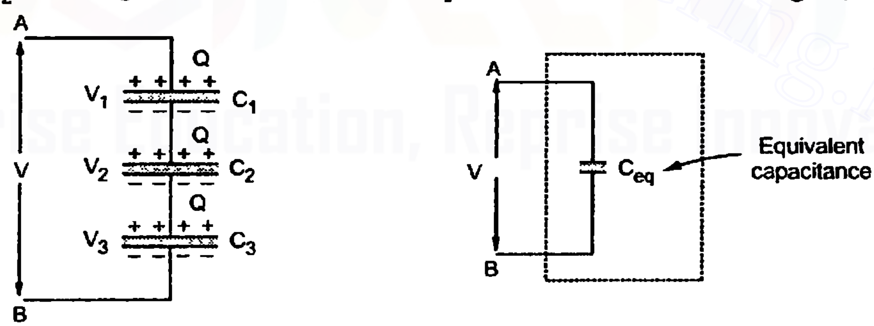
$$C = \frac{Q}{V} = \frac{\oint_{S} \varepsilon \bar{E} \cdot d\bar{S}}{-\int_{-}^{+} \bar{E} \cdot \overline{dL}} \text{ F} \qquad \text{... (3)}$$

If the charge Q is increased, then $\bar{E}$ and $\bar{D}$ get increased by same factor. The voltage V also increases by same factor. Thus the ratio Q to V remains constant as C. Hence capacitance is not the function of charge, field intensity, flux density and potential difference.

Key Point : ***The capacitance depends on the physical dimensions of the system and the properties of the dielectric such as permittivity of the dielectric.***

5.11 Capacitors in Series

Consider the three capacitors in series connected across the applied voltage V as shown in the Fig. 5.14. Suppose this pushes charge Q on C_1 then the opposite plate of C_1 must have the same charge. This charge which is negative must have been obtained from the connecting leads by the charge separation which means that the charge on the upper plate of C_2 is also Q. In short, **all the three capacitors have the same charge Q.**



(a) (b)

Fig. 5.14 Capacitors in series

$$\boxed{Q = C_1 V_1 = C_2 V_2 = C_3 V_3}$$

Giving , $$V_1 = \frac{Q}{C_1}; \qquad V_2 = \frac{Q}{C_2}; \qquad V_3 = \frac{Q}{C_3}$$

If an equivalent capacitor also stores the same charge, when applied with the same voltage, then it is obvious that,

$$C_{eq}' = \frac{Q}{V} \quad \text{or} \quad V = \frac{Q}{C_{eq}}$$

But, $$V = V_1 + V_2 + V_3$$

$\therefore$ $$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$\therefore$ $$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

It is easy to find V_1, V_2 and V_3 if Q is known.

$$\boxed{\text{For 'n' capacitors in series, } \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}}$$

Key Point : *For all the capacitors in series, the charge on all of them is always same, but the voltage across them is different.*

5.12 Capacitors in Parallel

Key Point: *When capacitors are in parallel, the same voltage exists across them, but charges are different.*

$$\boxed{Q_1 = C_1 V, \quad Q_2 = C_2 V, \quad Q_3 = C_3 V}$$

The total charge stored by the parallel bank of capacitors Q is given by,

$$Q = Q_1 + Q_2 + Q_3$$

$$= C_1 V + C_2 V + C_3 V = (C_1 + C_2 + C_3) V \qquad \dots (1)$$

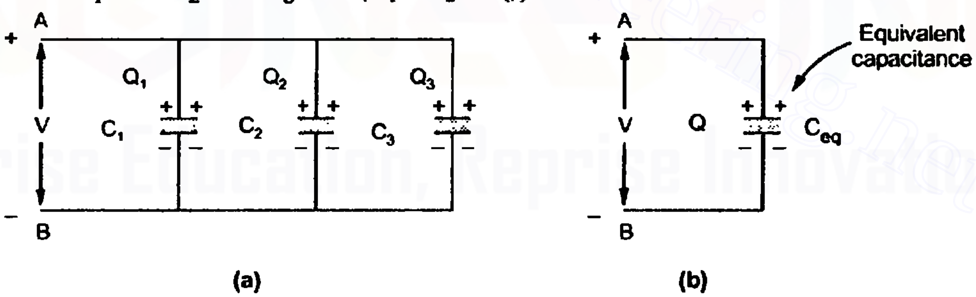


Fig. 5.15 Capacitors in parallel

An equivalent capacitor which stores the same charge Q at the same voltage V, will have

$$Q = C_{eq} V \qquad \dots (2)$$

Comparing equation (1) and equation (2),

As $$C_{eq} = C_1 + C_2 + C_3$$

$\therefore$ $$Q = C_1 V + C_2 V + C_3 V$$

It is easy to find Q_1, Q_2 and Q_3 if V is known.

$$\boxed{\text{For 'n' capacitors in parallel, } C_{eq} = C_1 + C_2 + \dots + C_n}$$

5.13 Parallel Plate Capacitor

A parallel plate capacitor is shown in the Fig. 5.16. It consists of two parallel metallic plates separated by distance 'd'. The space between the plates is filled with a dielectric of permittivity ε. The lower plate, plate 1 carries the positive charge and is distributed over it with a charge density $+\rho_S$. The upper plate, plate 2 carries the negative charge and is distributed over its surface with a charge density $-\rho_S$. The plate 1 is placed in z = 0 i.e. xy plane hence normal to it is z-direction. While upper plate 2 is in z = d plane, parallel to xy plane.

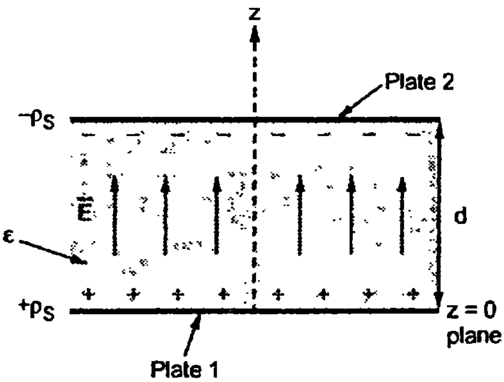


Fig. 5.16

Let A = Area of cross section of the plates in m^2.

$$\therefore \quad Q = \rho_S \, A \quad C \qquad \text{... (1)}$$

This is magnitude of charge on any one plate as charge carried by both is equal in magnitude. To find potential difference, let us obtain $\bar{E}$ between the plates.

Assuming plate 1 to be infinite sheet of charge,

$$\bar{E}_1 = \frac{\rho_S}{2\varepsilon}\bar{a}_N = \frac{\rho_S}{2\varepsilon}\bar{a}_z \quad \text{V/m} \qquad \text{... (2)}$$

The $\bar{E}_1$ is normal at the boundary between conductor and dielectric without any tangential component.

While for plate 2, we can write

$$\bar{E}_2 = \frac{-\rho_S}{2\varepsilon}\bar{a}_N = \frac{-\rho_S}{2\varepsilon}(-\bar{a}_z) \quad \text{V/m} \qquad \text{... (3)}$$

The direction of $\bar{E}_2$ is downwards i.e. in $-\bar{a}_z$ direction.

In between the plates,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{\rho_S}{2\varepsilon}\bar{a}_z + \frac{\rho_S}{2\varepsilon}\bar{a}_z = \frac{\rho_S}{\varepsilon}\bar{a}_z \qquad \text{... (4)}$$

The potential difference is given by,

$$V = -\int_{-}^{+} \bar{E}\cdot d\bar{L} = -\int_{\text{upper}}^{\text{lower}} \frac{\rho_S}{\varepsilon}\bar{a}_z \cdot d\bar{L}$$

Now $d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$ in Cartesian system.

$$\therefore \quad V = -\int_{z=d}^{z=0} \frac{\rho_S}{\varepsilon} \bar{a}_z \cdot [dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z]$$

$$= -\int_{z=d}^{z=0} \frac{\rho_S}{\varepsilon} dz = -\frac{\rho_S}{\varepsilon}[z]_d^0 = -\frac{\rho_S[-d]}{\varepsilon}$$

$$\therefore \quad V = \frac{\rho_S d}{\varepsilon} \text{ V}$$

The capacitance is the ratio of charge Q to voltage V,

$$C = \frac{Q}{V} = \frac{\rho_S A}{\frac{\rho_S d}{\varepsilon}} = \frac{\varepsilon A}{d} \text{ F} \quad \text{... (5)}$$

Thus if, $\varepsilon = \varepsilon_0 \varepsilon_r$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r A}{d} \text{ F}} \quad \text{... (6)}$$

It can be seen that the value of capacitance depends on,

1. The permittivity of the dielectric used.
2. The area of cross section of the plates.
3. The distance of separation of the plates.

It is not dependant on the charge or the potential difference between the plates.

5.14 Capacitance of a Co-axial Cable

Consider a co-axial cable or co-axial capacitor as shown in the Fig. 5.17.

Let a = Inner radius

b = Outer radius

The two concentric conductors are separated by dielectric of permittivity ε.

The length of the cable is L m.

The inner conductor carries a charge density $+\rho_L$ C/m on its surface then equal and opposite charge density $-\rho_L$ C/m exists on the outer conductor.

$$\therefore \quad Q = \rho_L \times L \quad \text{... (1)}$$

Assuming cylindrical co-ordinate system, $\bar{E}$ will be radial from inner to outer conductor, and for infinite line charge it is given by,

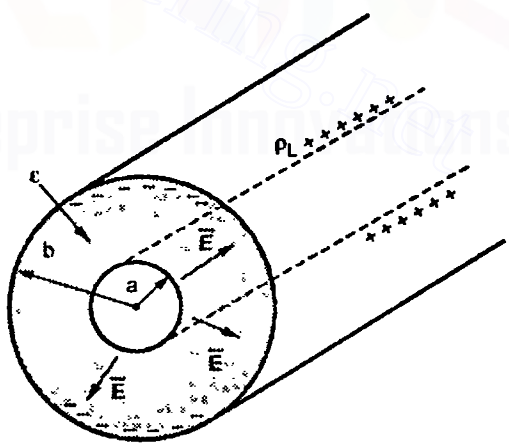


Fig. 5.17 Co-axial cable

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon r} \bar{a}_r \quad \text{... (2)}$$

$\bar{E}$ is directed from inner conductor to the outer conductor. The potential difference is work done in moving unit charge against $\bar{E}$ i.e. from r = b to r = a.

To find potential difference, consider $d\bar{L}$ in radial direction whch is $dr\,\bar{a}_r$

$$\therefore \quad V = -\int_{-}^{+} \bar{E} \cdot d\bar{L} = -\int_{r=b}^{r=a} \frac{\rho_L}{2\pi\varepsilon r}\bar{a}_r \cdot dr\,\bar{a}_r$$

$$= -\frac{\rho_L}{2\pi\varepsilon}\left[ln\, r\right]_b^a = -\frac{\rho_L}{2\pi\varepsilon}\, ln\left[\frac{a}{b}\right]$$

$$\therefore \quad V = \frac{\rho_L}{2\pi\varepsilon}\, ln\left[\frac{b}{a}\right] \text{V} \qquad \dots (3)$$

$$\therefore \quad C = \frac{Q}{V} = \frac{\rho_L \times L}{\frac{\rho_L}{2\pi\varepsilon}\, ln\left[\frac{b}{a}\right]}$$

$$\therefore \quad \boxed{C = \frac{2\pi\varepsilon L}{ln\left[\frac{b}{a}\right]} \text{ F}} \qquad \dots (4)$$

5.15 Spherical Capacitor

Consider a spherical capacitor formed of two concentric spherical conducting shells of radius a and b. The capacitor is shown in the Fig. 5.18.

The radius of outer sphere is 'b' while that of inner sphere is 'a'. Thus b > a. The region between the two spheres is filled with a dielectric of permittivity ε. The inner sphere is given a positive charge + Q while for the outer sphere it is − Q.

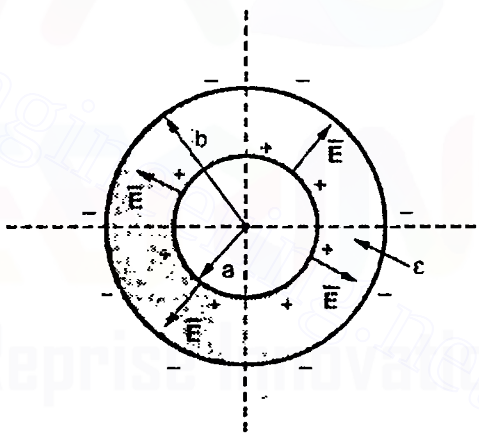


Fig. 5.18 Spherical capacitor

Considering Gaussian surface as a sphere of radius r, it can be obtained that $\bar{E}$ is in radial direction and given by,

$$\bar{E} = \frac{Q}{4\pi\varepsilon r^2}\bar{a}_r \text{ V/m} \qquad \dots (1)$$

The potential difference is work done in moving unit positive charge against the direction of $\bar{E}$ i.e. from r = b to r = a.

$$\therefore \quad V = -\int_{-}^{+} \bar{E} \cdot d\bar{L} = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\, \bar{a}_r \cdot d\bar{L} \qquad \dots (2)$$

Now $d\bar{L} = dr\,\bar{a}_r$ ... In radial direction

$$\therefore \quad V = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\bar{a}_r \cdot dr\,\bar{a}_r = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\,dr$$

$$= -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{r=b}^{r=a} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]_{r=b}^{r=a}$$

$$\therefore \quad V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right] V \qquad \text{... (3)}$$

Now
$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right]}$$

$$\therefore \quad \boxed{C = \frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]}\ F} \qquad \text{... (4)}$$

5.15.1 Capacitance of Single Isolated Sphere

Consider a single isolated sphere of radius 'a', given a charge + Q. It forms a capacitance with an outer plate which is infinitely large hence b = ∞.

The capacitance of such a single isolated spherical conductor can be obtained by substituting b = ∞ in the equation (4).

$$\therefore \quad C = \frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{\infty}\right]} \qquad \text{but } \frac{1}{\infty} = 0$$

$$\therefore \quad \boxed{C = 4\pi\varepsilon a \ F} \qquad \text{... (5)}$$

This is the case of a spherical conductor at a large distance from other conductors. Practically this fact is important in obtaining the stray capacitance of an isolated body.

5.15.2 Isolated Sphere Coated with Dielectric

Consider a single isolated sphere coated with a dielectric having permittivity ε_1, upto radius r_1. The radius of inner sphere is 'a' as shown in the Fig. 5.19. It is placed in a free space so outside sphere $\varepsilon = \varepsilon_0$. It carries a charge + Q.

So for $a < r < r_1, \varepsilon = \varepsilon_1$

and for $r > r_1, \varepsilon = \varepsilon_0$.

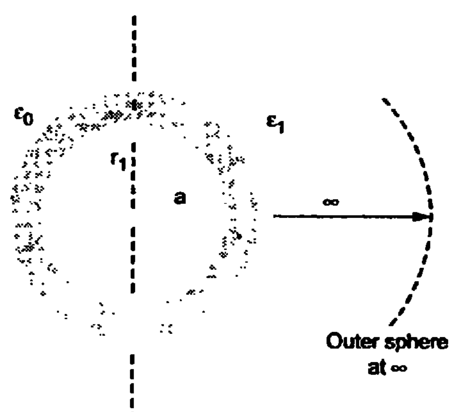


Fig. 5.19

The potential difference is work done in bringing unit positive charge from outer sphere r = ∞ to inner sphere r = a against $\bar{E}$. This is to be splitted in two as,

$$V = -\int_{-}^{+} \bar{E} \cdot d\bar{L} = -\int_{r=\infty}^{r=a} \bar{E} \cdot d\bar{L}$$

$$\therefore \quad V = -\int_{r=\infty}^{r=r_1} \bar{E} \cdot d\bar{L} - \int_{r=r_1}^{r=a} \bar{E} \cdot d\bar{L} \qquad \text{... (6)}$$

Now for $a < r < r_1$,

$$\bar{E}_1 = \frac{Q}{4\pi\varepsilon_1 r^2}\bar{a}_r \qquad \text{... (7)}$$

Now for $r_1 < r < \infty$,

$$\bar{E}_2 = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r \qquad \text{... (8)}$$

while $d\bar{L} = dr\,\bar{a}_r$ as $\bar{E}_1$ and E_2 are in radial direction.

$$\therefore \quad V = -\int_{r=\infty}^{r=r_1} \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r \cdot dr\,\bar{a}_r - \int_{r=r_1}^{r=a} \frac{Q}{4\pi\varepsilon_1 r^2}\bar{a}_r \cdot dr\,\bar{a}_r$$

$$= -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\int_{r=\infty}^{r=r_1}\frac{1}{r^2}dr + \frac{1}{\varepsilon_1}\int_{r=r_1}^{r=a}\frac{1}{r^2}dr\right]$$

$$= -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left(-\frac{1}{r}\right)_{r=\infty}^{r=r_1} + \frac{1}{\varepsilon_1}\left(-\frac{1}{r}\right)_{r=r_1}^{r=a}\right]$$

$$= -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left(-\frac{1}{r_1}+\frac{1}{\infty}\right) + \frac{1}{\varepsilon_1}\left(-\frac{1}{a}+\frac{1}{r_1}\right)\right]$$

$$= \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left(\frac{1}{r_1}\right) + \frac{1}{\varepsilon_1}\left(\frac{1}{a}\right) - \frac{1}{\varepsilon_1}\left(\frac{1}{r_1}\right)\right]$$

$$\therefore \quad V = \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}\right] V \qquad \text{... (9)}$$

$$\therefore \quad C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}\right]}$$

$$\therefore \quad \boxed{C = \frac{4\pi}{\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}\right]} F} \qquad \text{... (10)}$$

Now $$\frac{1}{C} = \frac{\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}\right]}{4\pi} \qquad \text{... (11)}$$

$$\therefore \quad \frac{1}{C} = \frac{\left(\frac{1}{a}-\frac{1}{r_1}\right)}{4\pi\varepsilon_1}+\frac{1}{4\pi\varepsilon_0 r_1} \quad \text{... (12)}$$

Now $$C_1 = \frac{4\pi\varepsilon_1}{\left(\frac{1}{a}-\frac{1}{r_1}\right)} = \text{Capacitance of spherical capacitor}$$

$$C_2 = 4\pi\varepsilon_0 r_1 = \text{Capacitance of isolated sphere}$$

$$\therefore \quad \frac{1}{C} = \frac{1}{C_1}+\frac{1}{C_2} \quad \text{... (13)}$$

$$\therefore \quad C = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} \quad \text{... (14)}$$

Thus total capacitance can be treated to be the two capacitors C_1 and C_2 in series.

5.16 Composite Parallel Plate Capacitor

The composite parallel plate capacitor is one in which the space between the plates is filled with more than one dielectric. Consider a composite capacitor with space filled with two separate dielectrics for the distances d_1 and d_2.

This is shown in the Fig. 5.20. **The dielectric interface is parallel to the conducting plates.**

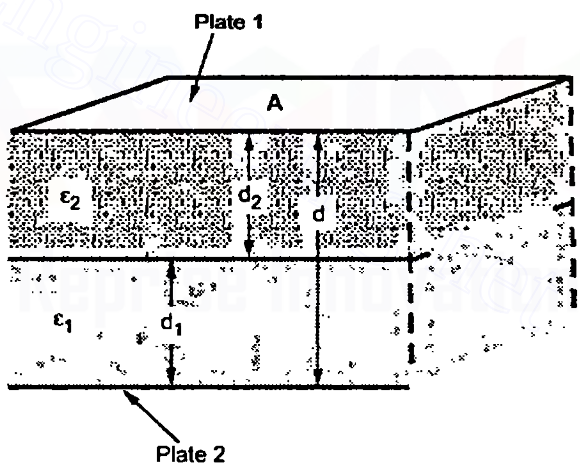


Fig. 5.20 Composite parallel plate capacitor

The space d_1 is filled with dielectric having permittivity ε_1 while space d_2 is filled with dielectric having permittivity ε_2.

Let Q = Charge on each plate

$\bar{E}_1$ = Field intensity in region d_1

$\bar{E}_2$ = Field intensity in region d_2

Both the intensities are uniform.

$$\therefore \quad V_1 = E_1 d_1$$

and $$V_2 = E_2 d_2$$

where E_1 and E_2 are the magnitudes of the two intensities.

$$\therefore \quad V = V_1 + V_2 = E_1 d_1 + E_2 d_2 \quad \text{... (1)}$$

At a dielectric - dielectric interface, the normal components of flux densities are equal i.e. $D_{N1} = D_{N2}$.

Now $D_1 = \varepsilon_1 E_1$ and $D_2 = \varepsilon_2 E_2$

Substituting in equation (1),

$$V = \frac{D_1}{\varepsilon_1} d_1 + \frac{D_2}{\varepsilon_2} d_2 \quad \text{... (2)}$$

The magnitude of surface charge is same on each plate hence

$$\rho_S = D_1 = D_2 \quad \text{... (3)}$$

Substituting in equation (2)

$$\therefore \quad V = \rho_S \left[\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right] \quad \text{... (4)}$$

Now $$C = \frac{Q}{V} = \frac{Q}{\rho_S \left[\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right]}$$

But $$Q = \rho_S \times A$$

$$\therefore \quad C = \frac{\rho_S A}{\rho_S \left[\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right]} = \frac{A}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}} \quad \text{... (5)}$$

$$\therefore \quad C = \frac{1}{\frac{d_1}{\varepsilon_1 A} + \frac{d_2}{\varepsilon_2 A}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \quad \text{... (6)}$$

where $$C_1 = \frac{\varepsilon_1 A}{d_1} \quad \text{and} \quad C_2 = \frac{\varepsilon_2 A}{d_2}$$

Thus the result can be generalized for a capacitor with n dielectrics as,

$$\boxed{C = \frac{A}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} + \frac{d_3}{\varepsilon_3} + \ldots + \frac{d_n}{\varepsilon_n}}} \quad \text{... (7)}$$

5.16.1 Dielectric Boundary Normal to the Plates

Consider the composite capacitor in which dielectric boundary is normal to the conducting plates.

The dielectric ε_1 occupying area A_1 of the plates while the dielectric ε_2 occupying area A_2, as shown in the Fig. 5.21.

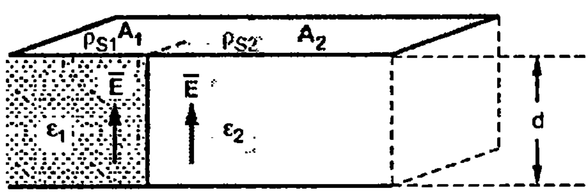


Fig. 5.21 Dielectric boundary normal to the plates

The total potential across the two plates is V and distance between the plates is d. Hence magnitude of $\bar{E}$ is,

$$E = \frac{V}{d} \quad \text{... (8)}$$

At the boundary, both $\overline{E}_1$ and $\overline{E}_2$ are tangential and for dielectric - dielectric interface tangential components are equal.

$$\therefore \quad E_{tan1} = E_{tan1} = E_1 = E_2 = \frac{V}{d} \quad \text{... (9)}$$

$$\text{Now} \quad D_1 = \varepsilon_1 E_1 \quad \text{and} \quad D_2 = \varepsilon_2 E_2$$

$$\therefore \quad D_1 = \frac{\varepsilon_1 V}{d} \quad \text{and} \quad D_2 = \frac{\varepsilon_2 V}{d} \quad \text{... (10)}$$

On the plate the charge is divided into two parts.

On area A_1, the charge density is $\rho_{S1} = D_1$ while on area A_2, the charge density is $\rho_{S2} = D_2$.

$$\therefore \quad Q = Q_1 + Q_2 = \rho_{S1} A_1 + \rho_{S2} A_2 = D_1 A_1 + D_2 A_2 \quad \text{... (11)}$$

$$\therefore \quad Q = \frac{\varepsilon_1 V A_1}{d} + \frac{\varepsilon_2 V A_2}{d} \quad \text{... (12)}$$

$$\text{Now} \quad C = \frac{Q}{V} = \frac{\frac{\varepsilon_1 V A_1}{d} + \frac{\varepsilon_2 V A_2}{d}}{V}$$

$$\therefore \quad \boxed{C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2} \quad \text{... (13)}$$

Thus if **dielectric boundary is parallel to the plates**, the arrangement is equivalent to **two capacitors in series** for which $C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$. While if the **dielectric boundary is normal to the plates**, the arrangement is equivalent to two **capacitors in parallel** for which $C_{eq} = C_1 + C_2$.

Example 5.12 : *Find the capacitance of a conducting sphere of 2 cm in diameter, covered with a layer of polyethelene with ε_r = 2.26 and 3 cm thick.*

Solution : The sphere is shown in the Fig. 5.22.

$$a = \text{Radius of sphere} = \frac{d}{2} = 1 \text{ cm}$$

$$r_1 = a + \text{Thickness} = 1 + 3 = 4 \text{ cm}$$

This forms two capacitors in series.

C_1 = Capacitor due to spherical arrangement of concentric spheres

$$= \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

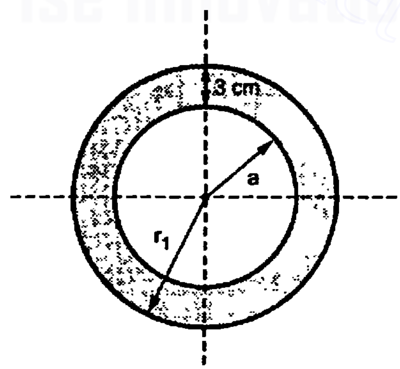


Fig. 5.22

Here a = 1 cm and b = r_1 = 4 cm.

$$\therefore \qquad C_1 = \frac{4\pi\times 2.26\times\varepsilon_0}{\left[\left(\frac{1}{1\times10^{-2}}\right)-\left(\frac{1}{4\times10^{-2}}\right)\right]} = 3.3527 \text{ pF}$$

And C_2 = Due to isolated sphere of r_1 = 4 cm = $4\pi\varepsilon r_1$

But between this sphere of radius r_1 and sphere at ∞, the dielectric is free space ε_0.

$$\therefore \qquad C_2 = 4\pi\varepsilon_0 r_1 = 4\pi\times 8.854\times10^{-12}\times 4\times10^{-2} = 4.4505 \text{ pF}$$

$$\therefore \qquad C_{eq} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} \qquad \text{... } C_1 \text{ and } C_2 \text{ in series}$$

$$= \frac{C_1 C_2}{C_1 + C_2}$$

$$= \frac{3.3527\times10^{-12}\times 4.4505\times10^{-12}}{10^{-12}\,[3.3527+4.4505]}$$

$$= \mathbf{1.9121 \text{ pF}}$$

Students can use direct formula for such a case,

$$C = \frac{4\pi}{\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{r_1}\right)+\frac{1}{\varepsilon_o r_1}\right]} \text{ F}$$

Example 5.13 : *A parallel plate capacitor with a separation of 1 cm has 29 kV applied, when air was the dielectric used. Assume that the dielectric strength of air as 30 kV/cm. A thin piece of glass with ε_r = 6.5 with a dielectric strength of 290 kV/cm with thickness 0.2 cm is inserted. Find whether glass will break or air ?*

Solution : The arrangement is shown in the Fig. 5.23.

There are two capacitors formed.

$$C_{air} = \frac{\varepsilon_0 A}{d_1}$$

$$= \frac{8.854\times10^{-12} \text{ A}}{0.8\times10^{-2}}$$

$$= 1.1067\times10^{-9} \text{ A} \quad \text{F}$$

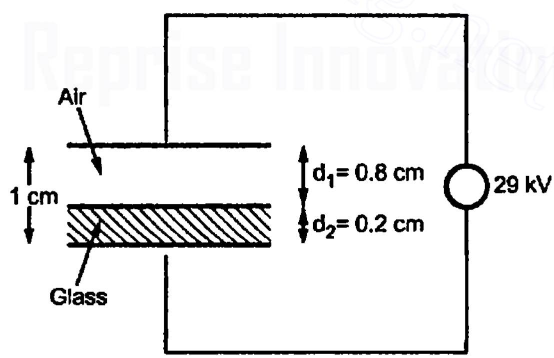


Fig. 5.23

$$C_{glass} = \frac{\varepsilon_0 \varepsilon_r A}{d_2} = \frac{8.854\times10^{-12}\times6.5\text{ A}}{0.2\times10^{-2}}$$

$$= 2.8775\times10^{-8}\text{ A} \quad \text{F}$$

$$\therefore \quad C_{eq} = C_{air} \text{ and } C_{glass} \text{ in series}$$

$$= \frac{C_{air}\times C_{glass}}{C_{air}+C_{glass}} = 1.0657\times10^{-9}\text{ A} \quad \text{F}$$

Let area of cross-section $A = 1\ m^2$.

$$\therefore \quad C_{eq} = 1.0657\times10^{-9}\text{ F}$$

$$\text{Total charge } Q = C_{eq}\times V = 1.0657\times10^{-9}\times29\times10^{3}$$

$$= 3.0905\times10^{-5}\text{ C}$$

$$\therefore \quad V_{air} = \frac{Q}{C_{air}} = \frac{3.0905\times10^{-5}}{1.1067\times10^{-9}} = 27.9259\text{ kV}$$

$$\text{And} \quad V_{glass} = V - V_{air} = 29\text{ kV} - 27.9259\text{ kV} = 1.074\text{ kV}$$

$$\therefore \quad E_{air} = \frac{V_{air}}{d_1} = \frac{27.9259\times10^{3}}{0.8\times10^{-2}} = 3.49\times10^{6}\text{ V/m}$$

$$= 3.49\times10^{4}\text{ V/cm} = 34.9\text{ kV/cm}$$

$$\text{And} \quad E_{glass} = \frac{V_{glass}}{d_2} = \frac{1.074}{0.2} = 5.37\text{ kV/cm}$$

The dielectric strength of air is 30 kV/cm and E_{air} is more than that hence air will breakdown.

The dielectric strength of glass is 290 kV/cm and E_{glass} is less than that hence **glass will not breakdown.**

Example 5.14 : *The width of the region containing ε_{R1} in the Fig. 5.24 is 1.2 m. Find ε_{R1} if $\varepsilon_{R2} = 2.5$ and the total capacitance is 60 nF.*

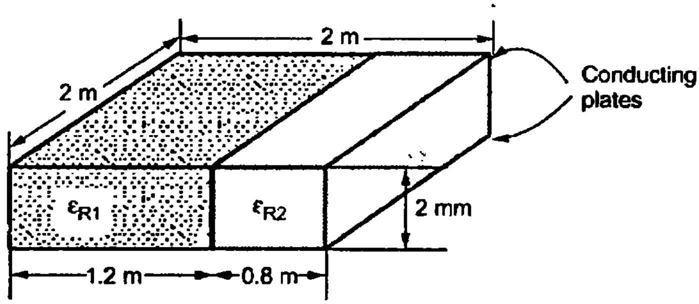


Fig. 5.24

Solution : In this case, the dielectric interface is normal to the plates. Hence two capacitors formed are in parallel.

Now $d = 2$ mm same for both the dielectrics

$$A_1 = 1.2 \times 2 = 2.4\ m^2 \quad \text{and} \quad A_2 = 0.8 \times 2 = 1.6\ m^2$$

$$\therefore \quad C_1 = \frac{\varepsilon_0 \varepsilon_{R1} A_1}{d} = \frac{8.854 \times 10^{-12} \varepsilon_{R1} \times 2.4}{2 \times 10^{-3}} = 1.062 \times 10^{-8} \varepsilon_{R1} \text{ F}$$

$$C_2 = \frac{\varepsilon_0 \varepsilon_{R2} A_2}{d} = \frac{8.854 \times 10^{-12} \times 2.5 \times 1.6}{2 \times 10^{-3}} = 1.7708 \times 10^{-8} \text{ F}$$

$$\therefore \quad C_{eq} = C_1 + C_2 = 1.7708 \times 10^{-8} + 1.062 \times 10^{-8} \varepsilon_{R1}$$

But $C_{eq} = 60$ nF

$$\therefore \quad 60 \times 10^{-9} = 1.7708 \times 10^{-8} + 1.062 \times 10^{-8} \varepsilon_{R1}$$

$$\therefore \quad 1.062 \times 10^{-8} \varepsilon_{R1} = 4.2292 \times 10^{-8}$$

$$\therefore \quad \varepsilon_{R1} = 3.9823$$

5.17 Energy Stored in a Capacitor

It is seen that capacitor can store the energy. Let us find the expression for the energy stored in a capacitor.

Consider a parallel plate capacitor as shown in the Fig. 5.25. It is supplied with the voltage V.

Let $\bar{a}_N$ is the direction normal to the plates.

$$\therefore \quad \bar{E} = \frac{V}{d} \bar{a}_N \qquad \text{... (1)}$$

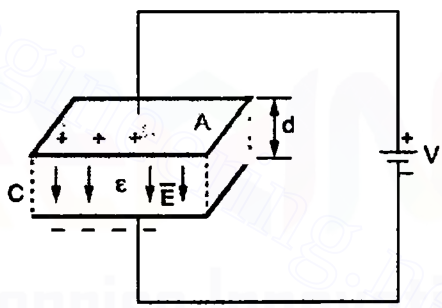


Fig. 5.25 Parallel plate capacitor

The energy stored is given by,

$$W_E = \frac{1}{2} \int_{vol} \bar{D} \cdot \bar{E}\, dv$$

$$= \frac{1}{2} \int_{vol} \varepsilon \bar{E} \cdot \bar{E}\, dv \qquad \text{but} \quad \bar{E} \cdot \bar{E} = |\bar{E}|^2$$

$$= \frac{1}{2} \int_{vol} \varepsilon |\bar{E}|^2\, dv \qquad \text{but} \quad |\bar{E}| = \frac{V}{d}$$

$$= \frac{1}{2} \varepsilon \frac{V^2}{d^2} \int_{vol} dv \qquad \text{but} \quad \int_{vol} dv = \text{Volume} = A \times d$$

$$= \frac{1}{2} \varepsilon \frac{V^2 A d}{d^2} = \frac{1}{2} \frac{\varepsilon A}{d} V^2$$

$$\boxed{W_E = \frac{1}{2} CV^2 \text{ J}}$$

If the dielectric is free space then there is increase in the stored energy if free space is replaced by other dielectric having $\varepsilon_r > 1$.

5.17.1 Energy-Density

As seen in earlier chapter, energy density is the energy stored per unit volume as volume tends to zero.

$$\therefore \quad W_E = \frac{1}{2}\varepsilon \int_{Vol} |\bar{E}|^2 \, dv$$

$$\therefore \quad W_E = \frac{1}{2}\varepsilon |\bar{E}|^2 \text{ J/m}^3 = \text{Energy density}$$

Using $|\bar{D}| = \varepsilon |\bar{E}|$ we can write,

$$\boxed{W_E = \frac{1}{2}\frac{|\bar{D}|^2}{\varepsilon} = \frac{1}{2}|\bar{D}||\bar{E}| \text{ J/m}^3}$$

Example 5.15 : *A pair of 200 mm long concentric cylindrical conductors of radii 50 mm and 100 mm, is filled with a dielectric with $\varepsilon = 10\varepsilon_0$. A voltage is applied between the conductors which establishes $\bar{E} = \frac{10^6}{r}\bar{a}_r$. Calculate :*

a) Capacitance b) Voltage applied c) Energy stored.

Solution : The arrangement is shown in the Fig. 5.26.

a) The capacitor is not a function of voltage V or $\bar{E}$, it depends on dielectric and physical dimensions. For coaxial conductors,

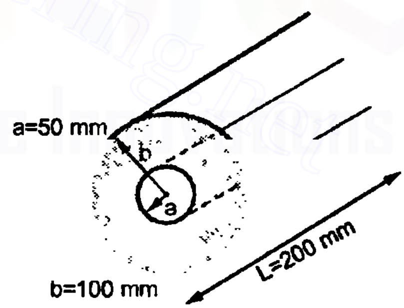


Fig. 5.26

$$C = \frac{2\pi\varepsilon L}{ln\left(\frac{b}{a}\right)}$$

$$= \frac{2\pi(10\varepsilon_0)\times 200\times 10^{-3}}{ln\left[\frac{100\times 10^{-3}}{50\times 10^{-3}}\right]}$$

$$= 160.518 \text{ pF}$$

b) $\bar{E}$ is function of r hence using,

$$V = -\int_{-}^{+} \bar{E}\cdot d\bar{L} = -\int_{r=b}^{r=a} \frac{10^6}{r}\bar{a}_r \cdot [dr\,\bar{a}_r]$$

$$= -\int_{r=b}^{r=a} \frac{10^6}{r} dr = -10^6 \left[ln\, r\right]_{r=100\text{ mm}}^{r=50\text{ mm}}$$

$$= -10^6 [ln 50\times10^{-3} - ln 100\times10^{-3}] = \mathbf{693.1471\ kV}$$

c) $$W_E = \frac{1}{2} C\, V^2 = \frac{1}{2}\times160.518\times10^{-12}\times(693.147\times10^3)^2$$

$$= 38.5606\ J$$

Alternatively, $$W_E = \int \frac{1}{2}\varepsilon|\bar{E}|^2 dv = \frac{\varepsilon}{2}\int \left|\frac{10^6}{r}\right|^2 r\, dr\, d\phi\, dz$$

$$= \frac{10^{12}\varepsilon}{2}\int_{z=0}^{L}\int_{\phi=0}^{2\pi}\int_{r=a}^{b}\frac{1}{r}\, dr\, d\phi\, dz$$

$$= \mathbf{38.56\ J}$$

5.18 Method of Images

The method of images is introduced by Lord Kelvin in 1848. The method is suitable to determine V, $\bar{E}$, $\bar{D}$ and ρ_S due to the charges in the presence of conductors. The conductors carry the charge only on the surface and surface is an equipotential surface. The method of images helps us to find V, $\bar{E}$, $\bar{D}$ and ρ_S due to the charges in the presence of conducting planes which are equipotential, without solving Poisson's or Laplace's equations.

5.18.1 The Image Theory

Consider a dipole field. The plane exists midway between the two charges, is a zero potential infinite plane. Such a plane may be represented by very thin conducting plane which is infinite. The conductor is an equipotential surface at a potential V = 0 and $\bar{E}$ is only normal to the surface. Thus if out of dipole, only positive charge is considered above a conducting plane then fields at all points in upper half of plane are same. In other words, if there is a charge above a conducting plane then same fields at the points above the plane can be obtained by replacing conducting plane by equipotential surface and an image of the charge at a symmetrical location below the plane. Such an image is negative of the original charge.

The images of various charge distributions are shown in the Fig. 5.27 (b). Where the conducting plane in the Fig. 5.27 (a) is replaced by an equipotential surface with V = 0. The charges may be point, line or volume charges.

The conditions to be satisfied to apply the method of images are,

1. The image charges must be located in the conducting region.
2. The image charges must be located such that on the conducting surface the potential is zero or constant.

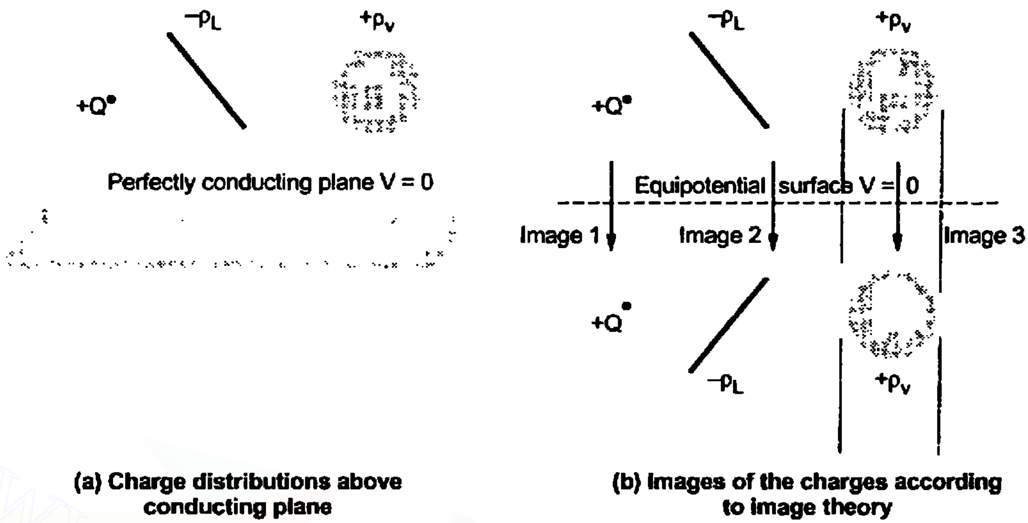


Fig. 5.27 Basics of method of images

The first condition is to satisfy Poisson's equation, while the other to satisfy the boundary conditions.

5.18.2 Method of Images for Point Charges

Consider a perfect conducting plane in xy plane, infinite in nature. The point charge $+Q$ is located at $z = h$, above the plane. It is required to obtain $\bar{E}$ at any point above the plane. Then replace the plane by the equipotential surface and get the image of Q, below the plane. The image charge is $-Q$, located at $z = -h$. The original charge and plane are shown in the Fig. 5.28 (a) while the image is shown in the Fig. 5.28 (b).

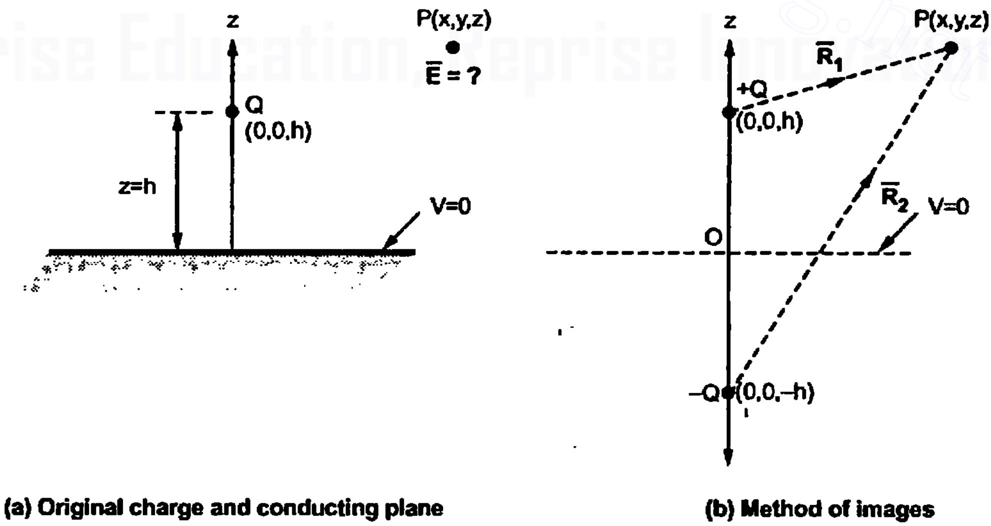


Fig. 5.28

Let us obtain $\bar{E}$ at P (x, y, z) using method of images.

$$\bar{E} = \bar{E}_1 + \bar{E}_2 \qquad \text{where } \bar{E}_1 \text{ due to } +Q,\ \bar{E}_2 \text{ due to } -Q$$

$$= \frac{Q}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{-Q}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2}$$

Now $\quad \bar{R}_1 = (x-0)\,\bar{a}_x + (y-0)\bar{a}_y + (z-h)\bar{a}_z$

$\therefore \quad \bar{a}_{R1} = \dfrac{\bar{R}_1}{|\bar{R}_1|} = \dfrac{(x)\bar{a}_x + (y)\bar{a}_y + (z-h)\bar{a}_z}{\sqrt{x^2+y^2+(z-h)^2}}$

And $\quad \bar{R}_2 = (x-0)\bar{a}_x + (y-0)\bar{a}_y + [z-(-h)]\bar{a}_z$

$\therefore \quad \bar{a}_{R2} = \dfrac{\bar{R}_2}{|\bar{R}_2|} = \dfrac{x\,\bar{a}_x + y\,\bar{a}_y + (z+h)\,\bar{a}_z}{\sqrt{x^2+y^2+(z+h)^2}}$

$$\therefore \quad \bar{E} = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{[x\,\bar{a}_x + y\,\bar{a}_y + (z-h)\,\bar{a}_z]}{(x^2+y^2+(z-h)^2)^{3/2}} - \frac{[x\,\bar{a}_x + y\,\bar{a}_y + (z+h)\,\bar{a}_z]}{(x^2+y^2+(z+h)^2)^{3/2}}\right\}$$

It can be seen that if z = 0 then $\bar{E}$ has only the Z component. This confirms that $\bar{E}$ is always normal to the conducting surface.

Let us obtain potential at P (x, y, z).

The potential due to the point charge is given by,

$$V = \frac{Q}{4\pi\varepsilon_0 R}$$

$$\therefore \quad V_P = V_1 + V_2 = \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{-Q}{4\pi\varepsilon_0 R_2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

$$R_1 = |\bar{R}_1| = \sqrt{x^2+y^2+(z-h)^2}$$

$$R_2 = |\bar{R}_2| = \sqrt{x^2+y^2+(z+h)^2}$$

$$\therefore \quad V_P = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{[x^2+y^2+(z-h)^2]^{1/2}} - \frac{1}{[x^2+y^2+(z+h)^2]^{1/2}}\right]$$

At z = 0, V_P = 0 V which confirms that surface of the conductor is equipotential surface with V = 0.

Similarly other parameters such as $\rho_S = D_N$ which is $\varepsilon_0 E_N$ and E_N is $\bar{E}$ with $z=0$, can be obtained. The total charge induced on the conducting surface also can be obtained from Gauss's law.

Example 5.16 : *A point charge of 25 nC located in free space at P (2, – 3, 5) and a perfectly conducting plane at z = 2. Find a) V at (3, 2, 4) b) $\bar{E}$ at (3, 2, 4) c) ρ_S at (3, 2, 2). Use method of images.*

Solution : The plane and point P are shown in the Fig. 5.29.

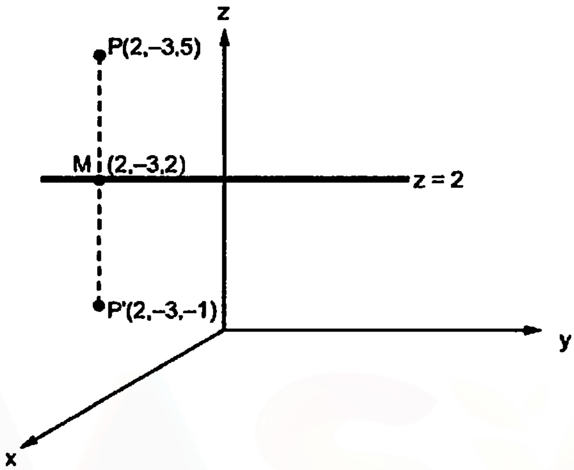


Fig. 5.29

If the perpendicular is dropped from P on z = 2 plane at M then M is (2, – 3, 2) hence perpendicular distance of P from z = 2 is 3. Now P′ is mirror image of P about z = 2 plane.

So co-ordinates of P′(2, – 3, –1) as the distance of P′ from z = 2 must be 3. The charge at P′ is – 25 nC.

a) V at (3, 2, 4)

$$V = V_1 + V_2$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2}$$

$$R_1 = \sqrt{(3-2)^2 + [2-(-3)]^2 + [4-5]^2} = 5.1961$$

$$R_2 = \sqrt{(3-2)^2 + [2-(-3)]^2 + [4-(-1)]^2} = 7.1414$$

$$\therefore \quad V = \frac{25\times10^{-9}}{4\pi\varepsilon_0}\left[\frac{1}{5.1961} - \frac{1}{7.1414}\right] = 11.7793 \text{ V}$$

b) $\bar{E}$ at (3, 2, 4)

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2}$$

$$\bar{R}_1 = (3-2)\bar{a}_x + [2-(-3)]\bar{a}_y + (4-5)\bar{a}_z = \bar{a}_z + 5\bar{a}_y - \bar{a}_z$$

$$|\bar{R}_1| = \sqrt{1^2+5^2+1^2} = \sqrt{27}$$

$$\bar{R}_2 = (3-2)\bar{a}_x + [2-(-3)]\bar{a}_y + [4-(-1)]\bar{a}_z = \bar{a}_z + 5\bar{a}_y + 5\bar{a}_z$$

$$|\bar{R}_2| = \sqrt{1^2+5^2+5^2} = \sqrt{51}$$

$$\therefore \quad \bar{E} = \frac{25\times10^{-9}}{4\pi\varepsilon_0}\left[\frac{\bar{R}_1}{R_1^2|\bar{R}_1|} - \frac{\bar{R}_2}{R_2^2|\bar{R}_2|}\right] = 224.693\left[\frac{\bar{R}_1}{R_1^3} - \frac{\bar{R}_2}{R_2^3}\right]$$

$$= 224.693\{7.12\times10^{-3}\,\bar{a}_x + 0.035\,\bar{a}_y - 7.12\times10^{-3}\,\bar{a}_z - 2.745\times10^{-3}\,\bar{a}_x - 0.0137\,\bar{a}_y - 0.0137\,\bar{a}_z\}$$

$$= 0.983\,\bar{a}_x + 4.785\,\bar{a}_y - 4.678\,\bar{a}_z \quad \text{V/m}$$

c) ρ_S at (3, 2, 2)

To find ρ_S, let us calculate $\bar{E}$ first at (3, 2, 2).

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2}$$

Now $\bar{R}_1 = (3-2)\bar{a}_x + [2-(-3)]\bar{a}_y + (2-5)\bar{a}_z = \bar{a}_z + 5\bar{a}_y - 3\bar{a}_z$

$\bar{R}_2 = (3-2)\,\bar{a}_x + [2-(-3)]\bar{a}_y + [2-(-1)]\bar{a}_z = \bar{a}_z + 5\bar{a}_y + 3\bar{a}_z$

Substituting $|\bar{R}_1|$, $|\bar{R}_2|$ in the equation of $\bar{E}$ we get,

$$\bar{E} = -6.51\,\bar{a}_z \quad \text{V/m}$$

$$\therefore \quad \bar{D} = \varepsilon_0\bar{E} = -57.64\,\bar{a}_z \quad \text{pC/m}^2$$

But $\rho_S = |\bar{D}_N| = |\bar{D}| = 57.64 \quad \text{pC/m}^2$

But as $\bar{D}$ is directed in $-\bar{a}_z$ direction, ρ_S must be negative,

$$\therefore \quad \rho_S = -57.64 \quad \text{pC/m}^2$$

Examples with Solutions

Example 5.17 : *A capacitor with two dielectrics is as follows :*

Plate area 100 cm², Dielectric 1 thickness = 3 mm, ε_{r1} = 3, Dielectric 2 thickness = 2 mm, ε_{r2} = 2. If a potential of 100 V is applied across the plates find the energy stored in each dielectric and potential gradient in each dielectric.

Solution : $C_1 = \dfrac{\varepsilon_1 A}{d_1}$ and $C_2 = \dfrac{\varepsilon_2 A}{d_2}$

And C_1 and C_2 are in series.

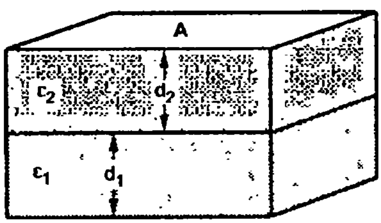


Fig. 5.30

$$\therefore \quad C_1 = \frac{\varepsilon_0\varepsilon_{r1}A}{d_1} = \frac{8.854\times10^{-12}\times3\times100\times10^{-4}}{3\times10^{-3}}$$

$$= 8.854\times10^{-11} \text{ F}$$

While $$C_2 = \frac{\varepsilon_0\varepsilon_{r2}A}{d_2} = \frac{8.854\times10^{-12}\times2\times100\times10^{-4}}{2\times10^{-3}}$$

$$= 8.854\times10^{-11} \text{ F}$$

$$\therefore \quad C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = 4.427 \times 10^{-11} \text{ F}$$

Now $\quad C_{eq} = \frac{Q}{V}$

$$\therefore \quad Q = C_{eq} V = 4.427 \times 10^{-11} \times 100 = 4.427 \times 10^{-9} \text{ C}$$

The charge Q remains same for C_1 and C_2.

$$\therefore \quad C_1 = \frac{Q}{V_1} \quad \text{hence } V_1 = 50 \text{ V}$$

While $\quad C_2 = \frac{Q}{V_2} \quad$ hence $V_2 = 50$ V

The energy stored in each dielectric is,

$$W_{E1} = W_{E2} = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} C_2 V_2^2 = 0.1106 \ \mu\text{J}$$

The potential gradients are,

$$E_1 = \frac{V_1}{d_1} = \frac{50}{3 \times 10^{-3}} = 16.667 \text{ kV/m}$$

and $\quad E_2 = \frac{V_2}{d_2} = \frac{50}{2 \times 10^{-3}} = 25$ kV/m

Example 5.18 : *The interface between two dielectrics is defined by $x = 0$ plane For dielectric 1, $x > 0$, $\varepsilon_{R_1} = 4$ while the dielectric 2, $x < 0$, $\varepsilon_{R_2} = 3$. If the electric flux density in region 1 is given by $\bar{D}_1 = 2\bar{a}_x + 3\bar{a}_y + 4\bar{a}_z$ C/m^2, find $\bar{D}_2$.*

Solution : The arrangement is shown in the Fig. 5.31.

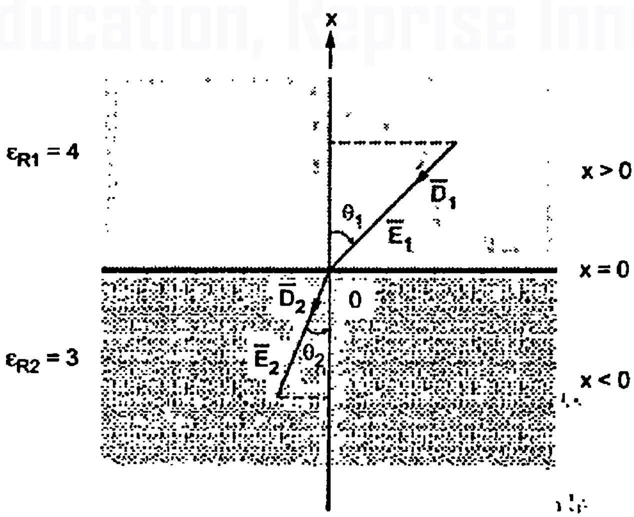


Fig. 5.31

The normal direction to x = 0 plane is $\bar{a}_x$ hence out of $\bar{D}_1$ the component of $\bar{a}_x$ is normal component of $\bar{D}_1$.

$$\therefore \quad \bar{D}_{N1} = 2\bar{a}_x$$

$$\therefore \quad \bar{D}_{tan1} = 3\,\bar{a}_y + 4\,\bar{a}_z$$

From the boundary conditions,

$$D_{N1} = D_{N2}$$

$$\therefore \quad \bar{D}_{N2} = 2\bar{a}_x$$

$$\text{And} \quad \frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{R1}}{\varepsilon_{R2}}$$

$$\therefore \quad \bar{D}_{tan2} = \bar{D}_{tan1}\left[\frac{4}{3}\right] = 4\bar{a}_y + 5.333\bar{a}_z$$

$$\therefore \quad \bar{D}_2 = \bar{D}_{N2} + \bar{D}_{tan2} = 2\,\bar{a}_x + 4\,\bar{a}_y + 5.333\,\bar{a}_z \ \text{C/m}^2$$

Example 5.19 : *A copper conductor having a 0.8 mm diameter and length 2 cm carries a current of 20 A. Find the electric field intensity, the voltage drop and resistance for 2 cm length. Assume conductivity of copper as 5.8×10^7 S/m.*

Solution : d = 0.8 mm, L = 2 cm, I = 20 A

$$|\bar{J}| = \frac{I}{S}$$

$$= \frac{I}{\frac{\pi}{4} d^2}$$

$$= \frac{20}{\frac{\pi}{4} \times [0.8 \times 10^{-3}]^2}$$

$$= 39.788 \times 10^6 \ \text{A/m}^2$$

$$\text{Now} \quad |\bar{J}| = \sigma |\bar{E}|$$

$$\therefore \quad |\bar{E}| = \frac{|\bar{J}|}{\sigma} = \frac{39.788 \times 10^6}{5.8 \times 10^7} = 0.686 \ \text{V/m}$$

$$\text{And} \quad V = EL = 0.686 \times 2 \times 10^{-2} = 0.0137 \ \text{V}$$

$$R = \frac{V}{I} = \frac{0.0137}{20} = 6.86 \times 10^{-4} \ \Omega$$

Example 5.20 : *A conducting sphere of radius 5 cm has a total charge of 1 μC. The sphere is surrounded by an inhomogeneous dielectric sphere $5 \le r \le 10$ cm in which relative permittivity varies as ε_r = 0.1/r. A second conducting spherical surface is at $r = 10$ cm. Calculate the potential difference and capacitance between the conductors.*

Solution : The arrangement is shown in the Fig. 5.32.

As $\varepsilon_r \propto \frac{1}{r}$, the standard formula for spherical capacitor cannot be used.

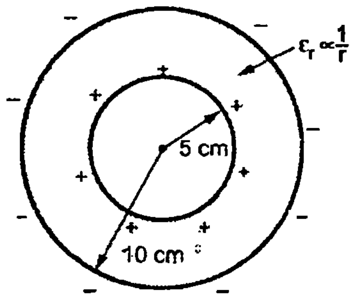


Fig. 5.32

In spherical conductor $\bar{E}$ at a radial distance r is given by,

$$\bar{E} = \frac{Q}{4\pi\varepsilon r^2}\bar{a}_r \text{ V/m}$$

$$\therefore \quad V = -\int_{-}^{+} \bar{E} \cdot d\bar{L}$$

$$= -\int_{r=10\text{ cm}}^{r=5\text{ cm}} \frac{Q}{4\pi\varepsilon r^2}\bar{a}_r \cdot dr\,\bar{a}_r \qquad \text{... Note } \varepsilon = \varepsilon_0\varepsilon_r$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_{r=0.1}^{r=0.05} \frac{1}{\left[\frac{0.1}{r}\right]r^2}dr = -\frac{Q}{4\pi\varepsilon_0}\int_{r=0.1}^{0.05}\frac{10}{r}dr$$

$$= -\frac{10Q}{4\pi\varepsilon_0}\{ln[r]\}_{0.1}^{0.05} = -\frac{10Q}{4\pi\varepsilon_0}ln\left[\frac{0.05}{0.1}\right] \qquad \text{... } Q = 1\ \mu C$$

$$= 62.298 \text{ kV}$$

And
$$C = \frac{Q}{V} = \frac{1\times10^{-6}}{62.298\times10^{3}} = 16.051 \text{ pF}$$

Example 5.21 : *The relative permittivity of dielectric in a parallel plate capacitor varies linearly from 4 to 8. If distance of separation of plates is 1 cm and area of cross section of plates is 12 cm², find the capacitance.*

Solution : The arrangement is shown in the Fig. 5.33.

The ε_r varies linearily from 4 to 8, along x direction. The equation for linear behaviour is,

$$\varepsilon_r = Kx + C$$

At $x = 0,\ \varepsilon_r = \varepsilon_{r1} = 4$

$$\therefore \quad 4 = 0 + C$$

$$\therefore \quad C = 4$$

At $x = 0.01,\ \varepsilon_r = \varepsilon_{r2} = 8$

$$\therefore \quad 8 = 0.01\,K + 4$$

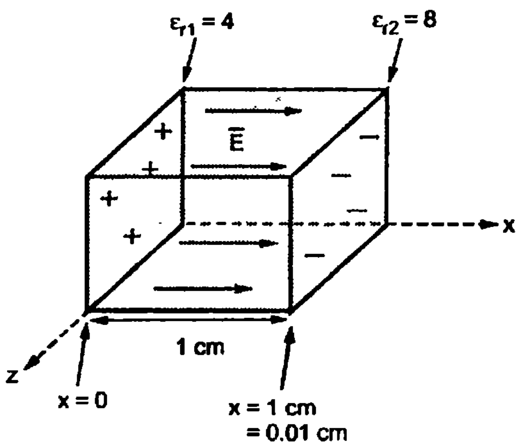


Fig. 5.33

$\therefore \quad K = 400$

$\therefore \quad \varepsilon_r = 400\,x + 4 \quad \ldots (1)$

Now let plate at x = 0 carries positive charge.

$\therefore \quad \bar{E}_1 = \frac{+\rho_S}{2\varepsilon}\,\bar{a}_x \quad$ ... due to plate at x = 0

And $\quad \bar{E}_2 = \frac{-\rho_s}{2\varepsilon}(-\bar{a}_x) \quad$ ... due to plate at x = 0.01

$\therefore \quad \bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{\rho_S}{\varepsilon}\,\bar{a}_x \quad$ ... between the plates

$$\therefore \quad V = -\int_{-}^{+} \bar{E} \cdot d\bar{L} = -\int_{x=0.01}^{0} \frac{\rho_S}{\varepsilon}\,\bar{a}_x \cdot dx\,\bar{a}_x$$

and $\quad \varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0\,[400\,x + 4] \quad$ ... from equation (1)

$$\therefore \quad V = -\int_{x=0.01}^{0} \frac{\rho_S}{\varepsilon_0 (400\,x+4)}\,dx = -\frac{\rho_S}{\varepsilon_0}\int_{x=0.01}^{0} \frac{1}{400\,x+4}\,dx$$

$$= -\frac{\rho_S}{\varepsilon_0}\,\frac{[\ln 400\,x+4]_{0.01}^{0}}{400}$$

$$= \frac{-\rho_S}{400\,\varepsilon_0}\,\ln\left[\frac{4}{8}\right]$$

$$= \frac{-\rho_S(-0.6931)}{400 \times 8.854 \times 10^{-12}}$$

$$= 195.715 \times 10^6\,\rho_S\ \text{V}$$

And $\quad Q = \rho_S\,A$

$$\therefore \quad C = \frac{Q}{V} = \frac{\rho_S\,A}{195.715 \times 10^6\,\rho_S} \quad \ldots A = 12\ \text{cm}^2$$

$$= \frac{12 \times 10^{-4}}{195.715 \times 10^6}$$

$$= \mathbf{6.1313\ pF}$$

Example 5.22 : *A spherical condenser has a capacity of 54 pF. It consists of two concentric spheres differing in radii by 4 cm and having air as dielectric. Find their radii.*

Solution : C = 54 pF and b − a = 4 cm, $\varepsilon_r = 1$

The spherical capacitor has a capacitance,

$$C = \frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)} \quad \text{where } b > a$$

$$\therefore \quad 54\times10^{-12} = \frac{4\pi\times 8.854\times10^{-12}(ab)}{(b-a)\times10^{-2}}$$

but $b - a = 4$ cm, 10^{-2} as $b - a$ in cm

$$\therefore \quad ab = 1.9413\times10^{-2}$$

Now $\quad b = 4\times10^{-2} + a$ ... From $b - a = 4$ cm

$$\therefore \quad a(4\times10^{-2}+a) = 1.9413\times10^{-2}$$

$$\therefore \quad a^2 + 0.04\,a - 0.019413 = 0$$

$$\therefore \quad a = \frac{-0.04\pm\sqrt{(0.04)^2-(4)(-0.019413)}}{2} = \mathbf{0.1207\ m}$$

... Neglecting negative value

And $\quad b = 4\times10^{-2} + a = \mathbf{0.1607\ m}$

Example 5.23 : *Let $A = 120\ cm^2$, $d = 5$ mm and $\varepsilon_R = 12$ for a parallel plate capacitor -*

a) Calculate the capacitance.

b) After connecting a 40 V battery across the capacitor, calculate E, D, Q and the total stored energy.

c) The source is now removed and the dielectric is carefully withdrawn from between the plates. Again calculate E, D, Q and the energy.

d) What is voltage between the plates.

Solution : a)

$$C = \frac{\varepsilon A}{d} = \frac{8.854\times10^{-12}\times12\times120\times10^{-4}}{5\times10^{-3}} = \mathbf{255\ pF}$$

b)

$$C = \frac{Q}{V}$$

$$\therefore \quad Q = CV = 255\times10^{-12}\times40 = \mathbf{10.2\ nC}$$

$$E = \frac{V}{d} = \frac{40}{5\times10^{-3}} = \mathbf{8\ kV/m}$$

$$D = \varepsilon E = \varepsilon_0\varepsilon_R\,E = \mathbf{0.85\ \mu C/m^2}$$

$$W_E = \frac{1}{2}CV^2 = \frac{1}{2}\times255\times10^{-12}\times(40)^2 = \mathbf{0.204\ \mu J}$$

c) Though source and dielectric is removed, Q on the surface remains same.

$$\therefore \quad Q = \mathbf{10.2\ nC}$$

$$\therefore \quad D = \rho_S = \frac{Q}{A} = \frac{10.2\times10^{-9}}{120\times10^{-4}} = \mathbf{0.85\ \mu C/m^2}$$

Now $$E = \frac{D}{\varepsilon_0} = 96 \text{ kV/m}$$ ... Now $\varepsilon = \varepsilon_0$.

and as V is now not same across the plates, calculate W_E as,

$$W_E = \frac{1}{2} CV^2 = \frac{1}{2} C\left[\frac{Q}{C}\right]^2 \quad \text{... As } V = \frac{Q}{C}$$

$$\therefore \quad W_E = \frac{1}{2}\frac{Q^2}{C}$$

and $$C = \frac{\varepsilon_0 A}{d} = 21.249 \text{ pF} \quad \text{... } \varepsilon = \varepsilon_0$$

$$\therefore \quad W_E = \frac{1}{2}\frac{(10.2\times10^{-9})^2}{21.249\times10^{-12}} = 2.448 \ \mu\text{J}$$

d) $$V = \frac{Q}{C} = \frac{10.2\times10^{-9}}{21.249\times10^{-12}} = 480 \text{ V}$$

Example 5.24 : *A 4 mF capacitor is charged by connecting it across 100 V/d.c. The supply is disconnected and another uncharged 2 mF capacitor is connected across it. If leakage charge is negligible, determine the potential between the plates.*

Solution : The arrangement is shown in the Fig. 5.34.

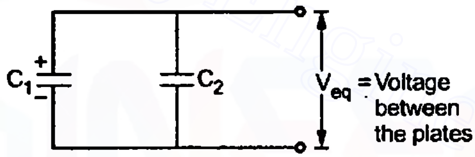


Fig. 5.34

Initially when C_1 is charged to 100 V d.c., the energy stored is,

$$E = \frac{1}{2} C_1 V^2 = \frac{1}{2}\times4\times10^{-3}\times100^2 = 20 \text{ J}$$

This energy must remain same while voltage across the two must be same as V_{eq}. So total energy in the new arrangement is,

$$E = \frac{1}{2} C_1 V_{eq}^2 + \frac{1}{2} C_2 V_{eq}^2$$

$$\therefore \quad 20 = \frac{1}{2}\times4\times10^{-3} V_{eq}^2 + \frac{1}{2}\times 10^{-3}\times V_{eq}^2$$

$$\therefore \quad V_{eq}^2 = 6666.6667$$

$$\therefore \quad V_{eq} = 81.6496 \text{ V}$$ ... Potential between the plates

Example 5.25 : *Find the capacitance of the system shown, if length of the plate is L m.* **[UPTU : 2002-03]**

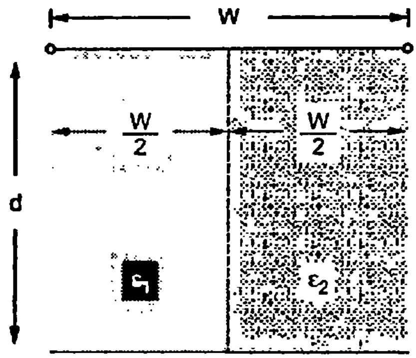


Fig. 5.35

Solution : The equivalent arrangement is two capacitors connected in parallel, as shown in the Fig. 5.36.

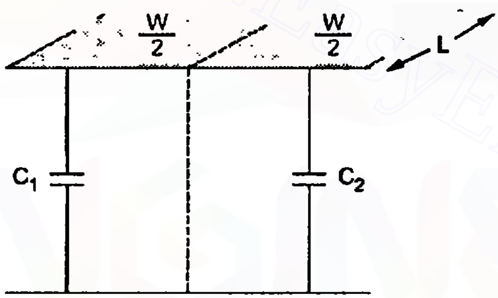


Fig. 5.36

For C_1, $A_1 = \frac{W}{2} \times L$

$$C_1 = \frac{\varepsilon_1 A_1}{d} = \frac{\varepsilon_1 \frac{W}{2} L}{d}$$

For C_2, $A_2 = \frac{W}{2} \times L$

$$C_2 = \frac{\varepsilon_2 A_2}{d} = \frac{\varepsilon_2 \frac{W}{2} L}{d}$$

$$\therefore \quad C_{eq} = C_1 + C_2 = \frac{\varepsilon_1 WL}{2d} + \frac{\varepsilon_2 WL}{2d} \quad \text{... Parallel capacitors}$$

$$\therefore \quad \boxed{C_{eq} = \frac{WL}{2d}(\varepsilon_1 + \varepsilon_2)} \quad \text{... Required capacitance}$$

Example 5.26 : *Find the capacitance of the system shown if length of the plate is L m.*

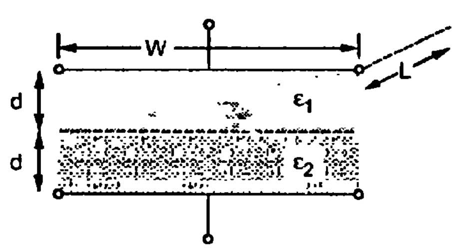


Fig. 5.37

Solution : Area A = W×L is common to both the capaciters. The equivalent arrangement is two capacitors in series as shown in the Fig. 5.38.

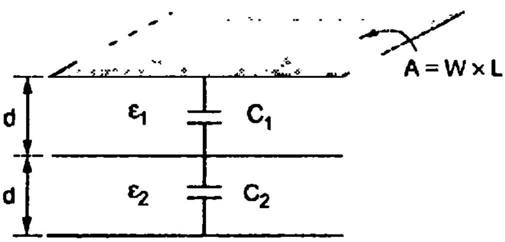


Fig. 5.38

$$C_1 = \frac{\varepsilon_1 A}{d} \quad \text{while } C_2 = \frac{\varepsilon_2 A}{d}$$

where $A = W \times L$

The equivalent capacitance of the

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \frac{\frac{\varepsilon_1 A}{d} \times \frac{\varepsilon_2 A}{d}}{\frac{\varepsilon_1 A}{d} + \frac{\varepsilon_2 A}{d}}$$

$$= \frac{\frac{A}{d}\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

$$\therefore \quad \boxed{C_{eq} = \frac{\left(\frac{A}{d}\right)}{\left[\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right]}} \quad \text{... Required capacitance}$$

Example 5.27 : *Find the total current in a circular conductor of radius 4 mm if the current density varies according to $J = (10^4 / r)\ A/m^2$.*

Solution : The current is given by,

$$I = \oint_S \bar{J} \cdot d\bar{S}$$

Assuming $\bar{J}$ given in $\bar{a}_z$ direction, $d\bar{S} = r\,dr\,d\phi\,\bar{a}_z$

$$\therefore \quad \bar{J} \cdot d\bar{S} = \frac{10^4}{r} \times r\,dr\,d\phi = 10^4\,dr\,d\phi$$

$$\therefore \quad I = \int_{\phi=0}^{2\pi} \int_{r=0}^{4\times10^{-3}} 10^4\,dr\,d\phi = 10^4\,[r]_0^{4\times10^{-3}}\,[\phi]_0^{2\pi}$$

$$= 10^4 \times 4 \times 10^{-3} \times 2 \times \pi = 80\pi\ A$$

Example 5.28 : *The electric field intensity in polystyrene ($\varepsilon_r = 2.55$) filling the space between the plates of a parallel plate capacitor is 10 kV/m. The distance between the plates is 1.5 mm. Calculate : i) The surface charge density of free charge on the plates ii) The potential difference between the plates.* **[UPTU : 2006-07, 5 Marks]**

Solution : $\varepsilon_r = 2.55$, E = 10 kV/m, d = 1.5 mm

i) $$D = \varepsilon_0 \varepsilon_r E = 8.854 \times 10^{-12} \times 2.55 \times 10 \times 10^3$$

$$= 2.2577 \times 10^{-7}\ C/m^2$$

$$\therefore \quad \rho_S = D = 2.2577 \times 10^{-7} \text{ C/m}^2$$

ii) $$E = \frac{V}{d} \quad \text{i.e.} \quad 10 \times 10^3 = \frac{V}{1.5 \times 10^{-3}}$$

$$\therefore \quad V = 15 \text{ V}$$

Example 5.29 : *The electric field intensity at a point on the surface of a conductor is given by*

$\bar{E} = 0.2\,\bar{a}_x - 0.3\,\bar{a}_y - 0.2\bar{a}_z$ V/m. Find the surface charge density at that point.

[UPTU : 2006-07, 5 Marks]

Solution : $E = 0.2\,\bar{a}_x - 0.3\,\bar{a}_y - 0.2\,\bar{a}_z$ V/m

Let the surface of the conductor is the interface between air and the conductor.

$$\bar{D}_N = \varepsilon_0 \bar{E}_N \quad \text{where} \quad \bar{E}_N = \bar{E}$$

$$\therefore \quad \bar{D}_N = 8.854 \times 10^{-12} [0.2\,\bar{a}_x - 0.3\,\bar{a}_y - 0.2\,\bar{a}_z]$$

$$= 1.7708\,\bar{a}_x - 2.6562\,\bar{a}_y - 1.7708\,\bar{a}_z \text{ pC/m}^2$$

$$\therefore \quad \rho_S = \bar{D}_N = 1.7708\,\bar{a}_x - 2.6562\,\bar{a}_y - 1.7708\,\bar{a}_z \text{ pC/m}^2$$

$$\therefore \quad \rho_S \text{ at that point} = \sqrt{(1.7708)^2 + (2.6562)^2 + (1.7708)^2}$$

$$= 3.6506 \text{ pC/m}^2$$

Example 5.30 : *Given the potential $V = 10\,(x^2 + xy)$ and a point $P(2, -1, 3)$ on a conductor to free space boundary find V, $\bar{E}$, $\bar{D}$ at point P.* **[UPTU : 2007-08, 5 Marks]**

Solution : $V = 10\,(x^2 + xy)$, $P\,(2, -1, 3)$

i) $$V = 10\,[2^2 + (2)(-1)] = 10\,[4 - 2] = 20 \text{ V}$$

ii) $$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$= -\,[(20x + 10\,y)\,\bar{a}_x + 10x\,\bar{a}_y + 0\,\bar{a}_z]$$

$$\therefore \quad \bar{E}_P = -\,(20 \times 2 - 10)\,\bar{a}_x + 10 \times 2\,\bar{a}_y] = -\,30\,\bar{a}_x - 20\,\bar{a}_y \text{ V/m}$$

ii) $$D = \varepsilon_0\,\bar{E} = 8.854 \times 10^{-12}\,[-30\,\bar{a}_x - 20\,\bar{a}_y]$$

$$= -\,0.2656\,\bar{a}_x - 0.1771\,\bar{a}_y \text{ nC/m}^2$$

Review Questions

1. *Derive the relation between I and $\bar{J}$.*
2. *Derive the relation between $\bar{J}$ and ρ_v.*
3. *State and explain continuity equation of current in integral form and point form.*
4. *Fxplain the classification of materials based on energy band theory.*
5. *Explain the properties of conductor.*
6. *Derive the Ohm's law in point form.*
7. *Obtain the expression for resistance of a conductor.*
8. *What is relaxation time ? Derive expression for it.*
9. *Explain the polarization in dielectric materials.*
10. *Explain the properties of dielectric materials.*
11. *Explain and derive the boundary conditions for a conductor free space interface.*
12. *Explain and derive the boundary conditions for a dielectric - dielectric interface.*
13. *Explain the law of refraction at dielectric-dielectric interface.*
14. *Explain the capacitance and derive its basic expression.*
15. *Derive the expression for capacitance of parallel plate capacitor.*
16. *Derive the expression for capacitance of co-axial cable.*
17. *Derive the expression for capacitance of spherical capacitor.*
18. *Derive the expression for capacitance of isolated sphere.*
19. *Derive the expression for capacitance of isolated sphere coated with dielectric.*
20. *Derive the expression for capacitance of composite parallel plate capacitor.*
21. *Explain the method of images.*
22. *Find the current density and E for an aluminium having drift velocity of electrons 5.3×10^{-4} m/s, conductivity 3.82×10^7 S/m and $\mu_e = 0.0014$ m²/V-s.* **[Ans. : 1.45×10^7 A / m², 0.379 V/m]**
23. *A conductor of uniform cross section and 150 m long has a voltage drop of 1.3 V and a current density of 4.65×10^5 A / m². What is the conductivity of the material in the conductor ?* **[Ans. : 5.37×10^7 S/m]**
24. *A potential field is given by $V = 150(x^2 - y^2)$. The point $P(4,-2,1)$ lies on the boundary of the conductor and free space. At P, obtain the magnitudes of*
    a) V *b) $\bar{E}$*
    c) E_N *d) E_{tan}*
    e) $\bar{D}$ and *f) ρ_S.*
    [Ans. : 1800 V, 1341.64 V/m, 1341.64 V/m, 0 V/m, 11.87 nC/m², 11.87 nC/m²]
25. *Given $\bar{J} = 10^4 \sin\theta \bar{a}_r$ A / m² in spherical system, find current passing the spherical shell of $r = 0.02$ m.* **[Ans. : 39.5 A]**
26. *Find the angle by which the direction of E changes, as it crosses the boundary between two dielectrics with dielectric constants 4 and 5. The incident angle is 50° with the normal.* **[Ans. : $\theta_2 = 56.1273°$, change by 6.1273°]**

27. *Find the capacitance of a parallel plate capacitor, if the plates are of area 1.5 m^2, the distance between the plates is 2 mm, potential gradient is 10^5 V/m and ρ_S as 2.5 $\mu C/m^2$.*

[Ans. : 18.75 nF]

[Hint : Potential gradient is $E = V/d$ and $Q = \rho_S \times A$]

28. *Find the capacitance of a parallel plate capacitor*

a) When the plates are of area 1 m^2, distance between the plates 1 mm, voltage gradient is 10^5 V/m and the $\rho_S = 2\,\mu C/m^2$.

b) When the stored energy is 5 mJ and the voltage across the plates is 5 V.

[Ans. : 20 nF, 0.4 μF]

29. *An electric field strength 1.2 V/m is entering a dielectric medium of $\varepsilon_R = 4$ from air. The orientation of $\bar{E}$ in air is 65° with respect to boundary. Determine the orientation of $\bar{E}$ in the dielectric and its strength in the dielectric.* **[Ans. : $\theta_2 = 70.35°$ with normal, 0.7308 V/m]**

30. *What is method of imges ? What conditions are to be satisfied to apply the method of images ?*

31. *Explain the method of images for point charges.*

University Questions

1. *Discuss the method of images to plain boundaries prolems.* **[UPTU : 2002-03, 2003-04, 5 Marks]**
2. *What is understood by boundary conditions in static electric field ? Why are the equipotential surfaces perpendiculat to the electric flux lines ?* **[UPTU: 2002-03, 5 Marks]**
3. *Discuss physically why*

   i) The electric field is entirely perpendicular on the surface of perfect conductor.

   ii) The surface of a perfect conductor is an equipotential surface. **[UPTU : 2003-04, 5 Marks]**
4. *Discuss the boundary condition for electric field.* **[UPTU : 2003-04, 10 Marks]**
5. *Explain convection current and conduction current. Derive ohm's law in point form.* **[UPTU : 2005-06, 5 Marks]**
6. *The electric field intensity in polystyrene ($\varepsilon_r = 2.55$) filling the space between the plates of a parallel plate capacitor is 10 kV/m.The distance beween the plates is 1.5 mm. Calculate : i) The surface charge density of free charge on the plates ii) The potential difference between the plates.* **[UPTU : 2006-07, 5 Marks]**
7. *Derive dielectric - dielectric boundary conditions.* **[UPTU : 2006-07, 5 Marks]**
8. *The electric field intensity at a point on the surface of a conductor is given by*

   $E = 0.2\,\bar{a}_x - 0.3\,\bar{a}_y - 0.2\bar{a}_z$ *V/m. Find the surface charge density at that point.* **[UPTU : 2006-07, 5 Marks]**
9. *State and prove continuity equation.* **[UPTU : 2007-08, 5 Marks]**
10. *Given the potential $V = 10\,(x^2 + xy)$ and a point P(2, − 1, 3) on a conductor to free space boundary find V, $\bar{E}$, $\bar{D}$ at point P.* **[UPTU : 2007-08, 5 Marks]**

11. *The Fig. 5.39 shows a capacitor consisting of two parallel, conducting plates separated by a distance d. The space between the plates contains two adjacent dielectrics, one with permittivity* ε_1 *and surface area* A_1 *and another with* ε_2 *and* A_2*. Find total capacitance C,* $C = C_1 + C_2$.

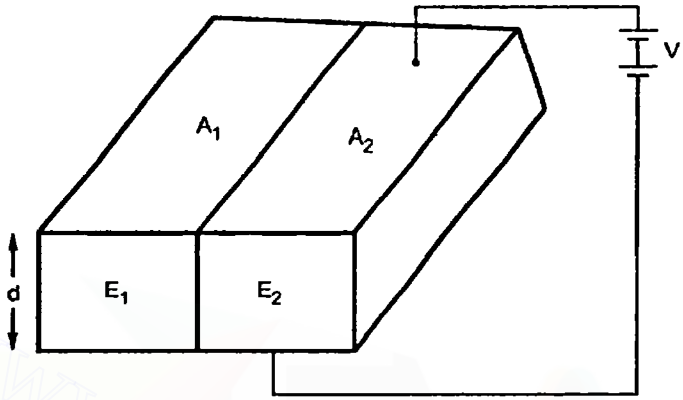


Fig. 5.39

[UPTU. : 2007-08, 5 Marks]

12. *Use Fig. 5.39 and find the electric fields* $\overline{E}_1$ *and* $\overline{E}_2$ *in two dielectric layers.*

[UPTU : 2007-08, 5 Marks]

13. *Show that the capacitance of a parallel plate capacitor is given by* $C = \dfrac{\varepsilon A}{d}$ *where* ε *is the permittivity of the dielectric in between the plates, A is the surface-area of the plates, and d is the distance between the plates of the capacitor.* **[UPTU : 2008-09, 4 Marks]**

14. *Derive an expression for the capacitance per unit length of a coaxial cable with permittivity* ε*, inner diameter d and outer diameter D.* **[UPTU : 2008-09, 6 Marks]**

15. *Define the following terms with suitable examples :* **[UPTU : 2008-09, 10 Marks]**

    i) The dielectric material

    ii) The conductors

    iii) The homogeneous medium

    iv) The linear medium

    v) The isotropic medium

6 Poisson's and Laplace's Equation

6.1 Introduction

In earlier chapters, the $\bar{E}$ and $\bar{D}$ in the given region are obtained using Coulomb's law and Gauss's law. Using these laws is easy, if the charge distribution or potential throughout the region is known. Practically it is not possible in many situations, to know the charge distribution or potential variation throughout the region. Practically charge and potential may be known at some boundaries of the region, only. From those values it is necessary to obtain potential and $\bar{E}$ throughout the region. Such electrostatic problems are called boundary value problems. To solve such problems, Poisson's and Laplace's equations must be known. This chapter derives the Poisson's and Laplace's equations and explains its use in few practical situations.

6.2 Poisson's and Laplace's Equations

From the Gauss's law in the point form, Poisson's equation can be derived. Consider the Gauss's law in the point form as,

$$\nabla \cdot \bar{D} = \rho_v \qquad \text{... (1)}$$

where $\bar{D}$ = Flux density and ρ_v = Volume charge density

It is known that for a homogeneous, isotropic and linear medium, flux density and electric field intensity are directly proportional. Thus,

$$\bar{D} = \varepsilon \bar{E} \qquad \text{... (2)}$$

$$\therefore \quad \nabla \cdot \varepsilon \bar{E} = \rho_v \qquad \text{... (3)}$$

From the gradient relationship,

$$\bar{E} = -\nabla V \qquad \text{... (4)}$$

Substituting (4) in (3),

$$\nabla \cdot \varepsilon(-\nabla V) = \rho_v \qquad \text{... (5)}$$

Taking $-\varepsilon$ outside as constant,

$$-\varepsilon[\nabla \cdot \nabla V] = \rho_v$$

$$\therefore \quad \nabla \bullet \nabla V = -\frac{\rho_v}{\varepsilon} \quad \dots (6)$$

Now $\nabla \bullet \nabla$ operation is called '**del squared**' operation and denoted as ∇^2.

$$\therefore \quad \boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \quad \dots (7)$$

This equation (7) is called **Poisson's equation.**

If in a certain region, volume charge density is zero $(\rho_v = 0)$, which is true for dielectric medium then the Poisson's equation takes the form,

$$\boxed{\nabla^2 V = 0} \quad \text{(For charge free region)}$$

This is special case of Poisson's equation and is called **Laplace's equation.** The ∇^2 operation is called the **Laplacian of V.**

Key Point: *Note that if $\rho_v = 0$, still in that region point charges, line charges and surface charges may exist at singular locations.*

The equation (7) is for homogeneous medium for which ε is constant. But if ε is not constant and the medium is inhomogeneous, then equation (5) must be used as Poisson's equation for inhomogeneous medium.

6.2.1 ∇^2 Operation in Different Co-ordinate Systems

The potential V can be expressed in any of the three co-ordinate systems as $V(x, y, z)$, $V(r, \phi, z)$ or $V(r, \theta, \phi)$. Depending upon it, the ∇^2 operation required for Laplace's equation must be used.

In **Cartesian co-ordinate system,**

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

$$\therefore \quad \nabla \bullet \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$\therefore \quad \boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0} \quad \dots (8)$$

The equation (9) is **Laplace's equation in cartesian form.**

In **cylindrical co-ordinate system,**

$$\boxed{\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0} \quad \dots (9)$$

The equation (10) is **Laplace's equation in cylindrical form.**

In spherical co-ordinate system,

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)+\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0} \quad \text{... (10)}$$

The equation (11) is **Laplace's equation in spherical form.**

6.3 Uniqueness Theorem

The boundary value problems can be solved by number of methods such as analytical, graphical, experimental etc. Thus there is a question that, is the solution of Laplace's equation solved by any method, unique ? The answer to this question is the uniqueness theorem, which is proved by contradiction method.

Assume that the Laplace's equation has two solutions say V_1 and V_2, both are function of the co-ordinates of the system used. These solutions must satisfy Laplace's equation. So we can write,

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0 \quad \text{... (1)}$$

Both the solutions must satisfy the boundary conditions as well. At the boundary, the potentials at different points are same due to equipotential surface then,

$$V_1 = V_2 \quad \text{... (2)}$$

Let the difference between the two solutions is V_d.

$$V_d = V_2 - V_1 \quad \text{... (3)}$$

Using Laplace's equation for the difference V_d,

$$\nabla^2 V_d = \nabla^2 (V_2 - V_1) = 0 \quad \text{... (4)}$$

$$\therefore \quad \nabla^2 V_2 - \nabla^2 V_1 = 0 \quad \text{... (5)}$$

On the boundary $V_d = 0$ from the equations (2) and (3).

Now the divergence theorem states that,

$$\int_{vol} \nabla \cdot \bar{A}\, dv = \oint_{S} \bar{A} \cdot d\bar{S} \quad \text{... (6)}$$

Let $\bar{A} = V_d \nabla V_d$ and from vector algebra,

$$\nabla \cdot (\alpha \bar{B}) = \alpha(\nabla \cdot \bar{B}) + \bar{B} \cdot (\nabla \alpha)$$

Now use this for $\nabla \cdot (V_d \nabla V_d)$ with $\alpha = V_d$ and $\nabla V_d = \bar{B}$.

$$\therefore \quad \nabla \cdot (V_d \nabla V_d) = V_d (\nabla \cdot \nabla V_d) + \nabla V_d \cdot (\nabla V_d)$$

But $\quad \nabla \cdot \nabla = \nabla^2$ hence,

$$\therefore \quad \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d \quad \text{... (7)}$$

Using equation (4),

$$\nabla \cdot (V_d \nabla V_d) = \nabla V_d \cdot \nabla V_d \quad \text{... (8)}$$

To use this in equation (6), let $\bar{A} = V_d \nabla V_d$ hence

$$\nabla \cdot V_d \nabla V_d = \nabla \cdot \bar{A} = \nabla V_d \cdot \nabla V_d$$

$$\int_{vol} \nabla V_d \cdot \nabla V_d \, dv = \oint_S V_d \nabla V_d \cdot d\bar{S} \qquad \text{... (9)}$$

But $V_d = 0$ on boundary, hence right hand side of equation (9) is zero.

$$\therefore \int_{vol} \nabla V_d \cdot \nabla V_d \, dv = 0 \qquad \text{... (10)}$$

This is volume integral to be evaluated on the volume enclosed by the boundary.

It is known that, $\bar{C} \cdot \bar{C} = |\bar{C}|^2$,

$$\therefore \int_{vol} |\nabla V_d|^2 \, dv = 0 \text{ as } \nabla V_d \text{ is vector.} \qquad \text{... (11)}$$

Now integration can be zero under two conditions,

i) The quantity under integral sign is zero.

ii) The quantity is positive in some regions and negative in other regions by equal amount and hence zero.

But square term can not be negative in any region hence, quantity under integral must be zero.

$$|\nabla V_d|^2 = 0$$

i.e.

$$\nabla V_d = 0 \qquad \text{... (12)}$$

As the gradient of $V_d = V_2 - V_1$ is zero means $V_2 - V_1$ is constant and not changing with any co-ordinates. But considering boundary it can be proved that $V_2 - V_1 = \text{constant} = \text{zero}$.

$$\therefore \quad V_2 = V_1 \qquad \text{... (13)}$$

This proves that both the solutions are equal and cannot be different.

Thus **Uniqueness Theorem** can be stated as :

If the solution of Laplace's equation satisfies the boundary condition then that solution is unique, by whatever method it is obtained.

The solution of Laplace's equation gives the field which is unique, satisfying the same boundary conditions, in a given region.

6.4 Procedure for Solving Laplace's Equation

The procedure to solve a problem involving Laplace's equation can be generalized as,

Step 1 : Solve the Laplace's equation using the method of integration. Assume constants of integration as per the requirement.

Step 2 : Determine the constants applying the boundary conditions given or known for the region. The solution obtained in step 1 with constants obtained using boundary conditions is an unique solution.

Step 3 : Then $\bar{E}$ can be obtained for the potential field V obtained, using gradient operation $-\nabla$ V.

Step 4 : For homogeneous medium, $\bar{D}$ can be obtained as $\varepsilon\bar{E}$.

Step 5 : At the surface, $\rho_S = D_N$ hence once $\bar{D}$ is known, the normal component D_N to the surface is known. Hence the charge induced on the conductor surface can be obtained as $Q = -\int \rho_S dS$.

Step 6 : Once the charge induced Q is known and potential V is known then the capacitance C of the system can be obtained.

If $\rho_v \neq 0$ then similar procedure can be adopted to solve the Poisson's equation.

Example 6.1 : *Determine whether or not the following potential fields satisfy the Laplace's equation :*

a) $V = x^2 - y^2 + z^2$ *b)* $V = r\cos\phi + z$ *c)* $V = r\cos\theta + \phi$ **[UPTU : 2003-04]**

Solution : a) $V = x^2 - y^2 + z^2$

$$\therefore \quad \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2}{\partial x^2}\left[x^2 - y^2 + z^2\right] + \frac{\partial^2}{\partial y^2}\left[x^2 - y^2 + z^2\right] + \frac{\partial^2}{\partial z^2}\left[x^2 - y^2 + z^2\right]$$

$$= \frac{\partial}{\partial x}[2x] + \frac{\partial}{\partial y}[-2y] + \frac{\partial}{\partial z}[2z] = 2 - 2 + 2 = 2$$

So $\nabla^2 V \neq 0$

Hence field V **does not satisfy Laplace's equation.**

b) $V = r\cos\phi + z$

In cylindrical co-ordinate system,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}[r\cos\phi + z] = \cos\phi$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}[r\cos\phi + z] = -r\sin\phi$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}[r\cos\phi + z] = 1$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}[r\cos\phi] = \frac{1}{r}\cos\phi$$

$$\frac{1}{r^2}\left[\frac{\partial^2 V}{\partial \phi^2}\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial \phi}(-r\sin\phi)\right] = -\frac{r\cos\phi}{r^2} = -\frac{\cos\phi}{r}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}[1] = 0$$

$$\therefore \quad \nabla^2 V = \frac{1}{r}\cos\phi - \frac{\cos\phi}{r} + 0 = 0$$

So this field **satisfies Laplace's equation.**

c) $V = r\cos\theta + \phi$

In spherical system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$r^2\frac{\partial V}{\partial r} = r^2\frac{\partial}{\partial r}[r\cos\theta+\phi] = r^2(\cos\theta)$$

$$\sin\theta\frac{\partial V}{\partial\theta} = \sin\theta\frac{\partial}{\partial\theta}[r\cos\theta+\phi] = \sin\theta[-r\sin\theta] = -r\sin^2\theta$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}[r\cos\theta+\phi] = \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}[1] = 0$$

$$\therefore \quad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\cos\theta\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(-r\sin^2\theta\right)$$

$$= \frac{1}{r^2}2r\cos\theta + \frac{1}{r^2\sin\theta}\left[-r\,2\sin\theta\cos\theta\right] = \frac{2}{r}\cos\theta - \frac{2}{r}\cos\theta$$

$$= 0$$

So this field **satisfies Laplace's equation.**

Example 6.2 : *Verify that the potential field given below satisfies the Laplace's equation.* $V = 2x^2 - 3y^2 + z^2$

Solution : Given field is in cartesian system,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}\left[2x^2-3y^2+z^2\right] + \frac{\partial^2}{\partial y^2}\left[2x^2-3y^2+z^2\right] + \frac{\partial^2}{\partial z^2}\left[2x^2-3y^2+z^2\right]$$

$$= \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial y}[-6y] + \frac{\partial}{\partial z}[2z] = 4-6+2 = 0$$

As $\nabla^2 V = 0$, the field **satisfies the Laplace's equation.**

Example 6.3 : *The region between two concentric right circular cylinders contains a uniform charge density ρ. Solve the Poisson's equation for the potential in the region.*

Solution : The cylinders are shown in the Fig. 6.1.

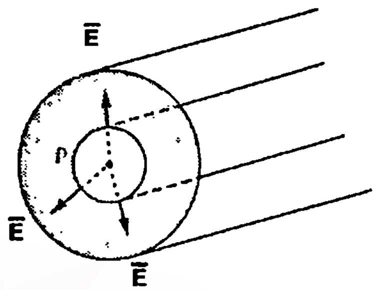


Fig. 6.1

Select the cylindrical co-ordinate system. In co-axial cable like structure, the electric field intensity $\bar{E}$ is in radial direction from inner to outer cylinder.

Hence $\bar{E}$ and V both are functions of only r and not of ϕ and z.

$\therefore$ $\frac{\partial V}{\partial r}$ is existing while $\frac{\partial V}{\partial \phi}$ and $\frac{\partial V}{\partial z}$ are zero.

According to Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\varepsilon}, \text{ here } \rho_v = \rho \text{ given}$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -\frac{\rho}{\varepsilon}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -\frac{\rho r}{\varepsilon}$$

Integrating both sides,

$$r\frac{\partial V}{\partial r} = \int -\frac{\rho r}{\varepsilon}\,dr = -\frac{\rho}{\varepsilon}\frac{r^2}{2}+C_1$$

where C_1 = Constant of integration

$$\therefore \quad \frac{\partial V}{\partial r} = -\frac{\rho}{\varepsilon}\frac{r}{2}+\frac{C_1}{r}$$

Integrating both sides,

$$\therefore \quad V = -\frac{\rho}{2\varepsilon}\left[\frac{r^2}{2}\right]+C_1[\ln r]+C_2$$

where $\quad C_2 = $ Constant of integration

$$\therefore \quad \boxed{V = -\frac{\rho r^2}{4\varepsilon} + C_1 \ln(r) + C_2}$$

Knowing the boundary conditions, C_1 and C_2 can be obtained.

Example 6.4 : *In a free space,* $\rho_v = \dfrac{200\,\varepsilon_0}{r^{2.4}}$

i) Use Poisson's equation, to find V as a function of r, if it is assumed that $r^2 E_r \to 0$ *as* $r \to 0$ *and* $V \to 0$ *as* $r \to \infty$. *Use spherical co-ordinate system.*

ii) Find potential V as a function of r using Gauss's law and line integral.

Solution : i) Poisson's equation states that,

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon_0} \quad \text{... as free space } \varepsilon = \varepsilon_0$$

$$\therefore \quad \nabla^2 V = -\frac{200\,\varepsilon_0}{r^{2.4}\,\varepsilon_0} = -\frac{200}{r^{2.4}}$$

From the conditions given it is clear that V is a function of r only and not the function of θ and ϕ.

$$\therefore \quad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right]$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = -\frac{200}{r^{2.4}}$$

$$\therefore \quad \frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = -200\, r^{-0.4}$$

Integrate, $r^2 \dfrac{\partial V}{\partial r} = -\int 200\, r^{-0.4}\, dr + C_1$

$$\therefore \quad r^2 \frac{\partial V}{\partial r} = -\frac{200\, r^{0.6}}{0.6} + C_1 = -333.33\, r^{0.6} + C_1 \qquad \text{... (1)}$$

As $\bar{E}$ is the function of r only we can write,

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\,\bar{a}_r = E_r\,\bar{a}_r$$

and

$$E_r = -\frac{\partial V}{\partial r} \qquad \text{... (2)}$$

$$\therefore \quad -r^2 E_r = -333.33\, r^{0.6} + C_1 \qquad \text{... (3)}$$

But as $r \to 0$, $r^2 E_r \to 0$ .... (given)

$$\therefore \quad 0 = 0+C_1$$

$$\therefore \quad C_1 = 0 \qquad \text{... (4)}$$

Using in equation (1), $r^2 \dfrac{\partial V}{\partial r} = -333.33\, r^{0.6}$

$$\therefore \quad \frac{\partial V}{\partial r} = -333.33\, r^{-1.4}$$

Integrate, $$V = -333.33 \int r^{-1.4}\, dr + C_2$$

$$= -333.33 \frac{r^{-0.4}}{(-0.4)} + C_2 = \frac{833.325}{(r)^{0.4}} + C_2 \qquad \text{... (5)}$$

Use $V \to 0$ as $r \to \infty$

$$\therefore \quad 0 = \frac{833.325}{(\infty)^{0.4}} + C_2 = 0 + C_2$$

$$\therefore \quad C_2 = 0 \qquad \text{... (6)}$$

$$\therefore \quad \boxed{V = \frac{833.325}{(r)^{0.4}}\ \text{V}}$$

ii) Let us verify this using Gauss's law.

$$\nabla \cdot \bar{D} = \rho_v$$

$$\therefore \quad \nabla \cdot \varepsilon_0 \bar{E} = \rho_v$$

$$\therefore \quad \nabla \cdot \bar{E} = \frac{\rho_v}{\varepsilon_0} = \frac{+200\,\varepsilon_0}{r^{2.4}\,\varepsilon_0} = \frac{200}{r^{2.4}}$$

where $\bar{E} = E_r \bar{a}_r$ and no other component exist.

Consider the radial component of $\bar{E}$ in spherical co-ordinate system and hence divergence of $\bar{E}$ is,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = \frac{200}{r^{2.4}}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 E_r\right) = \frac{200}{r^{0.4}}$$

Integrate, $r^2 E_r = 200 \dfrac{r^{-0.4+1}}{0.6} + C_1 = 333.33\, r^{0.6} + C_1$

But $r^2 E_r \to 0$ as $r \to 0$ hence $C_1 = 0$

$$\therefore \quad r^2 E_r = 333.33\, r^{0.6}$$

$$\therefore \quad \bar{E} = E_r \bar{a}_r = 333.33\, r^{-1.4}\, \bar{a}_r \quad \text{V/m}$$

Now $V = -\int \bar{E} \cdot d\bar{L}$ where $d\bar{L} = dr\,\bar{a}_r$ in radial direction

$\therefore \quad V = -\int 333.33\, r^{-1.4}\, \bar{a}_r \cdot dr\,\bar{a}_r = -333.33 \int r^{-1.4}\, dr$

$$= -333.33 \frac{r^{-0.4}}{-0.4} + C_2 = \frac{833.33}{(r)^{0.4}} + C_2$$

But $V = 0$ as $r \to \infty$ hence $C_2 = 0$

$$\therefore \quad \boxed{V = \frac{833.33}{(r)^{0.4}}\ V}$$

This is same as obtained above using Poisson's equation.

6.5 Calculating Capacitance using Laplace's Equation

As mentioned earlier, the Laplace's equation can be used to find the capacitance under various conditions. Let us discuss few examples of calculating capacitance using Laplace's equation.

Example 6.5 : *Solve the Laplace's equation for the potential field in the homogeneous region between the two concentric conducting spheres with radii a and b, such that $b > a$ if potential $V = 0$ at $r = b$ and $V = V_0$ at $r = a$. And find the capacitance between the two concentric spheres.* **[UPTU : 2002-03]**

Solution : The concentric conductors are shown in the Fig. 6.2.

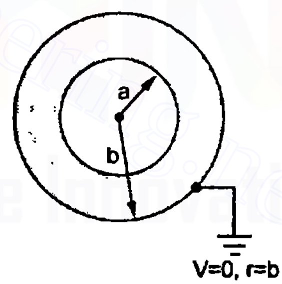


Fig. 6.2

At $r = b$, $V = 0$ hence the outer sphere is shown at zero potential.

The field intensity $\bar{E}$ will be only in radial direction hence V is changing only in radial direction as the radial distance r, and not the function of θ and ϕ.

According to Laplace's equation,

$$\nabla^2 V = 0$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \quad \text{... as V is function of r only}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

Integrate, $$r^2 \frac{\partial V}{\partial r} = \int 0 + C_1 = C_1 \quad \text{... (1)}$$

$$\therefore \quad \frac{\partial V}{\partial r} = \frac{C_1}{r^2} = C_1\, r^{-2}$$

Integrate, $$V = \int C_1\, r^{-2}\, dr + C_2 = \frac{C_1\, r^{-1}}{-1} + C_2$$

$$\therefore \quad V = -\frac{C_1}{r} + C_2 \qquad \text{... (2)}$$

Use the boundary conditions,

$V = 0$ at $r = b$ and $V = V_0$ at $r = a$

$$\therefore \quad 0 = -\frac{C_1}{b} + C_2 \quad \text{and} \quad V_0 = -\frac{C_1}{a} + C_2$$

Subtracting the two equations,

$$-V_0 = -\frac{C_1}{b} - \left(-\frac{C_1}{a}\right)$$

$$\therefore \quad -V_0 = C_1\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$\therefore \quad C_1 = \frac{-V_0}{\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{V_0}{\left[\frac{1}{b} - \frac{1}{a}\right]}$$

$$\therefore \quad C_2 = \frac{C_1}{b} = \frac{V_0}{b\left[\frac{1}{b} - \frac{1}{a}\right]}$$

$$\therefore \quad \boxed{V = -\frac{V_0}{r\left[\frac{1}{b} - \frac{1}{a}\right]} + \frac{V_0}{b\left[\frac{1}{b} - \frac{1}{a}\right]} \ V}$$

This is the potential field in the region between the two spheres.

Now $$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\,\bar{a}_r = -\frac{\partial}{\partial r}\left[\frac{-V_0}{r\left(\frac{1}{b} - \frac{1}{a}\right)}\right]\bar{a}_r \qquad \text{... } C_2 \text{ is not function of r}$$

$$\therefore \quad \boxed{\bar{E} = \frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)} \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\bar{a}_r = \frac{-V_0}{\left(\frac{1}{b} - \frac{1}{a}\right) r^2}\,\bar{a}_r \quad V/m}$$

$$\therefore \quad \bar{D} = \varepsilon \bar{E} = \frac{-\varepsilon V_0}{\left(\frac{1}{b} - \frac{1}{a}\right) r^2}\,\bar{a}_r = \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right) r^2} \ C/m^2$$

As per the boundary conditions between conductor and dielectric, the $\bar{D}$ is always normal to the surface hence $\bar{D}_N$.

$$\therefore \quad \rho_S = |\bar{D}_N| = |\bar{D}| = \frac{\varepsilon V_0}{\left(\frac{1}{a}-\frac{1}{b}\right) r^2} \ C/m^2$$

Now $\quad Q$ = Total charge on the surface of sphere of radius r

$$= \frac{\varepsilon V_0}{\left(\frac{1}{a}-\frac{1}{b}\right) r^2} \times \text{ Surface area of sphere of radius r}$$

$$= \frac{\varepsilon V_0}{\left(\frac{1}{a}-\frac{1}{b}\right) r^2} \times 4\pi r^2 = \frac{4\pi\varepsilon V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} \ C$$

Now $\quad C = \frac{Q}{V}$ where V = Potential between two spheres

$\therefore \quad V = V_0$ = Potential difference between two spheres

$$\therefore \quad \boxed{C = \frac{Q}{V_0} = \frac{\frac{4\pi\varepsilon V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}}{V_0} = \frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)} \ F}$$

This is the capacitance of a spherical capacitor.

Example 6.6 : *Use Laplace's equation to find the capacitance per unit length of a co-axial cable of inner radius 'a' m and outer radius 'b' m. Assume $V = V_0$ at $r = a$ and $V = 0$ at $r = b$.*

Solution : The co-axial cable is shown in the Fig. 6.3.

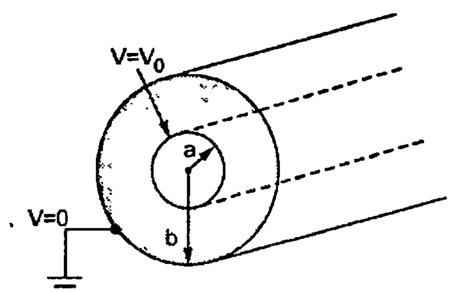


Fig. 6.3

Consider cylindrical co-ordinate system. The field intensity $\bar{E}$ is in radial direction from inner to outer cylinder hence V is a function of r only and not the function of ϕ and z.

Using Laplace's equation,

$$\nabla^2 V = 0$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \qquad \text{... V = f(r) only}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

Integrate, $$r\frac{\partial V}{\partial r} = \int 0 + C_1 = C_1 \qquad \text{... (1)}$$

$$\therefore \quad \frac{\partial V}{\partial r} = \frac{C_1}{r}$$

Integrate, $$V = \int \frac{C_1}{r} + C_2 = C_1\,[ln\, r] + C_2 \qquad \text{... (2)}$$

Using boundary conditions, V = 0 at r = b and $V = V_0$ at r = a,

$$0 = C_1\, ln(b) + C_2 \quad \text{and} \quad V_0 = C_1\, ln(a) + C_2$$

Subtracting, $$-V_0 = C_1\,\{ln(b) - ln(a)\} = C_1\left\{ln\left(\frac{b}{a}\right)\right\}$$

$$\therefore \quad C_1 = \frac{-V_0}{ln\left(\frac{b}{a}\right)} = \frac{V_0}{ln\left(\frac{a}{b}\right)}$$

and $$C_2 = -C_1\, ln(b) = \frac{-V_0\, ln(b)}{ln\left(\frac{a}{b}\right)}$$

$$\therefore \quad \boxed{V = \frac{V_0}{ln\left(\frac{a}{b}\right)}\, ln(r) - \frac{V_0\, ln(b)}{ln\left(\frac{a}{b}\right)} \quad V}$$

Now $$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\,\bar{a}_r$$

$$= -\frac{\partial}{\partial r}\left[\frac{V_0\, ln(r)}{ln\left(\frac{a}{b}\right)}\right]\bar{a}_r \qquad \text{.... } C_2 \text{ is not function of r}$$

$$\therefore \quad \boxed{\bar{E} = -\frac{V_0}{ln\left(\frac{a}{b}\right)}\left[\frac{\partial}{\partial r}\, ln(r)\right]\bar{a}_r = -\frac{V_0}{r\, ln\left(\frac{a}{b}\right)}\,\bar{a}_r \quad V/m}$$

$$\therefore \quad D = \varepsilon \bar{E} = \frac{-V_0 \varepsilon}{r \ln\left(\frac{a}{b}\right)} \bar{a}_r = \frac{V_0 \varepsilon}{r \ln\left(\frac{b}{a}\right)} \bar{a}_r \ C/m^2$$

Now $\bar{D}$ is existing normal to the surface as per the boundary conditions.

$$\therefore \quad \bar{D} = \bar{D}_N = \frac{V_0 \varepsilon}{r \ln\left(\frac{b}{a}\right)} \bar{a}_r$$

$$\therefore \quad \rho_S = |\bar{D}_N| = \frac{V_0 \varepsilon}{r \ln\left(\frac{b}{a}\right)} \ C/m^2$$

ρ_S exists on entire surface area of inner cylinder.

$$\therefore \quad Q = \rho_S \times \text{Surface area of inner cylinder}$$

$$= \frac{V_0 \varepsilon}{r \ln\left(\frac{b}{a}\right)} \times 2\pi r \times L = \frac{V_0 \varepsilon 2\pi L}{\ln\left(\frac{b}{a}\right)} \ C$$

The potential difference between the two cylinders is V_0. Thus $V = V_0$.

$$\therefore \quad C = \frac{Q}{V} = \frac{\frac{V_0 \varepsilon 2\pi L}{\ln\left(\frac{b}{a}\right)}}{V_0}$$

$$\therefore \quad C = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)} \ F$$

The capacitance per unit length i.e. L = 1 m is,

$$\boxed{C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)} \ F/m}$$

Examples with Solutions

Example 6.7 : *Two parallel conducting discs are separated by distance 5 mm at z = 0 and z = 5 mm. If V = 0 at z = 0 and V = 100 V at z = 5 mm, find the charge densities on the discs.*

Solution : The discs are shown in the Fig. 6.4.

Consider cylindrical co-ordinates. The potential V is the function of z alone and is independent of r and ϕ.

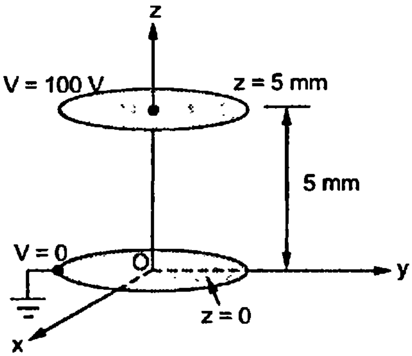


Fig. 6.4

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{... Laplace's equation}$$

Integrating, $\frac{\partial V}{\partial z} = \int 0\,dz + C_1 = C_1$

Integrating, $V = \int C_1 dz + C_2 = C_1 z + C_2$

At z = 0, V = 0 V and at z = 0.005 m, V = 100 V

$\therefore \quad 0 = C_1(0) + C_2$ thus $C_2 = 0$

and $\quad 100 = C_1 \times 0.005 + C_2$ thus $C_1 = 20 \times 10^3$

$\therefore \quad V = 20 \times 10^3 z$ V

Now $\quad \bar{E} = -\nabla V = -\frac{\partial V}{\partial z} \bar{a}_z = -\frac{\partial}{\partial z}[20 \times 10^3 z]\bar{a}_z$

$= -20 \times 10^3 \bar{a}_z$ V/m

$\therefore \quad \bar{D} = \varepsilon_0 \bar{E} = -8.854 \times 10^{-12} \times 20 \times 10^3 \bar{a}_z = -1.77 \times 10^{-7} \bar{a}_z$ C/m^2

- The $\bar{D}$ acts in the normal direction as per the boundary conditions. Thus $\bar{D} = \bar{D}_N$.

$\therefore \quad \bar{D}_N = -1.7708 \times 10^{-7} a_z$

$\therefore \quad \rho_S = |\bar{D}_N| = 1.7708 \times 10^{-7}$ C/m^2 = 177.08 nC/m^2

This is the magnitude of surface charge densities on the discs. So $\rho_S = \pm 177.08$ nC/m^2, positive on upper plate and negative on lower plate.

Example 6.8 : *Two large parallel conducting plates at x = – 1 cm and x = 3 cm are at the potentials V_a and V_b respectively. The region between the plates is filled with the dielectric carbon-di-oxide with $\varepsilon_r = 1$. Potential at x = – 0.5 cm is 70 V while at x = 1.5 cm it is 450 V. Calculate V_a and V_b and $\bar{E}$ between the plates.*

Solution : The plates are shown in the Fig. 6.5, which are parallel to y-z plane.

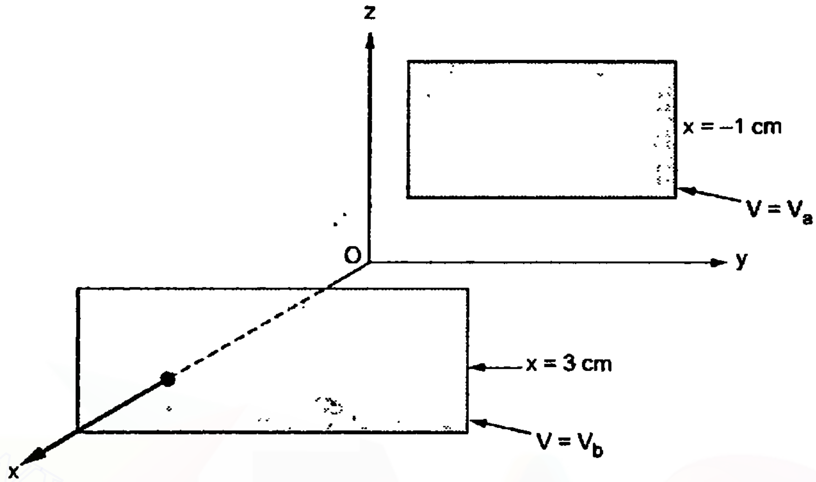


Fig. 6.5

The potential is changing with respect to x only and is independent of y and z.

$$\therefore \quad \nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$$

$$\text{Integrating,} \quad \frac{\partial V}{\partial x} = \int 0\, dx + C_1 = C_1$$

$$\text{Integrating,} \quad V = \int C_1\, dx + C_2 = C_1 x + C_2$$

At x = – 0.5 cm, V = 70 V and at x = 1.5 cm, V = 450 V

$$\therefore \quad 70 = -0.5\times10^{-2}\, C_1 + C_2 \quad \text{and} \quad 450 = +1.5\times10^{-2}\, C_1 + C_2$$

$$\text{Subtracting,} \quad -380 = -0.5\times10^{-2}\, C_1 - 1.5\times10^{-2} C_1$$

$$\therefore \quad -380 = -2\times10^{-2}\, C_1 \qquad \text{hence } C_1 = 19000$$

$$\therefore \quad C_2 = 165$$

$$\therefore \quad V = 19000\, x + 165 \text{ V}$$

$$\therefore \quad V_a = (V \text{ at } x = -1 \text{ cm}) = -19000\times1\times10^{-2} + 165$$

$$= -25 \text{ V}$$

$$\text{and} \quad V_b = (V \text{ at } x = 3 \text{ cm}) = 19000\times3\times10^{-2} + 165 = 735 \text{ V}$$

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial x}\bar{a}_x = -19000\,\bar{a}_x \quad \text{V/m}$$

Example 6.9 : *Find V at P (2, 1, 3) for the field of two co-axial conducting cones, with $V = 50$ V at $\theta = 30°$ and $V = 20$ V at $\theta = 50°$.*

Solution : V is a function of θ only and not the function of r and ϕ.

$$\therefore \quad \nabla^2 V = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \quad \text{... Laplace's equation}$$

$$\therefore \quad \frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] = 0$$

Integrating,

$$\sin\theta\frac{\partial V}{\partial\theta} = \int 0\, d\theta + C_1 = C_1$$

$$\therefore \quad \frac{\partial V}{\partial\theta} = \frac{C_1}{\sin\theta} = C_1 \operatorname{cosec}\theta$$

Integrating, $$V = \int C_1 \operatorname{cosec}\theta\, d\theta + C_2$$

$$= C_1\, ln\left[\tan\frac{\theta}{2}\right] + C_2$$

At $\theta = 30°$, V = 50 V and at $\theta = 50°$, V = 20 V

$$\therefore \quad 50 = C_1\, ln\left[\tan\frac{30°}{2}\right] + C_2 \quad \text{and} \quad 20 = C_1\, ln\left[\tan\frac{50°}{2}\right] + C_2$$

i.e. $50 = -1.3169\, C_1 + C_2$ and $20 = -0.7629\, C_1 + C_2$

Subtracting, $30 = -0.5539\, C_1$

$$\therefore \quad C_1 = -54.152, \qquad C_2 = -21.3125$$

$$\therefore \quad V = -54.152\, ln\left[\tan\frac{\theta}{2}\right] - 21.3125 \quad \text{V} \quad \text{... Use } \theta \text{ in degrees.}$$

For P (2, 1, 3), x = 2, y = 1, z = 3

$$\therefore \quad \theta = \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$$

$$= 36.6692°$$

$$\therefore \quad V_P = -54.152\, ln\left[\tan\frac{36.6992°}{2}\right] - 21.3125$$

$$= \mathbf{38.4489\ V}$$

Example 6.10 : *If V = 2 V at x = 1 mm and V = 0 at x = 0 and volume charge density ρ_v is $-10^6 \varepsilon_0$ C/m^3 constant throughout the region between x = 0 to x = 1 mm, calculate V at x = 0.5 mm and E_x at x = 1 mm in free space.*

Solution : As ρ_v is not zero, use Poisson's equation.

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = \frac{-\left[-10^6 \varepsilon_0\right]}{\varepsilon_0} = 10^6 \qquad \ldots \varepsilon = \varepsilon_0 \text{ as free space.}$$

As V is the function of x only and not of y and z,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 10^6$$

Integrating, $$\frac{\partial V}{\partial x} = \int 10^6\, dx + C_1 = 10^6\, x + C_1$$

Integrating, $$V = \int \left(10^6 x + C_1\right) dx + C_2 = \frac{10^6 x^2}{2} + C_1 x + C_2$$

At x = 0, V = 0 V hence, $0 = 0 + 0 + C_2$, $C_2 = 0$

At x = 1 mm = 1×10^{-3} m, V = 2 V

$$\therefore \quad 2 = \frac{10^6}{2}\left(1 \times 10^{-3}\right)^2 + C_1\left(1 \times 10^{-3}\right)$$

$$\therefore \quad C_1 = 1500$$

$$\therefore \quad V = 0.5 \times 10^6\, x^2 + 1500\, x \quad V$$

At x = 0.5 mm = 0.5×10^{-3} m,

$$V = 0.5 \times 10^6 \left[0.5 \times 10^{-3}\right]^2 + 1500 \times \left(0.5 \times 10^{-3}\right) = \mathbf{0.875\ V}$$

From V, $$\bar{E} = -\nabla V = -\frac{\partial V}{\partial x}\bar{a}_x$$

$$= -\frac{\partial}{\partial x}\left[0.5 \times 10^6\, x^2 + 1500\, x\right]\bar{a}_x$$

$$= \left[-1 \times 10^6\, x - 1500\right]\bar{a}_x$$

$$\therefore \quad E_x = -1 \times 10^6\, x - 1500 \text{ V/m}$$

At x = 1 mm = 1×10^{-3} m,

$$E_x = -1 \times 10^6 \times 1 \times 10^{-3} - 1500 = \mathbf{-2500\ V/m}$$

Example 6.11 : *Two long metal plates of width 1 m each, held at an angle of 10° by an insulated hinge (plates are electrically separated). Using Laplace's equation, determine potential function.*

Solution : The plates are shown in the Fig. 6.6.

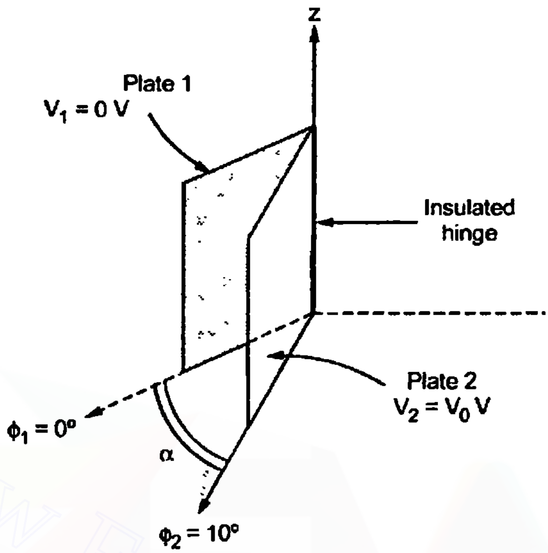


Fig. 6.6

Consider cylindrical co-ordinate system.

$$\phi_1 = 0° \text{ for plate 1}$$

$$\phi_2 = \alpha = 10° \text{ for plate 2}$$

The potential is a function of ϕ only and constant with r and z. Hence Laplace's equation in cylindrical sytem is,

$$\frac{1}{r}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Integrating, $\quad \frac{\partial V}{\partial \phi} = \int 0\, d\phi + C_1 = C_1$

Integrating, $\quad V = \int C_1\, d\phi + C_2 = C_1\phi + C_2 \quad$ ... (1)

At $\phi_1 = 0°$, $\quad V_1 = 0$ V i.e. $0 = C_1 \times 0 + C_2$, so $C_2 = 0$

$\therefore \quad V = C_1\phi$

Now $\quad V = V_0$ at $\phi = \phi_2 = \alpha$

$\therefore \quad C_1 = \frac{V_0}{\alpha}$

$\therefore \quad V = \frac{V_0}{\alpha}\phi$ V

For $\alpha = 10°$, $\quad V = \frac{V_0}{10}\phi$ V $\quad$ ... ϕ must be in degrees.

Example 6.12 : *Given the volume charge density $\rho_v = -2\times 10^7\ \varepsilon_0 \sqrt{x}$ C/m³ in free space, let V = 0 at x = 0 and V = 2 V at x = 2.5 mm. Find V at x = 1 mm.*

Solution : As $\rho_v \neq 0$, use Poisson's equation

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = -\frac{\left[-2\times 10^7\ \varepsilon_0 \sqrt{x}\right]}{\varepsilon_0} \qquad \ldots\ \varepsilon = \varepsilon_0 \text{ as free space}$$

Now V is a function of x alone, hence $\nabla^2 V = \frac{\partial^2 V}{\partial x^2}$.

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = 2\times 10^7\ x^{1/2}$$

Integrating,
$$\frac{\partial V}{\partial x} = \frac{2\times 10^7\ x^{3/2}}{\frac{3}{2}} + C_1 = 13.33\times 10^6 x^{1.5} + C_1$$

Integrating,
$$V = \int \left[13.33\times 10^6\ x^{1.5} + C_1\right] dx + C_2$$

$$= \frac{\left[13.33\times 10^6\ x^{2.5}\right]}{2.5} + C_1\ x + C_2$$

$$\therefore \quad V = 5.33\times 10^6 x^{2.5} + C_1\ x + C_2$$

At x = 0, V = 0 hence $0 = 0 + 0 + C_2,\ C_2 = 0$

At x = 2.5 mm, V = 2 V hence

$$2 = 5.33\times 10^6 (2.5\times 10^{-3})^{2.5} + C_1 (2.5\times 10^{-3})$$

$$\therefore \quad C_1 = 133.75$$

$$\therefore \quad V = 5.33\times 10^6\ x^{2.5} + 133.75\, x\ \ \text{V}$$

At x = 1 mm = 1×10^{-3} m,

$$V = 5.33\times 10^6 (1\times 10^{-3})^{2.5} + 133.75 (1\times 10^{-3})$$

$$= \mathbf{0.30229\ V}$$

Example 6.13 : *A capacitor of two large horizontal parallel plates has an internal separation 'd' between plates. A dielectric slab of relative permittivity ε_r and thickness a is placed on the lower plate of capacitor. Neglect edge effects. If the potential difference between the plates is ϕ, show that the electric-field intensity E_1 in the dielectric is $E_1 = \frac{\phi}{-\varepsilon_r d + a(\varepsilon_r - 1)}$ and capacitance C of the arrangement will be :*

$$\frac{\varepsilon_0 A}{d}\left[\frac{\varepsilon_r}{\left(1-\frac{a}{d}\right)\varepsilon_r+\frac{a}{d}}\right]$$ *where A → Area of the plate.*

[UPTU : 2005-06, 10 Marks]

Solution : Assume that the plates are placed parallel to x-y plane as shown in the Fig. 6.7.

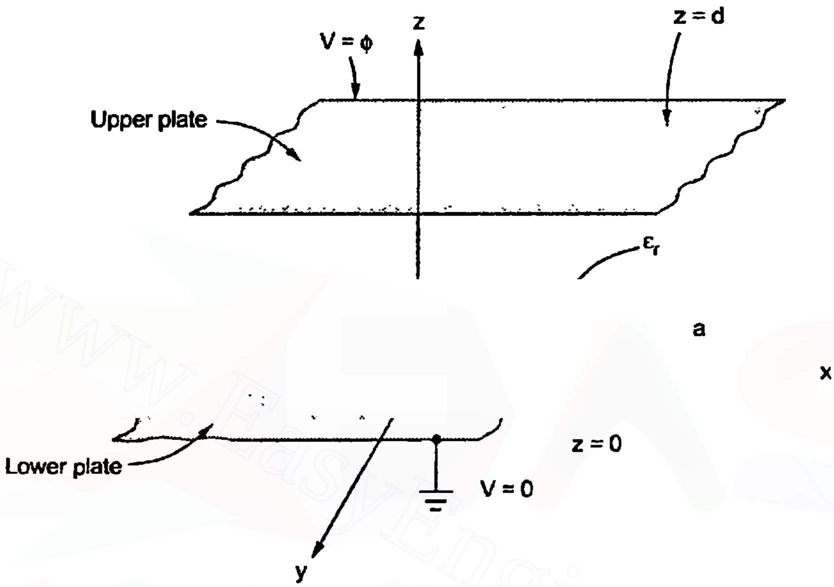


Fig. 6.7

The space between the plates is filled with two dielectrics,

1. For thickness 'a' with ε_r
2. For thickness 'd – a' with air ε_0

Using Laplace's equation in cartesian form

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 0$$

But V = f(z) only and not the functions of x and y.

$$\therefore \quad \frac{\partial^2 V}{\partial z^2} = 0$$

Integrating, $$\frac{\partial V}{\partial z} = \int 0 + A = A$$

Integrating, $$V = \int A\,dz + B = Az + B$$

Let the potential for z < a, $V_1 = A_1 z + B_1$ ...z < a ...(1)

Let the potential for z > a, $V_2 = A_2 z + B_2$ ...z > a ...(2)

At z = 0, V = 0 hence $B_1 = 0$ ...From equation (1)

At z = d, V = ϕ hence $B_2 = \phi - A_2 d$ ...From equation (2)

At z = a, $V_1 = V_2$

$$\therefore \quad A_1 a = A_2 a + B_2 = A_2 a + \phi - A_2 d$$

$$\therefore \quad A_1 = \frac{A_2(a-d)+\phi}{a} \qquad ...(3)$$

At the boundary, at z = a there are two perfect dielectrics giving,

$$D_{N1} = D_{N2} \quad \text{i.e.} \quad \varepsilon_1 E_{N1} = \varepsilon_2 E_{N2}$$

where $\varepsilon_1 = \varepsilon_0 \varepsilon_r$ and $\varepsilon_2 = \varepsilon_0$

$$\therefore \quad \varepsilon_r E_{N1} = E_{N2}$$

$$\therefore \quad \varepsilon_r \frac{dV_1}{dz} = \frac{dV_2}{dz}$$

$$\therefore \quad \varepsilon_r \frac{d}{dz}[A_1 z] = \frac{d}{dz}[A_2 z + B_2]$$

$$\therefore \quad \varepsilon_r A_1 = A_2 \qquad ...(4)$$

Solving equations (3) and (4),

$$\frac{A_2}{\varepsilon_r} = \frac{A_2(a-d)+\phi}{a}$$

$$\therefore \quad aA_2 = A_2 \varepsilon_r (a-d) + \varepsilon_r \phi$$

$$\therefore \quad A_2 [a - \varepsilon_r a + \varepsilon_r d] = \varepsilon_r \phi$$

$$\therefore \quad A_2 = \frac{\varepsilon_r \phi}{a(1-\varepsilon_r)+\varepsilon_r d} \quad \text{and} \quad A_1 = \frac{\phi}{a(1-\varepsilon_r)+\varepsilon_r d}$$

$$\therefore \quad B_2 = \phi - \frac{\varepsilon_r \phi d}{a(1-\varepsilon_r)+\varepsilon_r d}$$

$$= \frac{a(1-\varepsilon_r)\phi + \phi\varepsilon_r d - \varepsilon_r \phi d}{a(1-\varepsilon_r)+\varepsilon_r d}$$

$$= \frac{a(1-\varepsilon_r)\phi}{a(1-\varepsilon_r)+\varepsilon_r d}$$

$$\therefore \quad V_1 = \frac{\phi}{a(1-\varepsilon_r)+\varepsilon_r d} z, \quad V_2 = \frac{\varepsilon_r \phi}{a(1-\varepsilon_r)+\varepsilon_r d} z + \frac{a(1-\varepsilon_r)\phi}{a(1-\varepsilon_r)+\varepsilon_r d}$$

$$\bar{E}_1 = -\nabla V_1 = -\frac{\partial V_1}{\partial z}\bar{a}_z = -\frac{\phi}{a(1-\varepsilon_r)+\varepsilon_r d}\bar{a}_z$$

$$\boxed{\bar{E}_1 = \frac{\phi}{a(\varepsilon_r-1)-\varepsilon_r d}\bar{a}_z} \quad \text{... Proved}$$

$$C_1 = \frac{\varepsilon_1 A}{a} \quad \text{and} \quad C_2 = \frac{\varepsilon_0 A}{(d-a)}$$

Two capacitors are in series,

$$\therefore \quad C = \frac{C_1 C_2}{C_1+C_2} = \frac{\frac{\varepsilon_r\varepsilon_0 A}{a}\times\frac{\varepsilon_0 A}{d-a}}{\frac{\varepsilon_r\varepsilon_0 A}{a}+\frac{\varepsilon_0 A}{d-a}} = \frac{\varepsilon_0^2 A^2(\varepsilon_r)}{\varepsilon_0 A[\varepsilon_r(d-a)+a]}$$

$$= \frac{\varepsilon_r\varepsilon_0 A}{\varepsilon_r(d-a)+a} = \frac{\varepsilon_0 A}{d}\left[\frac{\varepsilon_r}{\varepsilon_r\left(1-\frac{a}{d}\right)+\frac{a}{d}}\right] \quad \text{...Proved}$$

Example 6.14 : *Two conducting cones ($\theta=\pi/10$ and $\theta=\pi/6$) of infinite extent are separated by an infinitesimal gap r = 0. If V ($\theta=\pi/10$) = 0 and V ($\theta=\pi/6$) = 50 V. Find V and E between the cones.*

[UPTU : 2006-07, 5 Marks]

Solution : The two cones are shown in the Fig. 6.8.

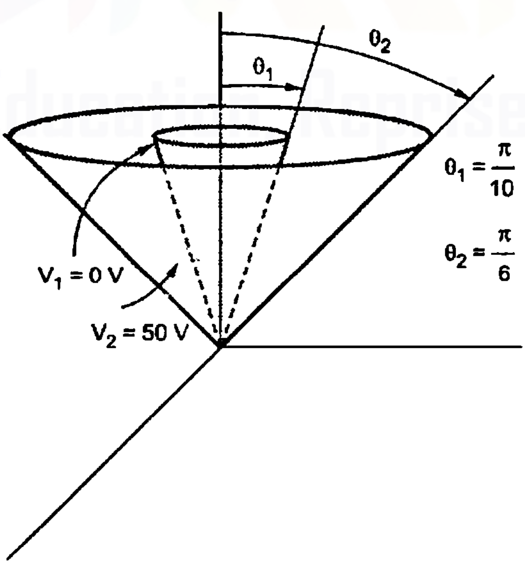


Fig. 6.8

The potential is constant with r and ϕ and is the function of θ only.

So Laplace's equation reduces to,

$$\frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

Integrating, $$\sin\theta \frac{dV}{d\theta} = \int 0 + A = A \quad ...(1)$$

Integrating $$\int \frac{dV}{d\theta} = \int \frac{A}{\sin\theta} d\theta + B = \int A \operatorname{cosec}\theta + B$$

$\therefore$ $$V = A\, ln\left[\tan\left(\frac{\theta}{2}\right)\right] + B \quad ...(2)$$

For $\theta_1 = \frac{\pi}{10}$, $V_1 = 0$ V

$\therefore$ $$0 = A\, ln\left[\tan\left(\frac{\pi/10}{2}\right)\right] + B \quad \text{i.e. } 0 = -1.8427\,A + B \quad ...(3)$$

For $\theta_2 = \frac{\pi}{6}$, $V_2 = 50$ V

$\therefore$ $$50 = A\, ln\left[\tan\left(\frac{\pi/6}{2}\right)\right] + B \quad \text{i.e. } 50 = -1.3169\,A + B \quad ...(4)$$

Solving equations (3) and (4), A = 95.09319, B = 175.2282

$$\boxed{V = 95.09319\; ln\left[\tan\left(\frac{\theta}{2}\right)\right] + 175.2282}$$

$$\bar{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta}\bar{a}_\theta \quad \text{...Other terms are zero}$$

$$= -\frac{1}{r}\frac{d}{d\theta}\left\{95.09319\, ln\left[\tan\left(\frac{\theta}{2}\right) + 175.2282\right]\right\}\bar{a}_\theta$$

$$= -\frac{1}{r}\left\{95.09319 \times \frac{1}{\tan\left(\frac{\theta}{2}\right)} \times \sec^2\left(\frac{\theta}{2}\right) \times \frac{1}{2}\right\}\bar{a}_\theta$$

$$= -\frac{95.09319}{r} \times \frac{\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} \times \frac{1}{\cos^2\left(\frac{\theta}{2}\right)}\bar{a}_\theta$$

$$= -\frac{95.09319}{r} \times \frac{1}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} \bar{a}_\theta$$

$$= -\frac{95.09319}{r\sin\theta} \bar{a}_\theta \text{ V/m}$$

Example 6.15 : *Determine $\bar{E}$ in spherical co-ordinates from Poisson's equation, assuming a uniform charge density ρ.* **[UPTU : 2006-07, 5 Marks]**

Solution : The Poisson's equation for charge density ρ is,

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

In spherical co-ordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = -\frac{\rho_v}{\varepsilon}$$

The charge density is uniform and is a function of r only.

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = -\frac{\rho_v}{\varepsilon} \qquad \text{...Other terms are neglected}$$

Integrating $\quad r^2\frac{\partial V}{\partial r} = \int -\frac{\rho_v r^2}{\varepsilon} + A = \frac{-\rho_v r^3}{3\varepsilon} + A$

Integrating $\quad \int\frac{\partial V}{\partial r} = \int\left[\frac{-\rho_v r}{3\varepsilon} + A\, r^{-2}\right] dr + B$

$$\therefore \quad V = -\frac{\rho_v r^2}{6\varepsilon} - \frac{A}{r} + B$$

$$\bar{E} = -\frac{\partial V}{\partial r}\bar{a}_r = -\frac{\partial}{\partial r}\left[\frac{-\rho_v r^2}{6\varepsilon} - \frac{A}{r} + B\right]\bar{a}_r$$

$$\therefore \quad \boxed{\bar{E} = \left[\frac{\rho_v r}{3\varepsilon} - \frac{A}{r^2}\right]\bar{a}_r} \qquad \text{... where A = Constant}$$

Example 6.16 : *Let $V = 2xy^2z^3$ and $\varepsilon = \varepsilon_o$. Given point is P(1, 3, – 1). Find V at point P. Also find out if V satisfies Laplace's equation.* **[UPTU : 2007-08, 5 Marks]**

Solution : $V = 2\,xy^2z^3$

$$\therefore \quad V_P = 2 \times 1 \times (3)^2 \times (-1)^3 = -18 \text{ V}$$

The Laplace's equation is $\nabla^2 V = 0$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial x}[2y^2z^3]+\frac{\partial}{\partial y}[4xyz^3]+\frac{\partial}{\partial z}[6xy^2z^2]$$

$$= 0 + 12\,xz^3 + 12\,xy^2z$$

As $\nabla^2 V \neq 0$, the given V **does not satisfy** Laplace's equation.

Review Questions

1. *Derive Poisson's and Laplace's equations.*
2. *State the applications of Poisson's and Laplace's equations.*
3. *State and prove Uniqueness theorem.*
4. *Write a note on product solution of Laplace's equation.*
5. *Obtain the capacitance of parallel plate capacitor using Laplace's equation.*
6. *Obtain the capacitance of spherical plate capacitor using Laplace's equation.*
7. *Obtain the capacitance of co-axial cable capacitor using Laplace's equation.*
8. *In cylindrical co-odinates, V = 75 V at r = 5 mm and V = 0 at r = 60 mm. Find the voltage at r = 130 mm, if the potential depends only on r.* **[Ans. : – 23.3366 V]**
9. *Long concentric and right conducting cylinders in free space, at r = 5 mm and r = 25 mm in cylindrical co-ordinates, have voltages 0 and V_o respectively. If $\bar{E} = -8.28\times10^3\,\bar{a}_r$ V/m at r = 15 mm, find V_o and ρ_S on the outer conductor by using Laplace's equation.* **[Ans. : V_o = 199.89 V, 44 nC/m²]**
10. *Find the potential V at the point P (2, 3, 4) for the field of two co-axial conducting cylinders, given V = 60 V at r = 3 m and V = 10 V at r = 5 m. [Hint : Find V as a function of r. Then for P (2, 3, 4), $r = \sqrt{x^2+y^2}$ using cartesian to cylindrical conversion]* **[Ans. : V = – 97.8 ln r + 167.53 V, 42 V]**
11. *Find the potential at P (1, 2, 3) for the field of two infinite radial conducting plates, given V = 40 V at $\phi = 15°$ and V = 15 V at $\phi = 40°$* **[Ans. : – 8.4426 V]**
12. *Find the potential between two co-axial cones using Laplace's equation with $V = V_1$ at $\theta = \theta_1$ and V = 0 at $\theta = \theta_2$.* **[Ans. : $V = V_1 \dfrac{\ln\left(\tan\frac{\theta}{2}\right)-\ln\left(\tan\frac{\theta_2}{2}\right)}{\ln\left(\tan\frac{\theta_1}{2}\right)-\ln\left(\tan\frac{\theta_2}{2}\right)}$]**
13. *Find V at $\theta = 20°$ for the field between two conducting cones with V = 0 V at $\theta = 30°$ and V = 100 V at $\theta = 10°$. Also calculate θ for a voltage to be 50 V.* **[Ans. : 37.4 V, θ = 17.41°]**
14. *Two semi-infinite conducting planes $\phi = 0$ and $\phi = \frac{\pi}{6}$ are separated by an infinitesimal insulating gap. If V = 0 V for $\phi = 0$ and V = 100 V, for $\phi = 100V$, find V and $\bar{E}$ in the region between the planes.* **[Ans. : $V = \frac{600}{\pi}\phi$ V, $\bar{E} = \frac{600}{\pi r}\bar{a}_r$ V/m]**

15. *Find a solution to Laplace's equation subject to the boundary conditions V = 100 V at z = 0.01 and V = – 33 V at z = 0.02.* **[Ans. : $V = -1.33 \times 10^4 z + 2.33 \times 10^2$ V]**

University Questions

1. *Consider two concentric spheres of radii a and b, a < b. The outer sphere is kept at a potential V_0 and the inner sphere at zero potential. Solve Laplace's equation in spherical co-ordinates to find the potential and electric field in the region between the two spheres.* **[UPTU : 2002-03, 5 Marks]**
2. *Derive Poisson's equation and discuss its application is electrostatics.* **[UPTU : 2003-04(A), 5 Marks]**
4. *Discuss the solution of Poisson's and Laplace's equation in one dimension.* **[UPTU : 2003-04(B), 10 Marks]**
5. *A capacitor of two large horizontal parallel plates has an internal separation 'd' between plates. A dielectric slab of relative permittivity ε_r and thickness a is placed on the lower plate of capacitor. Neglect edge effects. If the potential difference between the plates is ϕ, show that the electric - field intensity E_1 in the dielectric is $E_1 = \frac{\phi}{-\varepsilon_r d + a(\varepsilon_r - 1)}$ and capacitance C of the arrangement will be :*

   $$\frac{\varepsilon_0 A}{d}\left[\frac{\varepsilon_r}{\left(1-\frac{a}{d}\right)\varepsilon_r + \frac{a}{d}}\right]$$ *where A --> Area of the plate.*

   [UPTU : 2005-06, 10 Marks]
6. *Calcualte the potential at any point between two grounded semi-infinite parallel electrodes separated by a distance 'b' and a plane electrode at potential V_0.* **[UPTU : 2005-06, 10 Marks]**
7. *Two conducting cones $(\theta = \pi/10$ and $\theta = \pi/6)$ of infintie extent are separated by an infinitesimal gap r = 0. If $V(\theta = \pi/10) = 0$ and $V(\theta = \pi/6) = 50$ V. Find V and E between the cones.* **[UPTU : 2006-07, 5 Marks]**
8. *Determine $\bar{E}$ in spherical co-ordinates from Poisson's equation, assuming a uniform charge density ρ.* **[UPTU : 2006-07, 5 Marks]**
9. *Write down Poisson's and Laplace equation in cylindrical co-ordinates system. Let $V = 2xy^2z^3$ and $\varepsilon = \varepsilon_0$. Given point is P(1, 3, –1). Find V at point P. Also find out if V satisfies Laplace's equation.* **(UPTU : 2007-08, 5 Marks)**

□□□

7 Magnetostatics

7.1 Introduction

Uptill now static electrostatic fields are discussed. The electrostatic field exists due to the static charges i.e. charges at rest. The magnetic field exists due to a permanent magnet, which is a natural magnet. But in electromagnetic engineering a link between electric and magnetic field is required to be studied. Such a link is absent with magnetic field due to a natural magnet.

The scientist Oersted has discovered the relation between electric and magnetic fields in 1820. Scientist Oersted stated that when the charges are in motion, they are surrounded by a magnetic field. The charges in motion i.e. flow of charges constitutes an electric current. Thus a current carrying conductor is always surrounded by a magnetic field. If such a current flow is steady i.e. time invariant then the magnetic field produced is a steady magnetic field which is also a time invariant. The direct current (d.c.) is a steady flow of current hence magnetic field produced by a conductor carrying a d.c. current is a static steady magnetic field. The study of steady magnetic field, existing in a given space, produced due to the flow of direct current through a conductor is called **magnetostatics.** The various other concepts like e.m.f. induced, force experienced by a conductor, motoring action, transformer action etc are dependent on the magnetostatics. Hence the study of steady magnetic field i.e. magnetostatics plays an important role in the engineering electromagnetics. This chapter explains the steady magnetic field in free space due to the conductor carrying a direct current.

7.2 Magnetic Field and its Properties

Before beginning the study of steady magnetic fields, let us study the basic properties of the magnetic field. To understand these properties, consider a permanent magnet. It has two poles, north (N) and south (S). The region around a magnet within which the influence of the magnet can be experienced is called **magnetic field.** The existence of such a field can be experienced with the help of compass needle. Such a field is represented by imaginary lines around the magnet which are called **magnetic lines of force.** These are introduced by the scientist Michael Faraday. The direction of such lines is always from N pole to S pole, external to the magnet as shown in the Fig. 7.1. These lines of force are also called **magnetic lines of flux** or **magnetic flux lines.**

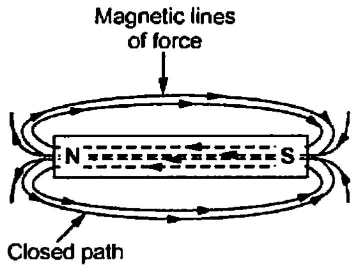


Fig. 7.1 Permanent magnet and magnetic lines of force

An important difference between electric flux lines and magnetic flux lines can be observed here. In case of electric flux, the flux lines originate from an isolated positive charge and diverge to terminate at infinity. While for a negative charge, electric flux lines converge on a charge, starting from infinity. But in case of magnetic flux, the poles exist in pairs only.

Key Point: *An isolated magnetic pole cannot exist.*

Hence every magnetic flux line starting from north pole must end at south pole and complete the path from south to north internal to the magnet.

Key Point: *Thus magnetic flux lines exist in the form of closed loop.*

This is true whether the field is due to permanent magnet or due to conductor carrying direct current.

7.2.1 Magnetic Field due to Current Carrying Conductor

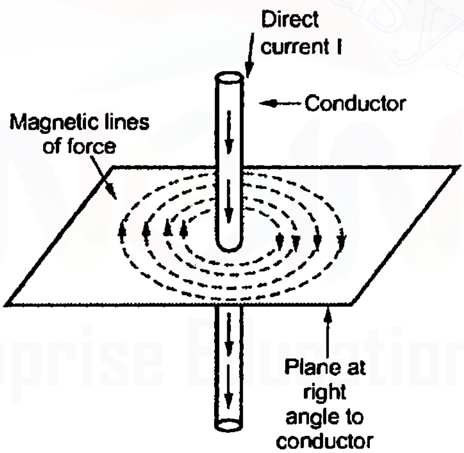


Fig. 7.2 Magnetic field due to conductor carrying direct current

When a straight conductor carries a direct current, it produces a magnetic field around it, all along its length. The lines of force in such a case are in the form of concentric circles in the planes at right angles to the conductor. This is shown in the Fig. 7.2. The direction of such magnetic flux can be experienced using a compass needle. The direction of concentric circles around, depends on the direction of current through the conductor. As long as direction of current is constant and current is time independent, magnetic lines of force are also constant, static and time independent, giving a steady magnetic field in the space around the conductor.

A right hand thumb rule is used to determine the direction of magnetic field around a conductor carrying a direct current. It states that, hold the current carrying conductor in the right hand such that the thumb pointing in the direction of current and parallel to the conductor, then curled fingers point in the direction of the magnetic lines of flux around it. The Fig. 7.3 explains the rule.

Practically the current carrying conductor is represented by a small circle i.e. top view of straight conductor while the direction of current through it is indicated by a 'cross' or a

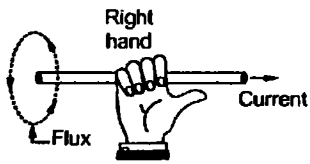


Fig. 7.3 Right hand thumb rule

'dot'. The cross indicates that the current direction is going into the plane of the paper away from the observer. The dot indicates that the current direction is coming out of the plane of the paper coming towards the observer. Using right hand thumb rule, the direction of magnetic flux around such a conductor is either clockwise or anticlockwise as shown in the Fig. 7.4.

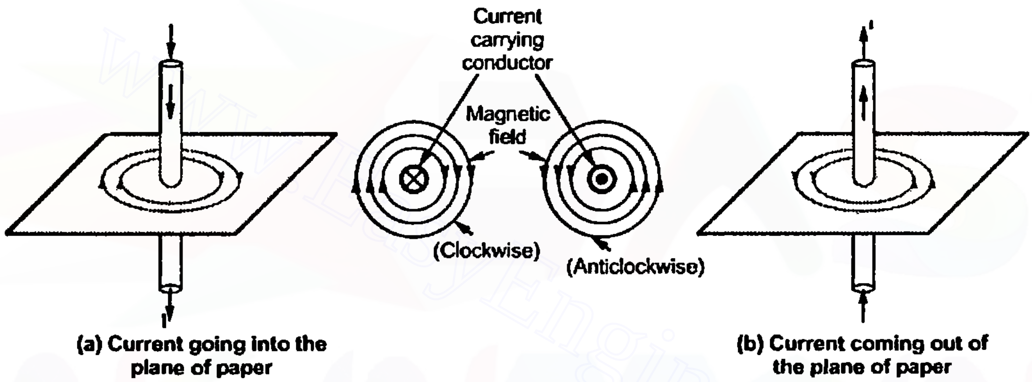


Fig. 7.4

Another method of identifying the direction of magnetic flux around a conductor is **right handed screw rule.** It states that, imagine a right handed screw to be along the conductor carrying current with its axis parallel to the conductor and tip pointing in the direction of the current flow. Then the direction of magnetic field is given by the direction in which the screw must be turned so as to advance in the direction of the current flow. The Fig. 7.5 illustrates this rule.

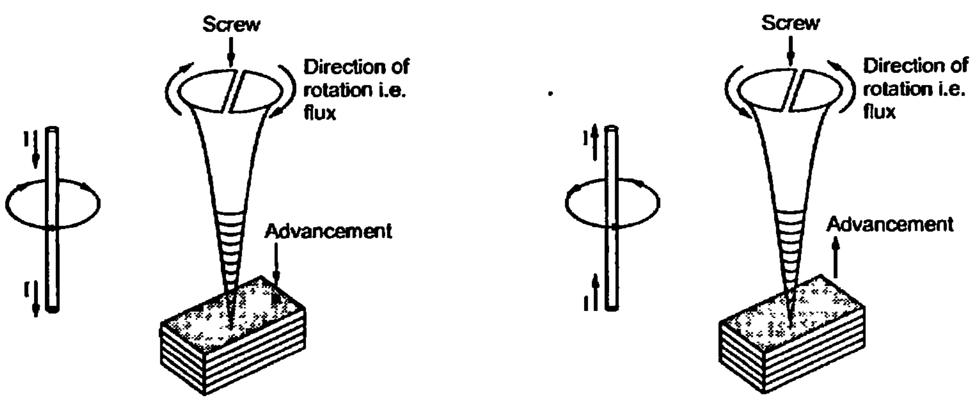


Fig. 7.5 Right handed screw rule

Thus the magnetic lines of force i.e. magnetic flux lines always form a closed loop and exist in the form of concentric circles, around a current carrying conductor. The total number of magnetic lines of force is called a **magnetic flux** denoted as ϕ. It is measured in weber (Wb). One weber means 10^8 lines of force.

7.2.2 Magnetic Field Intensity

The quantitative measure of strongness or weakness of the magnetic field is given by magnetic field intensity or magnetic field strength. The **magnetic field intensity** at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength, when placed at that point. The magnetic flux lines are measured in webers (Wb) while magnetic field intensity is measured in newtons/weber (N/Wb) or amperes per metre (A/m) or ampere-turns/metre (AT/m). It is denoted as $\bar{H}$. It is a vector quantity. This is similar to the electric field intensity $\bar{E}$ in electrostatics.

7.2.3 Magnetic Flux Density

The total magnetic lines of force i.e. magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called **magnetic flux density**. It is denoted as $\bar{B}$ and is a vector quantity. It is measured in weber per square metre. (Wb/m^2) which is also called Tesla (T). This is similar to the electric flux density $\bar{D}$ in electrostatics.

7.2.4 Relation between $\bar{B}$ and $\bar{H}$

In electrostatics, $\bar{E}$ and $\bar{D}$ are related to each other through permittivity ε of the region. In magnetostatics, the $\bar{B}$ and $\bar{H}$ are related to each other through the property of the region in which current carrying conductor is placed. It is called **permeability** denoted as μ. It is the ability or ease with which the current carrying conductor forces the magnetic flux through the region around it. For a free space, the permeability is denoted as μ_0 and its value is $4\pi\times10^{-7}$. As ε is measured in F/m, the permeability μ is measured in henries per metre (H/m). For any other region, a relative permeability is specified as μ_r and $\mu = \mu_0\mu_r$.

The $\bar{B}$ and $\bar{H}$ are related as,

$$\boxed{\bar{B} = \mu\,\bar{H} = \mu_0\mu_r\,\bar{H}}$$

For free space, $\bar{B} = \mu_0 H$

For all nonmagnetic media, $\mu_r = 1$ while for magnetic materials μ_r is greater than unity.

7.3 Biot-Savart Law

Consider a conductor carrying a direct current I and a steady magnetic field produced around it. The Biot-Savart law allows us to obtain the **differential magnetic field intensity** $d\bar{H}$, produced at a point P, due to a differential current element IdL. The current carrying conductor is shown in the Fig. 7.6.

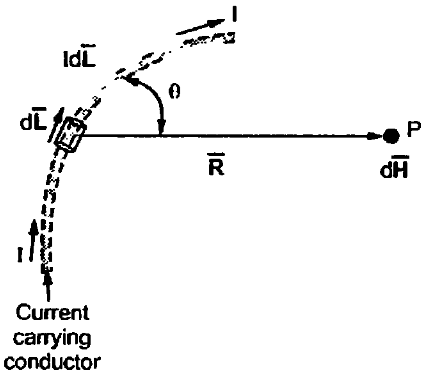


Fig. 7.6

Consider a differential length dL hence the differential current element is IdL. This is very small part of the current carrying conductor. The point P is at a distance R from the differential current element. The θ is the angle between the differential current element and the line joining point P to the differential current element.

The **Biot-Savart law** states that,

The magnetic field intensity $d\bar{H}$ produced at a point P due to a differential current element IdL is,

1. Proportional to the product of the current I and differential length dL.
2. The sine of the angle between the element and the line joining point P to the element.
3. And inversely proportional to the square of the distance R between point P and the element.

Mathematically, the Biot-Savart law can be stated as,

$$d\bar{H} \propto \frac{I\,dL\sin\theta}{R^2} \qquad \text{... (1)}$$

$$\therefore \quad \boxed{d\bar{H} = \frac{k\,I\,dL\sin\theta}{R^2}} \qquad \text{... (2)}$$

where k = Constant of proportionality

In SI units, $k = \frac{1}{4\pi}$

$$\therefore \quad d\bar{H} = \frac{I\,dL\sin\theta}{4\pi R^2} \qquad \text{... (3)}$$

Let us express this equation in vector form.

Let dL = Magnitude of vector length $d\bar{L}$ and

$\bar{a}_R$ = Unit vector in the direction **from differential current element to point P**

Then from rule of cross product,

$$d\bar{L}\times\bar{a}_R = dL\,|\bar{a}_R|\sin\theta = dL\sin\theta \qquad \text{...}\ |\bar{a}_R| = 1$$

Replacing in equation (3),

$$d\bar{H} = \frac{I\,d\bar{L}\times\bar{a}_R}{4\pi R^2} \text{ A/m} \quad \text{... (4)}$$

But $\quad \bar{a}_R = \dfrac{\bar{R}}{|\bar{R}|} = \dfrac{\bar{R}}{R}$

Hence, $\quad \boxed{d\bar{H} = \dfrac{I\,d\bar{L}\times\bar{R}}{4\pi R^3} \text{ A/m}} \quad \text{... (5)}$

The equations (4) and (5) is the mathematical form of Biot-Savart law.

According to the direction of cross product, the direction of $d\bar{H}$ is normal to the plane containing two vectors and in that normal direction which is along the progress of right handed screw, turned from $d\bar{L}$ through the smaller angle θ towards the line joining element to the point P. Thus the direction of $d\bar{H}$ is normal to the plane of paper. For the case considered, according to right handed screw rule, the direction of $d\bar{H}$ is going into the plane of the paper.

The entire conductor is made up of all such differential elements. Hence to obtain total magnetic field intensity $\bar{H}$, the above equation (4) takes the integral form as,

$$\boxed{\bar{H} = \oint \frac{I\,d\bar{L}\times\bar{a}_R}{4\pi R^2}} \quad \text{... (6)}$$

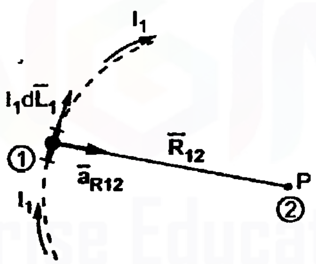


Fig. 7.7

The closed line integral is required to ensure that all the current elements are considered. This is because current can flow only in the closed path, provided by the closed circuit. If the current element is considered at point 1 and point P at point 2, as shown in the Fig. 7.7 then,

$$d\bar{H}_2 = \frac{I_1\,d\bar{L}_1\times\bar{a}_{R12}}{4\pi R_{12}^2} \text{ A/m} \quad \text{... (7)}$$

where $\quad I_1$ = Current flowing through dL_1 at point 1

$d\bar{L}_1$ = Differential vector length at point 1

$\bar{a}_{R12}$ = Unit vector in the direction from element at point 1 to the point P at point 2

$$\bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{R}_{12}}{R_{12}}$$

$\therefore \quad \boxed{\bar{H}_2 = \oint \dfrac{I_1\,d\bar{L}_1\times\bar{a}_{R12}}{4\pi R_{12}^2} \text{ A/m}} \quad \text{... (8)}$

This is called **integral form of Biot-Savart law.**

7.3.1 Biot-Savart Law Interms of Distributed Sources

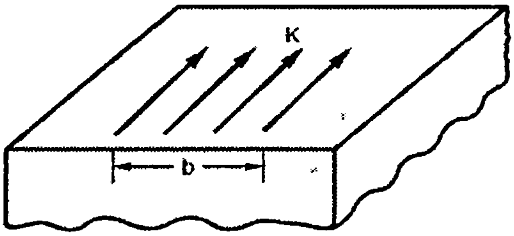


Fig. 7.8 Surface current density

Consider a surface carrying a uniform current over its surface as shown in the Fig. 7.8. Then the **surface current density** is denoted as $\bar{K}$ and measured in amperes per metre (A/m). Thus for uniform current density, the current I in any width b is given by I = Kb, where width b is perpendicular to the direction of current flow.

Thus if dS is the differential surface area considered of a surface having current density $\bar{K}$ then,

$$I\,d\bar{L} = \bar{K}\,dS \qquad \text{... (9)}$$

If the current density in a volume of a given conductor is $\bar{J}$ measured in A/m^2 then for a differential volume dv we can write,

$$I\,d\bar{L} = \bar{J}\,dv \qquad \text{... (10)}$$

Hence the Biot-Savart law can be expressed for surface current considering $\bar{K}$ dS while for volume current considering $\bar{J}$ dv.

$$\therefore \qquad \bar{H} = \int_S \frac{\bar{K}\times\bar{a}_R\,dS}{4\pi R^2} \text{ A/m} \qquad \text{... (11)}$$

and

$$\bar{H} = \int_{vol} \frac{\bar{J}\times\bar{a}_R\,dv}{4\pi R^2} \text{ A/m} \qquad \text{... (12)}$$

The Biot-Savart law is also called **Ampere's law for the current element.** Let us study now the various applications of Biot-Savart law.

Example 7.1 : *Find the incremental field strength at P_2 due to the current element of $2\pi\bar{a}_z\,\mu$ Am at P_1. The co-ordinates of P_1 and P_2 are (4,0,0) and (0,3,0) respectively.*

Solution : The two points P_1 and P_2 alongwith the $I_1 d\bar{L}_1$ current element at P_1 are shown in the Fig. 7.9.

According to the Biot-Savart law,

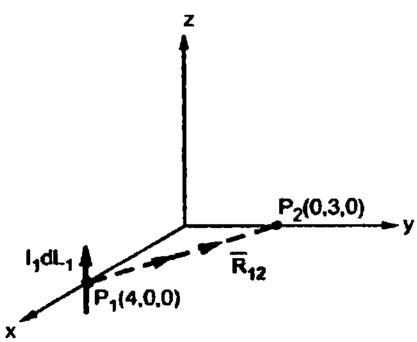


Fig. 7.9

$$d\bar{H}_2 = \frac{I_1\,d\bar{L}_1\times\bar{a}_{R12}}{4\pi R_{12}^2}$$

$$\bar{R}_{12} = (0-4)\bar{a}_x + (3-0)\bar{a}_y + 0\bar{a}_z$$

$$\therefore \quad \bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|}$$

$$= \frac{-4\bar{a}_x + 3\bar{a}_y}{\sqrt{16+9}} = \frac{-4\bar{a}_x + 3\bar{a}_y}{5}$$

While $I_1\,dL_1 = 2\pi\,\bar{a}_z\,\mu$Am

$$\therefore \quad I_1\, d\bar{L}_1 \times a_{R12} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & 2\pi \\ -\frac{4}{5} & \frac{3}{5} & 0 \end{vmatrix} = -\frac{4}{5}\times 2\pi\, \bar{a}_y - \frac{3}{5}\, 2\pi\, \bar{a}_x = -\frac{2\pi}{5}\left[3\bar{a}_x + 4\bar{a}_y\right]$$

$$\therefore \quad d\bar{H}_2 = \frac{-\frac{2\pi}{5}\left[3\bar{a}_x + 4\bar{a}_y\right]}{4\pi\times(5)^2} = -4\times 10^{-3}\left[3\bar{a}_x + 4\bar{a}_y\right] \;\mu A/m$$

$$= -12\, \bar{a}_x - 16\, \bar{a}_y \;\; nA/m$$

7.4 $\bar{H}$ due to Infinitely Long Straight Conductor

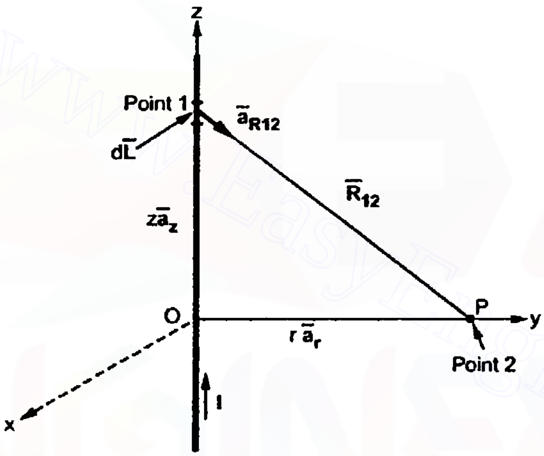


Fig. 7.10 $\bar{H}$ due to infinitely long straight conductor

Consider an infinitely long straight conductor, along z-axis. The current passing through the conductor is a direct current of I Amp. The field intensity $\bar{H}$ at a point P is to be calculated, which is at a distance 'r' from the z-axis. This is shown in the Fig. 7.10.

Consider small differential element at point 1, along the z-axis, at a distance z from origin.

$$\therefore \quad I\, d\bar{L} = I\, dz\, \bar{a}_z \qquad \text{... (1)}$$

The distance vector joining point 1 to point 2 is $\bar{R}_{12}$ and can be written as,

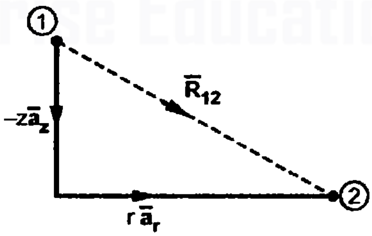


Fig. 7.11

$$\bar{R}_{12} = -z\,\bar{a}_z + r\,\bar{a}_r \qquad \text{... (2)}$$

$$\bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}} \qquad \text{... (3)}$$

$$\therefore\; d\bar{L}\times\bar{a}_{R12} = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ 0 & 0 & dz \\ r & 0 & -z \end{vmatrix} = r\, dz\, \bar{a}_\phi$$

While obtaining cross product, $|\bar{R}_{12}|$ is neglected for convenience and must be considered for further calculations.

$$\therefore \quad I\, d\bar{L}\times\bar{a}_{R12} = \frac{I\, r\, dz\, \bar{a}_\phi}{\sqrt{r^2+z^2}} \qquad \text{... (4)}$$

According to Biot-Savart law, $d\bar{H}$ at point 2 is,

$$d\bar{H} = \frac{I\, d\bar{L} \times \bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I\, r\, dz\, \bar{a}_\phi}{4\pi\sqrt{r^2+z^2}\,(\sqrt{r^2+z^2})^2}$$

$$= \frac{I\, r\, dz\, \bar{a}_\phi}{4\pi\left(r^2+z^2\right)^{3/2}} \qquad \text{... (5)}$$

Thus total field intensity $\bar{H}$ can be obtained by integrating $d\bar{H}$ over the entire length of the conductor.

$$\therefore \quad \bar{H} = \int_{z=-\infty}^{\infty} d\bar{H} = \int_{z=-\infty}^{\infty} \frac{I\, r\, dz\, \bar{a}_\phi}{4\pi\left(r^2+z^2\right)^{3/2}} \qquad \text{... (6)}$$

Put $z = r\tan\theta$, $z^2 = r^2\tan^2\theta$

and $dz = r\sec^2\theta\, d\theta$, $z = -\infty$, $\theta = -\frac{\pi}{2}$ and $z = +\infty$, $\theta = +\frac{\pi}{2}$

$$\therefore \quad \bar{H} = \int_{\theta=-\pi/2}^{\pi/2} \frac{I\, r\, r\sec^2\theta\, d\theta\, \bar{a}_\phi}{4\pi\left(r^2+r^2\tan^2\theta\right)^{3/2}}$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \frac{I\, r^2\sec^2\theta\, d\theta\, \bar{a}_\phi}{4\pi r^3 \sec^3\theta} \qquad \text{... } 1+\tan^2\theta = \sec^2\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \frac{I}{4\pi r}\frac{1}{\sec\theta}\, d\theta\, a_\phi = \frac{I}{4\pi r}\int_{\theta=-\pi/2}^{\pi/2} \cos\theta\, d\theta\, \bar{a}_\phi$$

$$= \frac{I}{4\pi r}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} \bar{a}_\phi = \frac{I}{4\pi r}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right]\bar{a}_\phi$$

$$= \frac{I}{4\pi r}\left[1-(-1)\right] a_\phi = \frac{2I}{4\pi r}\bar{a}_\phi$$

$$\therefore \quad \boxed{\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi \quad \text{A/m}} \qquad \text{... (7)}$$

$$\boxed{\bar{B} = \mu\bar{H} = \frac{\mu I}{2\pi r}\bar{a}_\phi \ \text{Wb/m}^2} \qquad \text{... (8)}$$

The following observations are important about $\bar{H}$:

1. The magnitude of magnetic field intensity $\bar{H}$ is not a function of ϕ or z. It is inversely proportional to r which is the perpendicular distance of the point from the conductor.

2. The direction of $\bar{H}$ is tangential i.e. circumferential along $\bar{a}_\phi$. This direction is going into the plane of the paper at point P.

3. The streamlines i.e. magnetic flux lines are in the form of concentric circles around the conductor. Thus if conductor is viewed from the top with I coming out of the paper towards observer, then the streamlines are anticlockwise.

7.5 $\bar{H}$ due to Straight Conductor of Finite Length

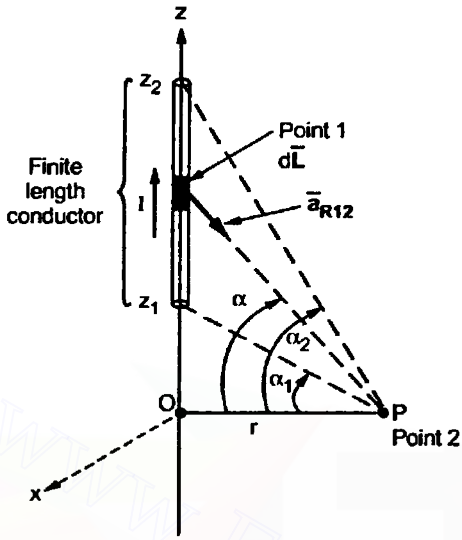


Fig. 7.12

Consider a conductor of finite length placed along z-axis, as shown in the Fig. 7.12.

It carries a direct current I. The perpendicular distance of point P from z-axis is r as shown in the Fig. 7.12. The conductor is placed such that its one end is at $z = z_1$ while other at $z = z_2$.

Consider a differential element $d\bar{L}$ along z-axis, at a distance z from origin.

$$\therefore \quad d\bar{L} = dz\,\bar{a}_z \qquad \text{... (1)}$$

The unit vector in the direction joining differential element to point P is $\bar{a}_{R12}$ and can be expressed as shown in the Fig. 7.13.

$$\therefore \quad a_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{-z\bar{a}_z + r\bar{a}_r}{\sqrt{(-z)^2 + (r)^2}}$$

$$= \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2 + z^2}} \qquad \text{... (2)}$$

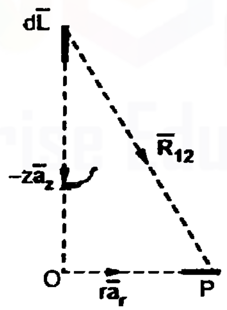


Fig. 7.13

$$\therefore \quad d\bar{L} \times \bar{a}_{R12} = r dz\,\bar{a}_\phi \qquad \text{... (3)}$$

This is same as obtained in the earlier section for infinitely long conductor.

$$\therefore \quad I\,d\bar{L} \times \bar{a}_{R12} = \frac{I\,r\,dz\,\bar{a}_\phi}{\sqrt{r^2 + z^2}} \qquad \text{... (4)}$$

According to Biot-Savart law, $d\bar{H}$ at point P is,

$$d\bar{H} = \frac{I\,d\bar{L} \times \bar{a}_{R12}}{4\pi R_{12}^2}$$

$$= \frac{I\,r\,dz\,\bar{a}_\phi}{4\pi\sqrt{r^2 + z^2}\,(\sqrt{r^2 + z^2})^2}$$

$$= \frac{I\,r\,dz\,\bar{a}_\phi}{4\pi\left(r^2 + z^2\right)^{3/2}} \qquad \text{... (5)}$$

The total $\bar{H}$ at P due to conductor of finite length can be obtained by integrating $d\bar{H}$ over $z = z_1$ to $z = z_2$.

$$\therefore \qquad \bar{H} = \int_{z_1}^{z_2} d\bar{H} = \int_{z_1}^{z_2} \frac{I\, r\, dz\, \bar{a}_\phi}{4\pi\left(r^2+z^2\right)^{3/2}} \qquad \text{... (6)}$$

Use $z = r\tan\alpha$, $\quad z^2 = r^2\tan^2\alpha$

$dz = r\sec^2\alpha\, d\alpha$

For $z = z_1$, $\quad z_1 = r\tan\alpha_1$
For $z = z_2$, $\quad z_2 = r\tan\alpha_2$ $\Big\}$ From the Fig. 7.12.

$$\therefore \qquad \alpha_1 = \tan^{-1}(z_1/r) \quad \text{and} \quad \alpha_2 = \tan^{-1}(z_2/r) \qquad \text{... (7)}$$

$$\therefore \qquad \bar{H} = \int_{\alpha_1}^{\alpha_2} \frac{I\, r\, r\sec^2\alpha\, d\alpha\, \bar{a}_\phi}{4\pi\left[r^2+r^2\tan^2\alpha\right]^{3/2}} = \int_{\alpha_1}^{\alpha_2} \frac{I\, d\alpha\, \bar{a}_\phi}{4\pi(\sec\alpha)\, r}$$

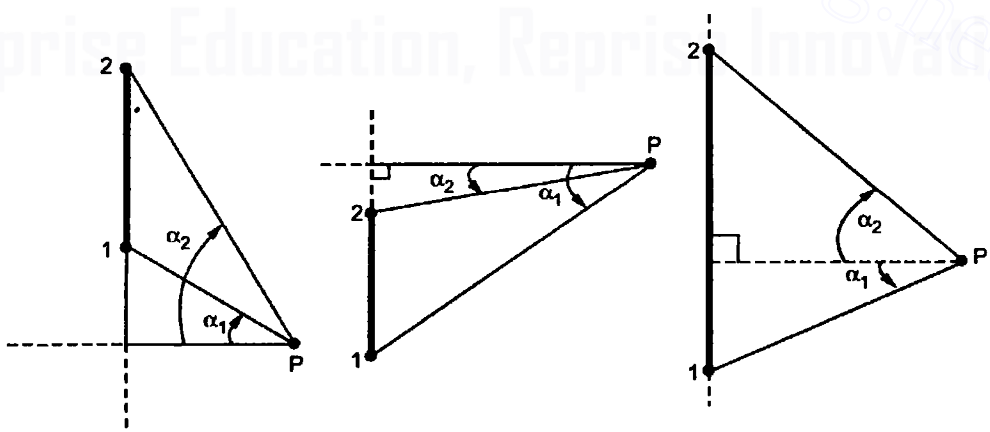
$$= \frac{I}{4\pi r}\int_{\alpha_1}^{\alpha_2} \cos\alpha\, d\alpha\, \bar{a}_\phi = \frac{I}{4\pi r}\left[\sin\alpha\right]_{\alpha_1}^{\alpha_2} \bar{a}_\phi$$

$$\therefore \qquad \boxed{\bar{H} = \frac{I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\bar{a}_\phi \quad \text{A/m}} \qquad \text{... (8)}$$

$$\boxed{\bar{B} = \mu\bar{H} = \frac{\mu I}{4\pi r}\left[\sin\alpha_2 - \sin\alpha_1\right]\bar{a}_\phi \ \ \text{Wb/m}^2} \qquad \text{... (9)}$$

7.5.1 Sign Convention for α_1 and α_2

If both the ends of conductor are above point P, then α_1 and α_2 are positive. If both the ends of conductor are below point P, then both α_1 and α_2 are negative. While if one end of the conductor is above P and other below then α_1 is negative and α_2 positive. This is shown in the Fig. 7.14.



(a) Both α_1, α_2 positive (b) Both α_1, α_2 negative (c) α_1 negative, α_2 positive

Fig. 7.14

The result given by equation (8) can be used directly to obtain $\bar{H}$ caused by current filaments which are arranged as the sequence of straight lines.

Very important note : While using this result, if segment carrying current I is not along z axis then the direction of $\bar{H}$ can not be $\bar{a}_\phi$. It depends on in which plane segment carrying current is placed. The magnitude of $\bar{H}$ is $\frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1]$ but direction is **always normal to the plane containing the source and to be decided by right handed screw rule.**

7.6 $\bar{H}$ at the Centre of a Circular Conductor

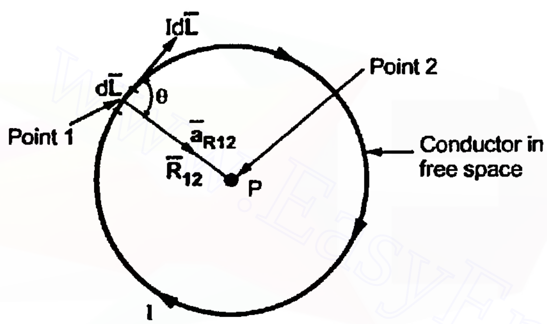


Fig. 7.15

Consider the current carrying conductor arranged in a circular form as shown in the Fig. 7.15.

The $\bar{H}$ at the centre of the circular loop is to be obtained. The conductor carries the direct current I.

Consider the differential length $d\bar{L}$ at a point 1.

The direction of $d\bar{L}$ at a point 1 is tangential to the circular conductor at point 1.

Let θ = Angle between $I\, d\bar{L}$ and $\bar{a}_{R12}$

$\bar{a}_{R12}$ = Unit vector in the direction of $\bar{R}_{12}$

$\bar{R}_{12}$ = Distance vector joining differential current element at point 1 to point P at point 2 which is centre of circle.

Using the definition of cross product,

$$\therefore \quad I\, d\bar{L} \times \bar{a}_{R12} = I\left|d\bar{L}\right|\left|\bar{a}_{R12}\right| \sin\theta\, \bar{a}_N = I\, dL \sin\theta\, \bar{a}_N \qquad \text{... (1)}$$

$\bar{a}_N$ = Unit vector normal to the plane containing $d\bar{L}$ and $\bar{a}_{R12}$ i.e. normal to the plane in which the circular conductor is lying

According to Biot-Savart law, the differential magnetic field intensity $d\bar{H}$ at point P is,

$$d\bar{H} = \frac{I\, d\bar{L} \times \bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I\, dL \sin\theta\, \bar{a}_N}{4\pi R^2} \qquad \text{... } R = R_{12} = \text{Radius}$$

Hence total magnetic field intensity $\bar{H}$ at point P can be obtained by integrating $d\bar{H}$ around the circular closed path.

$$\therefore \quad \bar{H} = \oint d\bar{H} = \oint \frac{I\, dL \sin\theta\, \bar{a}_N}{4\pi R^2} = \frac{I \sin\theta\, \bar{a}_N}{4\pi R^2} \oint dL \qquad \ldots (2)$$

But $$\oint dL = \text{Circumference of the circle} = 2\pi R \qquad \ldots (3)$$

$$\therefore \quad \bar{H} = \frac{I \sin\theta\, 2\pi R\, \bar{a}_N}{4\pi R^2} = \frac{I \sin\theta}{2R}\, \bar{a}_N \qquad \ldots (4)$$

As $I d\bar{L}$ is tangential to the circle and R_{12} is the radius, angle θ must be 90°.

$$\therefore \quad \boxed{\bar{H} = \frac{I \sin 90°}{2R}\bar{a}_N = \frac{I}{2R}\bar{a}_N \quad \text{A/m}} \qquad \ldots (5)$$

$\bar{a}_N = \bar{a}_z$ if the circular loop is placed in xy plane

Now $\bar{B} = \mu_0 \bar{H}$ ... for free space

The flux density $\bar{B}$ at centre of the circular conductor carrying direct current I, placed in a free space is given by,

$$\boxed{\bar{B} = \frac{\mu_0 I}{2R}\bar{a}_N \quad \text{Wb/m}^2} \qquad \ldots (6)$$

7.7 $\bar{H}$ on the Axis of a Circular Loop

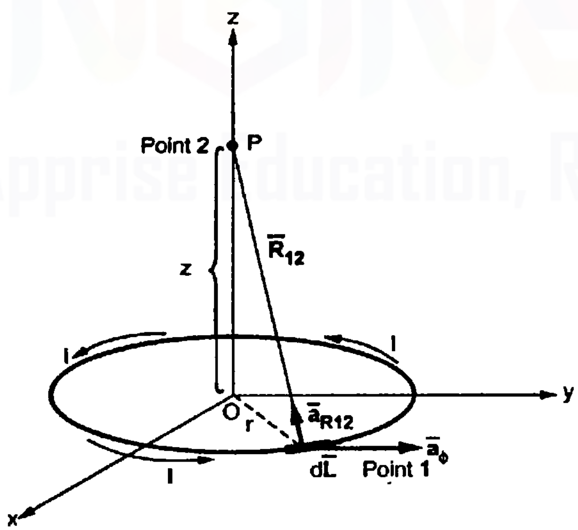


Fig. 7.16

Consider a circular loop carrying a direct current I, placed in xy plane, with z-axis as its axis as shown in the Fig. 7.16. The magnetic field intensity $\bar{H}$ at point P is to be obtained. The point P is at a distance z from the plane of the circular loop, along it's axis.

The radius of the circular loop is r. Consider the differential length $d\bar{L}$ of the circular loop as shown in the Fig. 7.16.

In the cylindrical co-ordinate system,

$$d\bar{L} = dr\, \bar{a}_r + r\, d\phi\, \bar{a}_\phi + dz\, \bar{a}_z$$

But $d\bar{L}$ is in the plane for which r is constant and z = 0 = constant plane. The $I\,d\bar{L}$ is tangential at point 1 in $\bar{a}_\phi$ direction.

$$\therefore \quad I\,d\bar{L} = I\,r\,d\phi\,\bar{a}_\phi \qquad \text{... (1)}$$

The unit vector $\bar{a}_{R12}$ is in the direction along the line joining differential current element to the point P.

$$\therefore \quad \bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} \qquad \text{... (2)}$$

From the Fig. 7.17, it can be observed that,

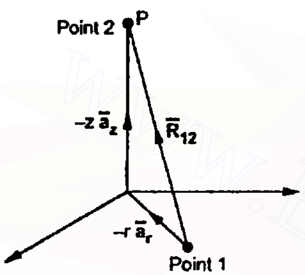


Fig. 7.17

$$\bar{R}_{12} = -r\,\bar{a}_r + z\,\bar{a}_z \qquad \text{... from 1 to 2}$$

$$\therefore \quad |\bar{R}_{12}| = \sqrt{(-r)^2+(z)^2} = \sqrt{r^2+z^2}$$

$$\therefore \quad \bar{a}_{R12} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}} \qquad \text{... (3)}$$

$$\text{Now} \quad d\bar{L} \times \bar{a}_{R12} = \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ 0 & rd\phi & 0 \\ -r & 0 & z \end{vmatrix} = zr\,d\phi\,\bar{a}_r + r^2\,d\phi\,\bar{a}_z$$

Note that while calculating cross product $|R_{12}|$ is neglected for convenience, which must be considered in further calculations.

According to Biot-Savart law, the differential field strength $d\bar{H}$ at point P is given by,

$$d\bar{H} = \frac{I\,d\bar{L}\times\bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I\left[z\,r\,d\phi\bar{a}_r + r^2 d\phi\bar{a}_z\right]}{4\pi\sqrt{r^2+z^2}\,(\sqrt{r^2+z^2})^2} \qquad \text{... (4)}$$

Note that $|\bar{R}_{12}|$ neglected while obtaining the cross product is considered in $d\bar{H}$.

The total $\bar{H}$ is to be obtained by integrating $d\bar{H}$ over the circular loop i.e. for $\phi=0$ to 2π.

Note : It can be observed that though $d\bar{H}$ consists of two components $\bar{a}_r$ and $\bar{a}_z$, due to radial symmetry all $\bar{a}_r$ components are going to cancel each other. So $\bar{H}$ exists only along the axis in $\bar{a}_z$ direction. Let us prove this mathematically.

$$\bar{H} = \int_{\phi=0}^{2\pi} \frac{I\left[z\,r\,\bar{a}_r + r^2\,\bar{a}_z\right]d\phi}{4\pi\left(r^2+z^2\right)^{3/2}} \qquad \text{... (5)}$$

$$= \frac{I}{4\pi}\left\{\int_{\phi=0}^{2\pi} \frac{z\,r\,d\phi}{\left(r^2+z^2\right)^{3/2}}\bar{a}_r + \int_{\phi=0}^{2\pi} \frac{r^2\,\bar{a}_z\,d\phi}{\left(r^2+z^2\right)^{3/2}}\right\} \qquad \text{... (6)}$$

Consider first integral to prove that its value is zero due to radial symmetry.

$$\int_{\phi=0}^{2\pi} \frac{z\, r\, d\phi}{\left(r^2+z^2\right)^{3/2}} \bar{a}_r = \int_{\phi=0}^{2\pi} \frac{z\, r\, d\phi}{\left(r^2+z^2\right)^{3/2}} \left[\cos\phi \bar{a}_x + \sin\phi \bar{a}_y\right]$$

The unit vector $\bar{a}_r$ is expressed in rectangular co-ordinate system as $\cos\phi\bar{a}_x + \sin\phi\bar{a}_y$.

Now $$\int_{\phi=0}^{2\pi} \cos\phi\, d\phi = [\sin\phi]_{\phi=0}^{2\pi} = \sin 2\pi - \sin 0 = 0$$

And $$\int_{\phi=0}^{2\pi} \sin\phi\, d\phi = [-\cos\phi]_0^{2\pi} = -\cos 2\pi - [-\cos 0] = -1+1 = 0$$

$$\therefore \quad \int_{\phi=0}^{2\pi} \frac{z\, r\, d\phi}{\left(r^2+z^2\right)^{3/2}} \bar{a}_r = 0$$

This proves that $\bar{H}$ at P can not have any radial component.

$$\therefore \quad \bar{H} = \frac{I}{4\pi} \int_{\phi=0}^{2\pi} \frac{r^2\, d\phi}{\left(r^2+z^2\right)^{3/2}} \bar{a}_z = \frac{I\, r^2 \bar{a}_z}{4\pi\left(r^2+z^2\right)^{3/2}} \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{I\, r^2 \bar{a}_z\, [\phi]_0^{2\pi}}{4\pi\left(r^2+z^2\right)^{3/2}} = \frac{I\, r^2\, 2\pi \bar{a}_z}{4\pi\left(r^2+z^2\right)^{3/2}}$$

$$\therefore \quad \bar{H} = \frac{I\, r^2}{2\left(r^2+z^2\right)^{3/2}} \bar{a}_z \quad \text{A/m} \qquad \ldots (7)$$

where r = Radius of the circular loop

z = Distance of point P along the axis

Note : If point P is shifted at the centre of the circular loop i.e. z = 0, we get the result obtained in earlier section.

$$\boxed{\bar{H} = \frac{I\, r^2}{2\left(r^2\right)^{3/2}} \bar{a}_z = \frac{I}{2r} \bar{a}_z \quad \text{A/m}}$$

where $\bar{a}_z$ is the unit vector normal to xy plane in which the circular loop is lying.

Example 7.2 : *A current filament carries a current of 10 A in the $\bar{a}_z$ direction on the z-axis. Find the magnetic field intensity $\bar{H}$ at point P (1,2,3) due to this filament if it extends from,*

a) z = −∞ to ∞ b) z = 0 to 5 m c) z = 5 to ∞.

Express answers in cartesian co-ordinates.

Solution : The arrangements are shown in the Fig. 7.18.

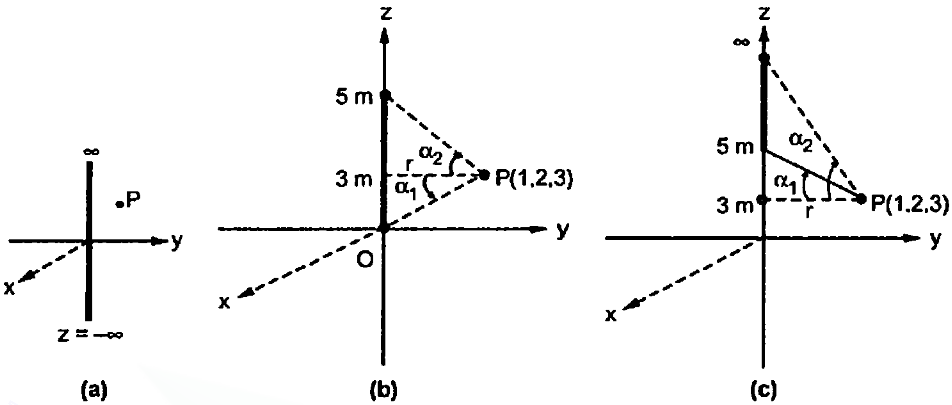


Fig. 7.18

Case a : It is infinitely long straight conductor.

$$\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi, \qquad P(1, 2, 3), \quad I = 10\text{ A}$$

Now $\quad r = \sqrt{x^2+y^2} = \sqrt{1+4} = \sqrt{5}\text{ m}$

$$\therefore \quad \bar{H} = \frac{10}{2\pi\times\sqrt{5}}\bar{a}_\phi = 0.7117\,\bar{a}_\phi \text{ A/m}$$

To find x component, take dot product with $\bar{a}_x$.

$$\therefore \quad H_x = \bar{H}\cdot\bar{a}_x = 0.7117\ \bar{a}_\phi\cdot\bar{a}_x = -0.7117\sin\phi$$

Similarly $\quad H_y = \bar{H}\cdot\bar{a}_y = 0.7117\ \bar{a}_\phi\cdot\bar{a}_y = +0.7117\cos\phi$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{1} = 63.43° \quad \text{... For point P}$$

$$\therefore \quad H_x = -0.6365, \quad H_y = 0.3183$$

$$\therefore \quad \bar{H} = -0.6365\,\bar{a}_x + 0.3183\,\bar{a}_y \text{ A/m}$$

Case b : It is a finite length conductor with $z_1 = 0$ and $z_2 = 5$ m.

$$r = \sqrt{x^2+y^2} = \sqrt{1+4} = \sqrt{5}\text{ m}$$

$$\alpha_1 = \tan^{-1}\frac{3}{\sqrt{5}} = 53.3°$$

but negative as that end is below point P.

$$\therefore \quad \alpha_1 = -53.3°$$

$$\alpha_2 = \tan^{-1}\frac{2}{\sqrt{5}} = 41.81°$$

Now $\bar{H} = \frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1]\bar{a}_\phi$

$= \frac{10}{4\pi\times\sqrt{5}}[\sin 41.81° - \sin(-53.3°)]\bar{a}_\phi = 0.5225\ \bar{a}_\phi$

$\therefore\quad H_x = \bar{H}\cdot\bar{a}_x = 0.5225\,(\bar{a}_\phi\cdot\bar{a}_x) = 0.5225\,(-\sin\phi)$

and $\quad H_y = \bar{H}\cdot\bar{a}_y = 0.5225\,(\bar{a}_\phi\cdot\bar{a}_y) = 0.5225\,(\cos\phi)$

$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{1} = 63.43°$ ... For point P

$\therefore\quad H_x = -0.4673,\quad H_y = 0.2337$

$\therefore\quad \bar{H} = \mathbf{-0.4673\ \bar{a}_x + 0.2337\ \bar{a}_y\ \ A/m}$

Case c : It is a conductor from z = 5 to z = ∞.

$r = \sqrt{x^2+y^2} = \sqrt{1+4} = \sqrt{5}\ \text{m}$

$\alpha_1 = \tan^{-1}\frac{2}{r} = \tan^{-1}\frac{2}{\sqrt{5}} = 41.81°$

$\alpha_2 = \tan^{-1}\frac{\infty}{r} = 90°$

Both α_1 and α_2 are positive as above point P.

$\therefore\quad \bar{H} = \frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1]\bar{a}_\phi$

$= \frac{10}{4\pi\times\sqrt{5}}[\sin 90 - \sin 41.81]\bar{a}_\phi = 0.1186\ \bar{a}_\phi$

$\therefore\quad H_x = \bar{H}\cdot\bar{a}_x = 0.1186\,(\bar{a}_\phi\cdot\bar{a}_x) = 0.1186\,(-\sin\phi)$

and $\quad H_y = \bar{H}\cdot\bar{a}_y = 0.1186\,(\bar{a}_\phi\cdot\bar{a}_y) = 0.1186\,(\cos\phi)$

$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{1} = 63.43°$ ... For point P

$\therefore\quad H_x = -0.106,\quad H_y = 0.053$

$\therefore\quad \bar{H} = \mathbf{-0.106\ \bar{a}_x + 0.053\ \bar{a}_y\ \ A/m}$

Example 7.3 : *Find the magnetic flux density at the centre 'O' of a square of sides equal to 5 m and carrying 10 amperes of current.*

Solution : The square is placed in the xy plane as shown in the Fig. 7.19.

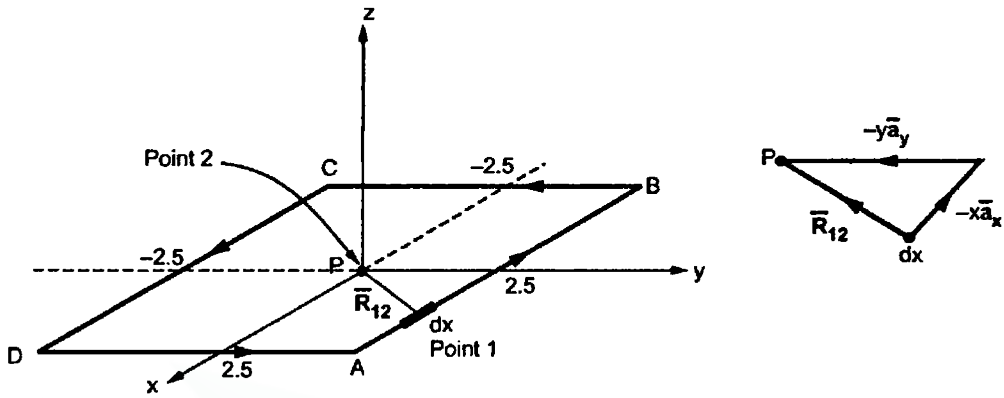


Fig. 7.19

Consider differential element dx along AB of the square.

$$\therefore \quad d\bar{L} = dx\,\bar{a}_x$$

The $\bar{R}_{12}$ joining differential element to point P is,

$$\bar{R}_{12} = -x\,\bar{a}_x - y\,\bar{a}_y$$

$$\therefore \quad |\bar{R}_{12}| = \sqrt{x^2+y^2}$$

$$\therefore \quad \bar{a}_{R12} = \frac{-x\,a_x - y\,a_y}{\sqrt{x^2+y^2}}$$

$$\therefore \quad d\bar{L}\times\bar{a}_{R12} = \begin{vmatrix} a_x & a_y & a_z \\ dx & 0 & 0 \\ -x & -y & 0 \end{vmatrix} = -y\,dx\,\bar{a}_z$$

According to Biot-Savart law,

$$d\bar{H} = \frac{I\,d\bar{L}\times\bar{a}_{R12}}{4\pi R_{12}^2} = \frac{I(-y\,dx)\,\bar{a}_z}{4\pi\sqrt{x^2+y^2}\,(\sqrt{x^2+y^2})^2} \quad \text{... Considering } |\bar{R}_{12}|$$

$$= \frac{10\times(-2.5)\,dx\,\bar{a}_z}{4\pi\left(x^2+2.5^2\right)^{3/2}} \quad \text{... } y = 2.5 \text{ for segment AB}$$

$$\therefore \quad \bar{H} = \int_{x=2.5}^{-2.5} \frac{-25\,dx\,\bar{a}_z}{4\pi\left(x^2+2.5^2\right)^{3/2}} = 2\int_{x=2.5}^{0} \frac{-25\,dx\,\bar{a}_z}{4\pi\left(x^2+2.5^2\right)^{3/2}}$$

Put $x = 2.5\tan\theta$, $\quad dx = 2.5\sec^2\theta\; d\theta$

Limits, $x = 2.5,\ \theta = 45°$ and $x = 0,\ \theta = 0°$

$$\therefore \quad \bar{H} = -\frac{25\times 2}{4\pi}\int_{\theta=45^\circ}^{0^\circ}\frac{2.5\sec^2\theta\, d\theta\,\bar{a}_z}{(2.5)^3(1+\tan^2\theta)^{3/2}} = -0.6366\int_{\theta=45^\circ}^{0^\circ}\frac{1}{\sec\theta}d\theta\,\bar{a}_z$$

$$= -0.6366\int_{\theta=45}^{0^\circ}\cos\theta\, d\theta\,\bar{a}_z = -0.6366\,[\sin\theta]_{45^\circ}^{0^\circ}\,\bar{a}_z = -0.6366\,[0-\sin 45^\circ]\bar{a}_z$$

$$= 0.4501\,\bar{a}_z \quad A/m$$

This $\bar{H}$ is due to the segment AB of the square. All sides will produce same $\bar{H}$ at point P.

$$\therefore \quad \bar{H}_{total} = 4\bar{H} = 4\times 0.4501\,\bar{a}_z = 1.8\,\bar{a}_z \;\; A/m$$

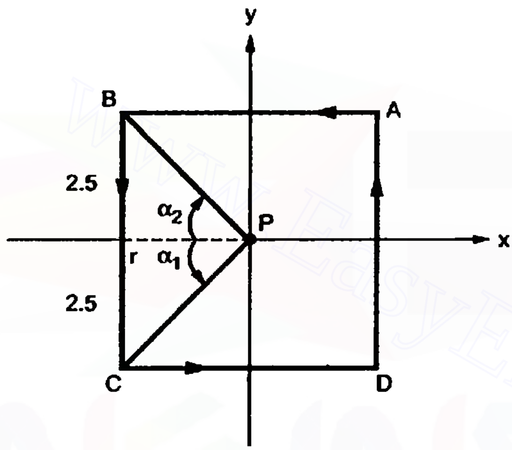


Fig. 7.20

Alternative method : Consider one side of a square as shown in the Fig. 7.20, in xy plane. Consider segment BC, which is finite length of the conductor. As B is above P, α_1 is negative and α_2 is positive.

$$\alpha_1 = \tan^{-1}\frac{2.5}{2.5} = 45^\circ, \text{ but}$$

$$\alpha_1 = -45^\circ$$

$$\alpha_2 = +45^\circ$$

$$\therefore \quad |\bar{H}| = \frac{I}{4\pi r}\sin\alpha_2 - \sin\alpha_1 = \frac{10}{4\pi\times 2.5}[\sin(45^\circ) - \sin(-45^\circ)]$$

$$= 0.4501 \text{ A/m}$$

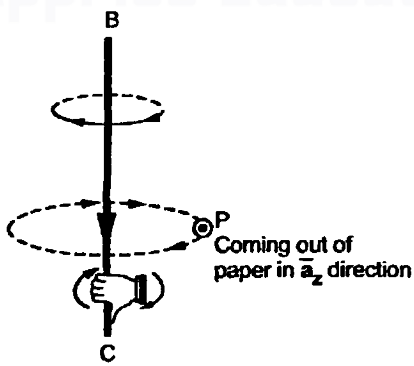


Fig. 7.21

Important note : As BC segment is not along z-axis while using formula derived earlier do not use direction as $\bar{a}_\phi$. Remember that $\bar{H}$ direction is normal to the plane containing the source. In this case, square is in xy plane normal to which is $\bar{a}_z$ hence direction of $\bar{H}$ is $\bar{a}_z$, as shown in the Fig. 7.21 by right handed screw rule.

$$\therefore \quad \bar{H} = 0.4501\,\bar{a}_z \;\; A/m$$

$$\therefore \quad \bar{H}_{total} = 4\bar{H} = 1.8\,\bar{a}_z \;\; A/m$$

Example 7.4 : *Find the magnetic field intensity at point P for the circuit shown in the Fig. 7.22.*

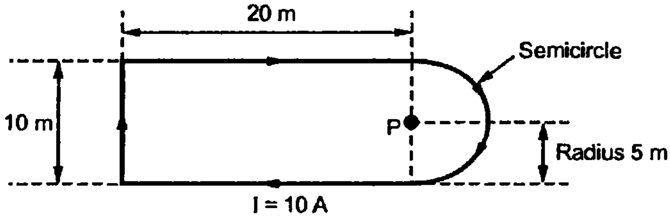


Fig. 7.22

Solution : Consider the various sections of the loop.

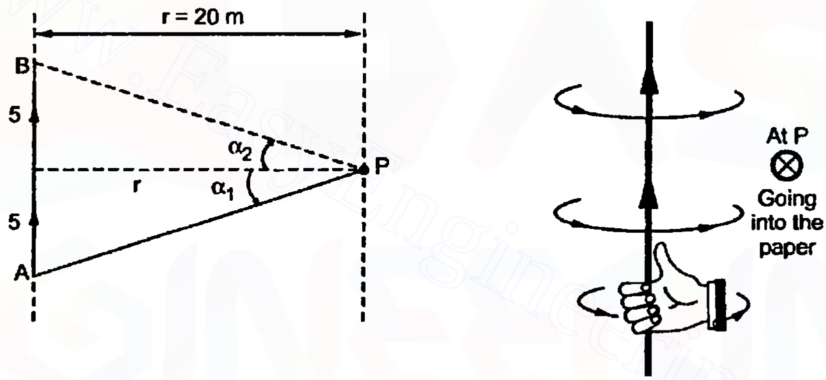


Fig. 7.23

Section I : From the Fig. 7.23,

$$\alpha_2 = \tan^{-1}\frac{5}{20} = 14.036°$$

$$\alpha_1 = \tan^{-1}\frac{5}{20} = 14.036° \qquad \text{but } \alpha_1 = -14.036°$$

But α_1 is negative as point A is below P. If the loop is placed in xy plane, direction of $\bar{H}$ at P is going into the paper, normal to xy plane according to right hand thumb rule. This is $-\bar{a}_z$ direction.

$$\therefore \qquad \bar{H}_1 = \frac{I}{4\pi r}(\sin\alpha_2 - \sin\alpha_1)\,[-\bar{a}_z]$$

$$= \frac{10}{4\pi \times 20}[\sin 14.036 - \sin(-14.036)][-\bar{a}_z] = -0.0193\,\bar{a}_z \text{ A/m}$$

Section II : Consider section BC as shown in the Fig. 7.24.

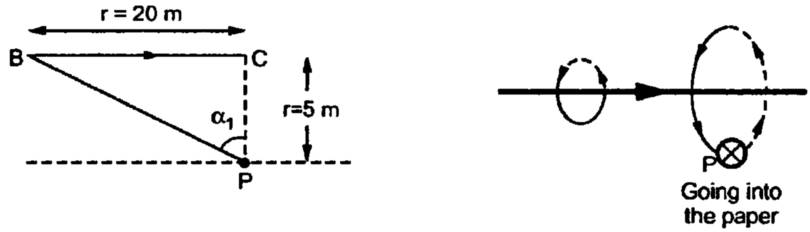


Fig. 7.24

From the Fig. 7.24, $\alpha_2 = 0°$ as point C and P are colinear.

$$\alpha_1 = \tan^{-1}\frac{20}{5} = 75.96°$$

But α_1 is negative as point B is below P, $\alpha_1 = -75.96°$. The direction of $\bar{H}$ at P is going into the paper according to the right hand thumb rule i.e. $-\bar{a}_z$ as the circuit is placed in xy plane.

$$\therefore \quad \bar{H}_2 = \frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1](-\bar{a}_z)$$

$$= \frac{10}{4\pi\times 5}[\sin 0° - \sin(-75.96°)](-\bar{a}_z) = -0.1544\ \bar{a}_z \text{ A/m.}$$

Section III : The semicircular loop CDE as shown in the Fig. 7.25.

The $\bar{H}$ at the centre of a circular loop is given by,

$$\bar{H} = \frac{I}{2R}\bar{a}_N \qquad \text{... (Refer section 7.6)}$$

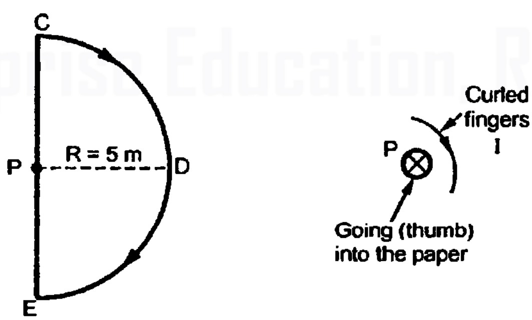


Fig. 7.25

Hence for a semicircular section CDE,

$$\bar{H}_3 = \frac{\left(\frac{I}{2R}\right)}{2}(-\bar{a}_z) = \frac{10}{4\times 5}(-\bar{a}_z)$$

$$= -0.5\ a_z \text{ A/m}$$

Section IV : The section EA is exactly similar to the section II BC and hence $\bar{H}_4$ due to EA is equal in magnitude and direction at P, to that of $\bar{H}_2$.

$$\therefore \quad \bar{H}_4 = \bar{H}_2 = -0.1544\ \bar{a}_z \text{ A/m}$$

Hence the total $\bar{H}$ at P is,

$$\bar{H} = H_1 + \bar{H}_2 + \bar{H}_3 + \bar{H}_4 = -[0.0193 + 0.1544 + 0.5 + 0.1544]\bar{a}_z$$

$$= -0.8281 \, a_z \text{ A/m}$$

7.8 Ampere's Circuital Law

In electrostatics, the Gauss's law is useful to obtain the $\bar{E}$ in case of complex problems. Similarly in the magnetostatics, the complex problems can be solved using a law called **Ampere's circuital law or Ampere's work law.**

The Ampere's circuital law states that,

The line integral of magnetic field intensity $\bar{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

The mathematical representation of Ampere's circuital law is,

$$\boxed{\oint \bar{H} \cdot d\bar{L} = I} \qquad \text{... (1)}$$

The law is very helpful to determine $\bar{H}$ when the current distribution is symmetrical.

7.8.1 Proof of Ampere's Circuital Law

Consider a long straight conductor carrying direct current I placed along z axis as shown in the Fig. 7.26. Consider a closed circular path of radius r which encloses the straight conductor carrying direct current I. The point P is at a perpendicular distance r from the conductor. Consider $d\bar{L}$ at point P which is in $\bar{a}_\phi$ direction, tangential to circular path at point P.

$$\therefore \qquad d\bar{L} = r \, d\phi \, \bar{a}_\phi \qquad \text{... (2)}$$

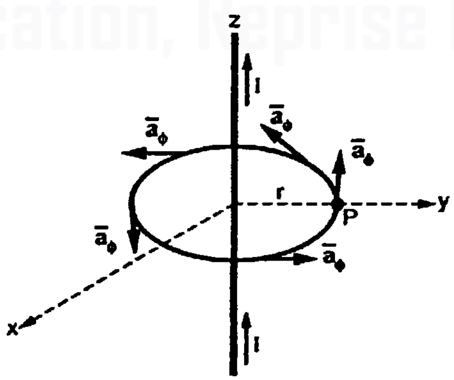


Fig. 7.26

While $\bar{H}$ obtained at point P, from Biot-Savart law due to infinitely long conductor is,

$$\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi \qquad \text{... (3)}$$

$$\therefore \quad \bar{H}\bullet d\bar{L} = \frac{I}{2\pi r}\bar{a}_\phi \bullet r\, d\phi\, \bar{a}_\phi$$

$$= \frac{I}{2\pi r} r\, d\phi = \frac{I}{2\pi} d\phi \qquad \text{.....} \ \bar{a}_\phi \bullet \bar{a}_\phi = 1$$

Integrating $\bar{H}\bullet d\bar{L}$ over the entire closed path,

$$\oint \bar{H}\bullet d\bar{L} = \int_{\phi=0}^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi}[\phi]_0^{2\pi} = \frac{I\,2\pi}{2\pi}$$

$$= I = \text{Current carried by conductor}$$

This proves that the integral $\bar{H}\bullet d\bar{L}$ along the closed path gives the direct current enclosed by that closed path.

Key Point: *The path enclosing the direct current I need not be a circular and it may be any irregular shape. The law does not depend on the shape of the path but the path must enclose the direct current once. This path selected is called Amperian path similar to the Gaussian surface used while applying Gauss's law.*

7.8.2 Steps to Apply Ampere's Circuital Law

Follow the steps given to apply Ampere's circuital law :

Step 1 : Consider a closed path preferrably symmetrical such that it encloses the direct current I once. This is Amperian path.

Step 2 : Consider differential length $d\bar{L}$ depending upon the co-ordinate system used.

Step 3 : Identify the symmetry and find in which direction $\bar{H}$ exists according to the co-ordinate system used.

Step 4 : Find $\bar{H}\bullet d\bar{L}$, the dot product. Make sure that $d\bar{L}$ and $\bar{H}$ in same direction.

Step 5 : Find the integral of $\bar{H}\bullet d\bar{L}$ around the closed path assumed. And equate it to current I enclosed by the path.

Solving this for the $\bar{H}$ we get the required magnetic field intensity due to the direct current I.

To apply Ampere's circuital law the following conditions must be satisfied,

1. The $\bar{H}$ is either tangential or normal to the path, at each point of the closed path.

2. The magnitude of $\bar{H}$ must be same at all points of the path where $\bar{H}$ is tangential.

Thus identifying symmetry and identifying the components of $\bar{H}$ present, plays an important role while applying the Ampere's circuital law.

7.9 Applications of Ampere's Circuital Law

Let us steady the various cases and the application of Ampere's circuital law to obtain $\bar{H}$.

7.9.1 $\bar{H}$ due to Infinitely Long Straight Conductor

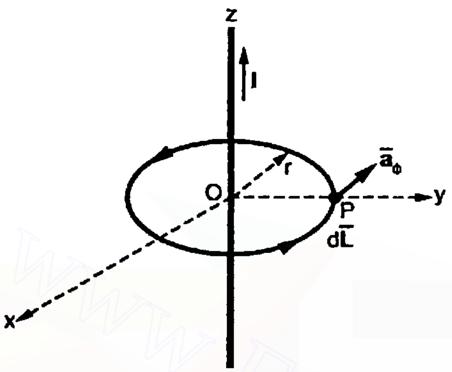


Fig. 7.27

Consider an infinitely long straight conductor placed along z-axis, carrying a direct current I as shown in the Fig. 7.27. Consider the Amperian closed path, enclosing the conductor as shown in the Fig. 7.27. Consider point P on the closed path at which $\bar{H}$ is to be obtained. The radius of the path is r and hence P is at a perpendicular distance r from the conductor.

The magnitude of $\bar{H}$ depends on r and the direction is always tangential to the closed path i.e. $\bar{a}_\phi$. So $\bar{H}$ has only component in $\bar{a}_\phi$ direction say H_ϕ.

Consider elementary length $d\bar{L}$ at point P and in cylindrical co-ordinates it is $r\,d\phi$ in $\bar{a}_\phi$ direction.

$$\therefore \quad \bar{H} = H_\phi\,\bar{a}_\phi \quad \text{and} \quad d\bar{L} = r\,d\phi\,\bar{a}_\phi$$

$$\therefore \quad \bar{H}\cdot d\bar{L} = H_\phi\,\bar{a}_\phi \cdot r\,d\phi\,\bar{a}_\phi = H_\phi\,r\,d\phi$$

According to Ampere's circuital law,

$$\oint \bar{H}\cdot d\bar{L} = I$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\,r\,d\phi = I$$

$$\therefore \quad H_\phi\,r \int_{\phi=0}^{2\pi} d\phi = I$$

$$\therefore \quad H_\phi\,r(2\pi) = I$$

$$\therefore \quad H_\phi = \frac{I}{2\pi r}$$

Hence $\bar{H}$ at point P is given by,

$$\boxed{\bar{H} = H_\phi\,\bar{a}_\phi = \frac{I}{2\pi r}\,\bar{a}_\phi \quad \text{A/m}}$$

7.9.2 $\bar{H}$ due to a Co-axial Cable

Consider a co-axial cable as shown in the Fig. 7.28. Its inner conductor is solid with radius a, carrying direct current I. The outer conductor is in the form of concentric cylinder whose inner radius is b and outer radius is c. This cable is placed along z axis. The current I is uniformly distributed in the inner conductor. While – I is uniformly distributed in the outer conductor.

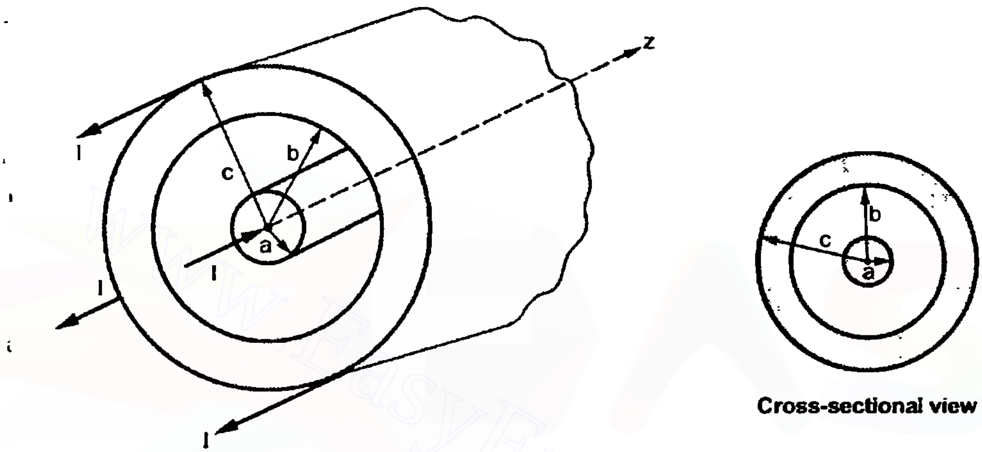


Fig. 7.28 Co-axial cable

The space between inner and outer conductor is filled with dielectric say air. The calculation of $\bar{H}$ is divided corresponding to various regions of the cable.

Region 1 : Within the inner conductor, r < a. Consider a closed path having radius r < a. Hence it encloses only part of the conductor as shown in the Fig. 7.29.

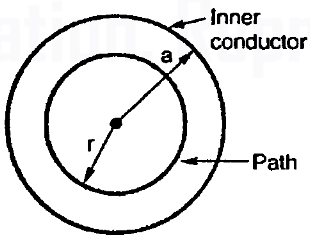


Fig. 7.29

The area of cross-section enclosed is πr^2 m^2.

The total current flowing is I through the area πa^2. Hence the current enclosed by the closed path is,

$$I' = \frac{\pi r^2}{\pi a^2} I = \frac{r^2}{a^2} I \qquad \text{... (1)}$$

The $\bar{H}$ is again only in $\bar{a}_\phi$ direction and depends only on r.

$$\therefore \quad \bar{H} = H_\phi \bar{a}_\phi$$

So consider $d\bar{L}$ in the $\bar{a}_\phi$ direction which is $r\,d\phi$.

$$\therefore \quad d\bar{L} = r d\phi\, \bar{a}_\phi$$

$$\therefore \quad \bar{H} \cdot d\bar{L} = H_\phi \bar{a}_\phi \cdot r d\phi \bar{a}_\phi = H_\phi\, r\, d\phi \qquad \text{... (2)}$$

According to Ampere's circuital law,

$$\oint \bar{H} \cdot d\bar{L} = I' \qquad \text{... Current enclosed}$$

$$\therefore \quad \oint H_\phi\, r\, d\phi = \frac{r^2}{a^2} I$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\, r\, d\phi = \frac{r^2}{a^2} I$$

$$\therefore \quad H_\phi\, r[2\pi] = \frac{r^2}{a^2} I$$

$$\therefore \quad H_\phi = \frac{r^2}{2\pi r a^2} I = \frac{r}{2\pi a^2} I$$

$$\boxed{\bar{H} = \frac{I r}{2\pi a^2} \bar{a}_\phi \ \text{A/m}}$$

$\therefore$ ... r < a within conductor

Region 2 : Within a < r < b consider a circular path which encloses the inner conductor carrying direct current I. This is the case of infinitely long conductor along z-axis. Hence $\bar{H}$ in this region is,

$$\boxed{\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi \ \text{A/m}} \qquad \text{... } (a < r < b)$$

Region 3 : Within outer conductor, b < r < c

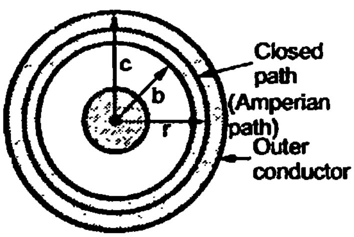


Fig. 7.30

Consider the closed path as shown in the Fig. 7.30. The current enclosed by the closed path is only the part of the current – I, in the outer conductor. The total current – I is flowing through the cross section $\pi(c^2 - b^2)$ while the closed path encloses the cross section $\pi(r^2 - b^2)$.

Hence the current enclosed by the closed path of outer conductor is,

$$I' = \frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)}(-I) = -\frac{(r^2 - b^2)}{(c^2 - b^2)} I \qquad \text{... (3)}$$

Key Point: *Note that the closed path also encloses the inner conductor hence the current I flowing though it.*

$$\therefore \quad I'' = I = \text{Current in inner conductor enclosed} \quad \ldots (4)$$

Total current enclosed by the closed path is,

$$I_{enc} = I' + I'' = -\frac{(r^2 - b^2)}{\left(c^2 - b^2\right)} I + I$$

$$= I\left[1 - \frac{(r^2 - b^2)}{(c^2 - b^2)}\right] = I\left[\frac{c^2 - r^2}{c^2 - b^2}\right] \quad \ldots (5)$$

According to Ampere's circuital law,

$$\oint \bar{H} \cdot d\bar{L} = I_{enc}$$

Now $\bar{H}$ is again in $\bar{a}_\phi$ direction only and is a function of r only.

$$\therefore \quad \bar{H} = H_\phi \bar{a}_\phi \quad \text{and} \quad d\bar{L} = r\, d\phi\, \bar{a}_\phi$$

$$\therefore \quad \bar{H} \cdot d\bar{L} = H_\phi \bar{a}_\phi \cdot r d\phi \bar{a}_\phi = H_\phi\, r\, d\phi$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\, r\, d\phi = I_{enc}$$

$$\therefore \quad H_\phi\, r[2\pi] = I\left[\frac{c^2 - r^2}{c^2 - b^2}\right]$$

$$\therefore \quad H_\phi = \frac{I}{2\pi r}\left[\frac{c^2 - r^2}{c^2 - b^2}\right]$$

$$\therefore \quad \boxed{\bar{H} = H_\phi \bar{a}_\phi = \frac{I}{2\pi r}\left[\frac{c^2 - r^2}{c^2 - b^2}\right] \bar{a}_\phi \quad \text{A/m}} \quad \ldots b < r < c$$

Region 4 : Outside the cable, r > c.

Consider the closed path with r > c such that it encloses both the conductors i.e. both currents + I and – I.

Thus the total current enclosed is,

$$I_{enc} = +I - I = 0 \text{ A}$$

$$\therefore \quad \oint \bar{H} \cdot d\bar{L} = 0 \quad \ldots \text{Ampere's circuital law}$$

$$\therefore \quad \boxed{\bar{H} = 0 \quad \text{A/m}} \quad \ldots r > c$$

The magnetic field does not exist outside the cable. The variation of $\bar{H}$ against r is shown in the Fig. 7.31.

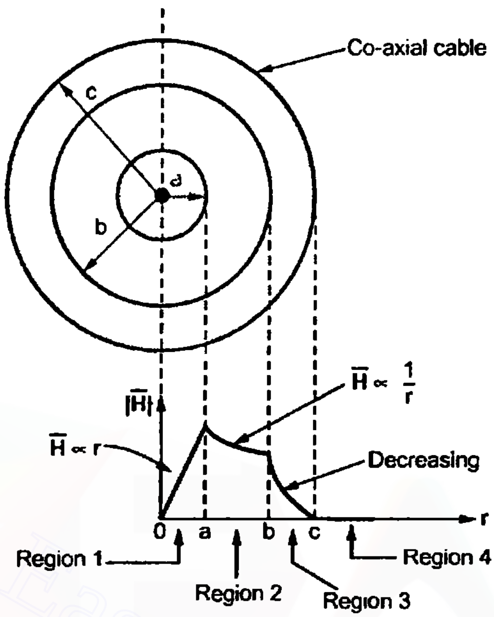


Fig. 7.31 Variation of $\bar{H}$ against r in co-axial cable

7.9.3 $\bar{H}$ due to Infinite Sheet of Current

Consider an infinite sheet of current in the z = 0 plane. The surface current density is $\bar{K}$. The current is flowing in positive y direction hence $\bar{K} = K_y\,\bar{a}_y$. This is shown in the Fig. 7.32.

Consider a closed path 1-2-3-4 as shown in the Fig. 7.32. The width of the path is b while the height is a. It is perpendicular to the direction of current hence in xz plane.

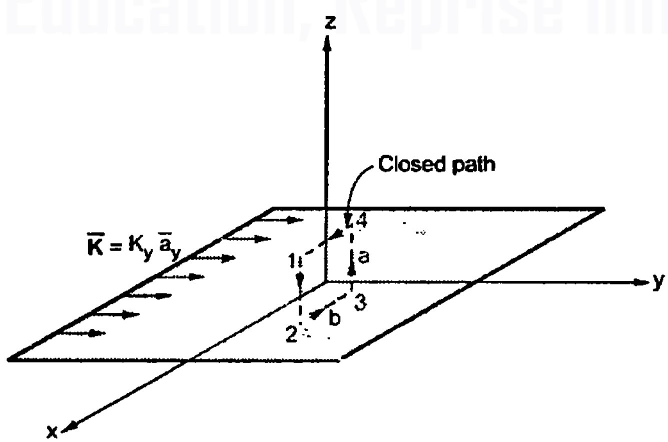


Fig. 7.32

The current flowing across the distance b is given by $K_y b$.

$$\therefore \qquad I_{enc} = K_y b \qquad \text{... (6)}$$

Consider the magnetic lines of force due to the current in $\bar{a}_y$ direction, according to right hand thumb rule. These are shown in the Fig. 7.33.

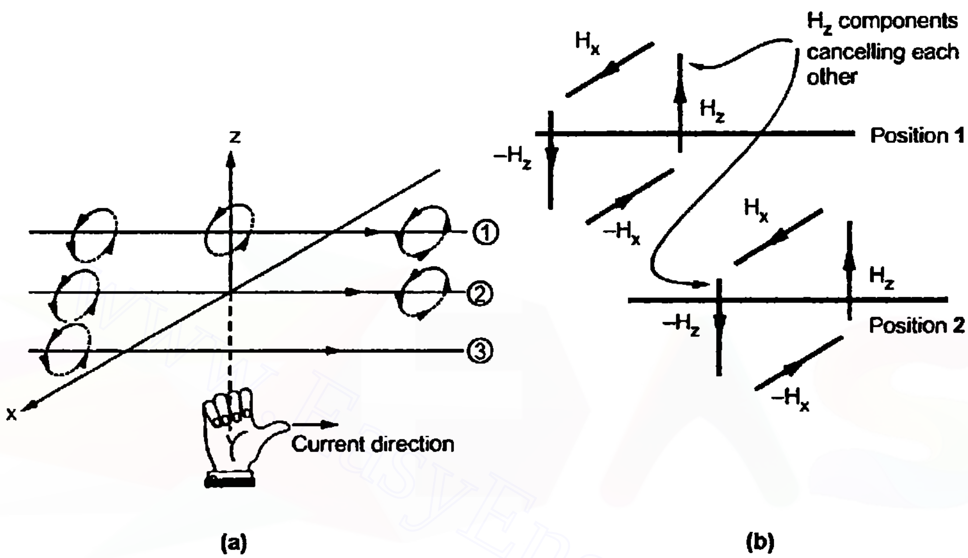


Fig. 7.33

In Fig. 7.33 (b), it is clear that in between two very closely spaced conductors, the components of $\bar{H}$ in z direction are oppositely directed ($-H_z$ for position 1 and $+H_z$ for position 2 between the two positions). All such components cancel each other and hence $\bar{H}$ can not have any component in z direction.

As current is flowing in y direction, $\bar{H}$ can not have component in y direction.

So $\bar{H}$ has only component in x direction.

$$\therefore \qquad \bar{H} = H_x\, \bar{a}_x \qquad \text{... for } z > 0 \qquad \text{... (7 (a))}$$

$$= -H_x\, \bar{a}_x \qquad \text{... for } z < 0 \qquad \text{... (7 (b))}$$

Applying Ampere's circuit law,

$$\oint \bar{H} \cdot d\bar{L} = I_{enc} \qquad \text{... (8)}$$

Evaluate the integral along the path 1-2-3-4-1.

For path 1-2, $\quad d\bar{L} = dz\, \bar{a}_z$,

For path 3-4, $\quad d\bar{L} = dz\, \bar{a}_z$

But $\bar{H}$ is in x direction while $\bar{a}_x \cdot \bar{a}_z = 0$.

Hence along the paths 1-2 and 3-4, the integral $\oint \bar{H} \cdot dL = 0$.

Consider path 2-3 along which $d\bar{L} = dx\,\bar{a}_x$.

$$\therefore \quad \int_2^3 H \cdot dL = \int_2^3 (-H_x\,\bar{a}_x)\cdot(dx\bar{a}_x) = H_x \int_2^3 dx = b\,H_x$$

The path 2-3 is lying in $z < 0$ region for which $\bar{H}$ is $-H_x\,\bar{a}_x$. And limits from 2 to 3, positive x to negative x hence effective sign of the integral is positive.

Consider path 4-1 along which $d\bar{L} = dx\,\bar{a}_x$ and it is in the region $z > 0$ hence $\bar{H} = H_x\,\bar{a}_x$.

$$\therefore \quad \int_4^1 H \cdot dL = \int_1^1 (H_x\,\bar{a}_x)\cdot(dx\bar{a}_x) = H_x \int_1^1 dx = b\,H_x$$

$$\therefore \quad \oint \bar{H}\cdot d\bar{L} = b\,H_x + b\,H_x = 2\,b\,H_x \qquad \text{... (9)}$$

Equating this to I_{enc} in equation (6),

$$2\,b\,H_x = K_y\,b$$

$$\therefore \quad H_x = \frac{1}{2}K_y \qquad \text{... (10)}$$

Hence,
$$\bar{H} = \frac{1}{2}K_y\,\bar{a}_x \quad \text{for} \quad z > 0 \qquad \text{... (11 (a))}$$

$$= -\frac{1}{2}K_y\,\bar{a}_x \quad \text{for} \quad z < 0 \qquad \text{... (11 (b))}$$

In general, for an infinite sheet of current density $\bar{K}$ A/m we can write,

$$\boxed{\bar{H} = \frac{1}{2}\,\bar{K}\times\bar{a}_N} \qquad \text{... (12)}$$

where $\bar{a}_N$ = Unit vector normal from the current sheet to the point at which $\bar{H}$ is to be obtained.

Example 7.5 : *Obtain the expression for $\bar{H}$ in all the regions if a cylindrical conductor carries a direct current I and its radius is 'R' m. Plot the variation of $\bar{H}$ against the distance r from the centre of the conductor.* **[UPTU : 2003-2004]**

Solution : Let the cylindrical conductor of radius R, carries a uniform direct current of I A. It is placed along z-axis and has infinite length. $\bar{H}$ is to be obtained considering two regions.

Region 1 : Within the conductor, $r < R$.

Consider the closed path of radius r within the conductor as shown in the Fig. 7.34.

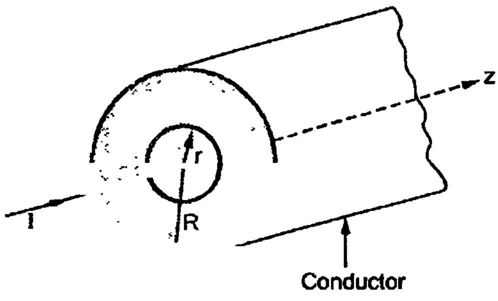


Fig. 7.34

As current I flows uniformly, it flows across the cross-sectional area of πR^2.

While the closed path encloses only part of the current which passes across the cross-sectional area of πr^2.

Hence current enclosed by the path,

$$\therefore \quad I_{enc} = I\frac{\pi r^2}{\pi R^2} = I\frac{r^2}{R^2} \quad \text{... (1)}$$

$\bar{H}$ has only $\bar{a}_\phi$ component and it is the function of r only hence,

$$\bar{H} = H_\phi \bar{a}_\phi \quad \text{and} \quad d\bar{L} = r\,d\phi\,\bar{a}_\phi \text{ in } \bar{a}_\phi \text{ direction}$$

$$\therefore \quad \bar{H} \cdot d\bar{L} = H_\phi \bar{a}_\phi \cdot r\,d\phi\,\bar{a}_\phi = H_\phi\, r\, d\phi \quad \text{... (2)}$$

According to Ampere's circuital law,

$$\oint \bar{H} \cdot d\bar{L} = I_{enc}$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\, r\, d\phi = I\frac{r^2}{R^2}$$

$$\therefore \quad H_\phi\, r(2\pi) = I\frac{r^2}{R^2}$$

$$\therefore \quad H_\phi = \frac{I}{2\pi r} \times \frac{r^2}{R^2} = \frac{I\,r}{2\pi R^2}$$

$$\therefore \quad \bar{H} = \frac{I\,r}{2\pi R^2}\bar{a}_\phi \ \text{A/m} \qquad \text{... for } r < R$$

Inside the conductor $\bar{H} \propto r$ and on the surface $r = R$ hence $\bar{H}$ becomes,

$$\bar{H} = \frac{I}{2\pi R}\bar{a}_\phi \quad \text{on the surface of conductor}$$

Region 2 : Outside the conductor, r > R.

The conductor is infinite length along z-axis carrying direct current I hence using the earlier result,

$$\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi \quad \text{for} \quad r > R$$

So outside the conductor, $\bar{H} \propto \frac{1}{r}$.

The graphical variation of $\bar{H}$ against r measured from centre of the conductor is shown in the Fig. 7.35.

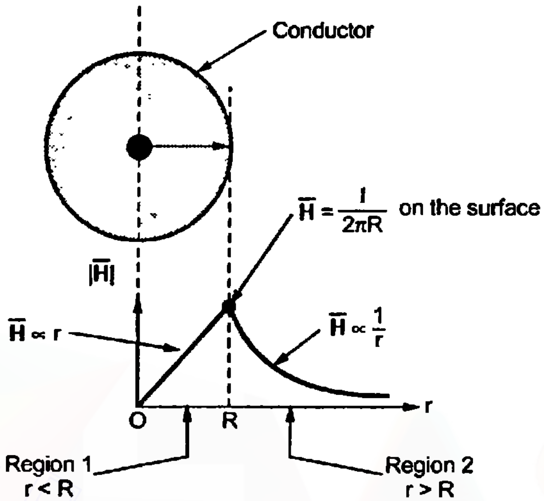


Fig. 7.35

Example 7.6 : *The plane y = 1 carries current density $\bar{K} = 40\bar{a}_z$ A/m. Find H at A (0,0,0) and B(1,5,–2).*

Solution : The sheet is located at y = 1 on which $\bar{K}$ is in $\bar{a}_z$ direction. The sheet is infinite and is shown in the Fig. 7.36.

The $\bar{H}$ will be in x direction.

a) Point A (0, 0, 0)

$\bar{a}_N = -\bar{a}_y$ normal to current sheet at Point A

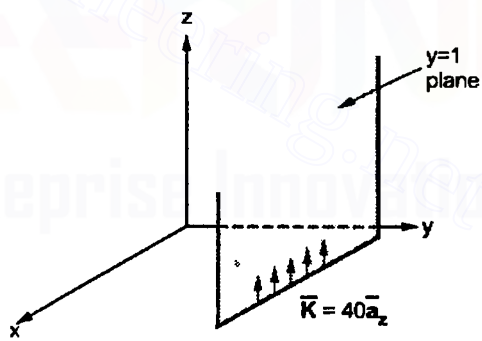


Fig. 7.36

P

$$\therefore \quad \bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_N$$

$$= \frac{1}{2}\left[40\bar{a}_z \times -\bar{a}_y\right]$$

Now $\quad \bar{a}_z \times \bar{a}_y = -\bar{a}_x$

$$\therefore \quad \bar{H} = \frac{1}{2}[+40]\bar{a}_x = 20\,\bar{a}_x \quad \text{A/m}$$

b) Point B (1, 5, –2)

This is to the right of the plane as y = 5 for B.

$\therefore \quad \bar{a}_N = \bar{a}_y$ normal to sheet at point B

$$\therefore \quad \bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_N = \frac{1}{2}\left[40\bar{a}_z \times \bar{a}_y\right] = -20\,\bar{a}_x \quad \text{A/m}$$

Example 7.7 : *In the region $0 < r < 0.5$ m, in cylindrical co-ordinates, the current density is,*

$\bar{J} = 4.5\, e^{-2r}\, \bar{a}_z\ A/m^2$

And $\bar{J} = 0$ elsewhere. Use Amperes circuital law to find $\bar{H}$.

Solution : The current from current density is given by,

$$I = \oint \bar{J} \cdot d\bar{S}$$

$$d\bar{S} = r\,dr\,d\phi\,\bar{a}_z \quad \text{normal to } \bar{a}_z \text{ as } \bar{J} \text{ is in } \bar{a}_z$$

$$\therefore \quad I = \int_{\phi=0}^{2\pi}\int_{r=0}^{r} 4.5\,e^{-2r}\,\bar{a}_z \cdot r\,dr\,d\phi\,\bar{a}_z$$

$$= 4.5 \int_{\phi=0}^{2\pi}\int_{r=0}^{r} r\,e^{-2r}\,dr\,d\phi$$

Using integration by parts,

$$= 4.5 \int_{\phi=0}^{2\pi} d\phi \left\{ r\int e^{-2r}\,dr - \int 1 \int e^{-2r}\,dr\,dr \right\}_0^r$$

$$= 4.5\,(2\pi)\left\{ \frac{r\,e^{-2r}}{-2} - \int \frac{e^{-2r}}{-2}\,dr \right\}_0^r$$

$$= 9\pi \left\{ \frac{r\,e^{-2r}}{-2} + \frac{1}{2}\frac{e^{-2r}}{-2} \right\}_0^r$$

$$= 9\pi \left\{ \frac{-r\,e^{-2r}}{2} - \frac{1}{4}e^{-2r} + \frac{1}{4} \right\} = \frac{9\pi}{4}\left\{ 1 - 2r\,e^{-2r} - e^{-2r} \right\} A$$

For $r = 0.5$, $I = 7.068\,[1 - 0.3678 - 0.3678] = 1.8676$ A

Consider a closed path with $r \geq 0.5$ such that the enclosed current I is 1.8676 A.

According to Ampere's circuital law,

$$\oint \bar{H} \cdot d\bar{L} = I$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\, r\, d\phi = I \qquad \ldots \bar{H} = H_\phi\,\bar{a}_\phi,\ d\bar{L} = r\,d\phi\,\bar{a}_\phi$$

$$\therefore \quad 2\pi r H_\phi = 1.8676$$

$$\therefore \quad H_\phi = \frac{1.8676}{2\pi r} = \frac{0.2972}{r}$$

$$\therefore \quad \bar{H} = \frac{0.2972}{r}\,\bar{a}_\phi \ \text{A/m} \quad \text{for } r \geq 0.5 \text{ m}$$

Example 7.8 : *A 'z' directed current distribution is given by,*
$\bar{J} = (r^2 + ur)$ for $r \leq a$.
Find $\bar{B}$ at any point $r \geq a$ using Ampere's circuital law.

Solution : As current density is given,

$$I = \oint \bar{J} \cdot d\bar{S}, \quad \bar{J} \text{ is in } \bar{a}_z \text{ direction}$$

$$\therefore \quad d\bar{S} = r\,dr\,d\phi\,\bar{a}_z$$

$$\therefore \quad I = \int_{\phi=0}^{2\pi} \int_{r=0}^{r} (r^2 + u\,r)\, r\,dr\,d\phi \qquad \ldots\ \bar{a}_z \cdot \bar{a}_z = 1$$

$$= [\phi]_0^{2\pi} \int_{r=0}^{r} [r^3 + u\,r^2]\,dr = 2\pi\left[\frac{r^4}{4} + \frac{u\,r^3}{3}\right]_0^r$$

$$= 2\pi\left[\frac{r^4}{4} + \frac{u\,r^3}{3}\right] \text{A}$$

For r = a,
$$I = 2\pi\left[\frac{a^4}{4} + \frac{u\,a^3}{3}\right] \text{A}$$

Consider a closed path with $r \geq a$ which is Amperian path.

$$\therefore \quad \bar{H} = H_\phi\,\bar{a}_\phi \quad \text{while} \quad d\bar{L} = r\,d\phi\,\bar{a}_\phi$$

According to Ampere's circuital law,

$$\oint \bar{H} \cdot d\bar{L} = I, \quad \text{As path has } r \geq a \text{, it encloses total I.}$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\,r\,d\phi = 2\pi\left[\frac{a^4}{4} + \frac{u\,a^3}{3}\right]$$

$$\therefore \quad 2\pi r H_\phi = \frac{2\pi}{12}[4\,a^4 + 3\,ua^3]$$

$$\therefore \quad H_\phi = \frac{1}{12\,r}[4\,a^4 + 3\,u\,a^3]$$

$$\therefore \quad \bar{H} = \frac{0.0833}{r}[4\,a^4 + 3\,u\,a^3]\,\bar{a}_\phi \quad \text{A/m}$$

Assuming permeability of medium as μ

$$\therefore \quad \bar{B} = \mu\,\bar{H} = \frac{0.0833\mu}{r}[3\,a^4 + 4\,u\,a^3]\,\bar{a}_\phi \quad \text{Wb/m}^3$$

7.10 Curl

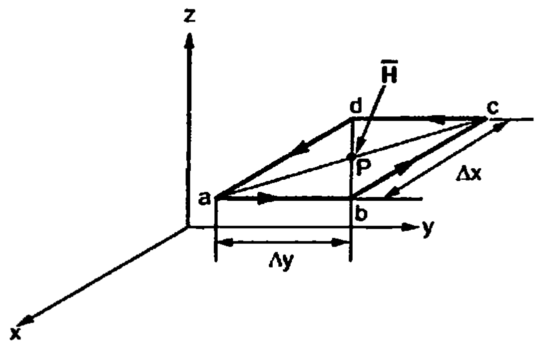


Fig. 7.37 Differential surface element

In electrostatics, the Gauss's law is applied to the differential volume element to develope the concept of divergence. Similarly in magnetostatics, the Ampere's circuit law is to be applied to the differential surface element to develope the concept of a curl.

Consider the differential surface element having sides Δx and Δy plane, as shown in the Fig. 7.37. The unknown current has produced $\bar{H}$ at the centre of this incremental closed path.

The total magnetic field intensity at the point P which is centre of the small rectangle is,

$$\bar{H} = H_{x0}\,\bar{a}_x + H_{y0}\,\bar{a}_y + H_{z0}\,\bar{a}_z \quad \text{... (1)}$$

While the total current density is given by,

$$\bar{J} = J_x\,\bar{a}_x + J_y\,\bar{a}_y + J_z\,\bar{a}_z \quad \text{... (2)}$$

To apply Ampere's circuit law to this closed path, let us evaluate the closed line integral of $\bar{H}$ about this path in the direction abcda. According to right hand thumb rule the current is in $\bar{a}_z$ direction.

Along path a-b, $\bar{H} = H_y\,\bar{a}_y$ and $d\bar{L} = \Delta y\ \bar{a}_y$

$$\therefore \qquad \bar{H} \cdot d\bar{L} = H_y\,\bar{a}_y \cdot \Delta y\,\bar{a}_y = H_y\,\Delta y \quad \text{... (3)}$$

The intensity H_y along a-b can be expressed interms of H_{y0} existing at P and the rate of change of H_y in the x direction with x. The distance in x direction of a-b from point P is $(\Delta x/2)$. Hence $\bar{H} \cdot d\bar{L}$ along a-b can be expressed as,

$$\left(\bar{H} \cdot d\bar{L}\right)_{a-b} = \left[H_{y0} + \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}\right]\Delta y \quad \text{... (4)}$$

For path b-c, $\bar{H}$ is in $-\bar{a}_x$ direction hence $-H_x\,\bar{a}_x$ and $d\bar{L} = \Delta x\,\bar{a}_x$.

$$\therefore \qquad \bar{H} \cdot d\bar{L} = -H_x\,\Delta x \quad \text{... (5)}$$

Now H_x can be expressed interms of H_{x0} at point P and rate of change of H_x in y direction with y.

$$\therefore \qquad H_x = H_{x0} + \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}$$

The distance of bc from P is $\Delta y/2$.

$$\therefore \quad (\bar{H} \cdot d\bar{L})_{b-c} = -\left[H_{x0} + \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}\right]\Delta x \quad \text{... (6)}$$

For path c-d, $\bar{H}$ is in $-\bar{a}_y$ direction hence $-H_y\,\bar{a}_y$ and $d\bar{L} = \Delta y\,\bar{a}_y$.

$$\therefore \quad \bar{H} \cdot d\bar{L} = -H_y\,\Delta y \quad \text{... (7)}$$

But H_y can be expressed interms of H_{y0} and rate of change of H_y in negative x direction. The distance of cd from point P is $(\Delta x/2)$ in **negative x direction.**

$$\therefore \quad H_y = H_{y0} - \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}$$

$$\therefore \quad (\bar{H} \cdot d\bar{L})_{c-d} = -\left[H_{y0} - \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}\right]\Delta y \quad \text{... (8)}$$

For the path d-a, $\bar{H}$ is in + $\bar{a}_x$ direction hence $H_x\,\bar{a}_x$ and $d\bar{L} = \Delta x\,\bar{a}_x$.

$$\therefore \quad \bar{H} \cdot d\bar{L} = H_x\,\Delta x \quad \text{... (9)}$$

But H_x can be expressed interms of H_{x0} and rate of change of H_x in negative y direction. The distance of da from point P is $(\Delta y/2)$ in **negative y direction.**

$$\therefore \quad H_x = H_{x0} - \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}$$

$$\therefore \quad (\bar{H} \cdot d\bar{L})_{d-a} = \left[H_{x0} - \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}\right]\Delta x \quad \text{... (10)}$$

Total $\bar{H} \cdot d\bar{L}$ can be obtained by adding the equation (4), (6), (8) and (10).

$$\therefore \quad \bar{H} \cdot d\bar{L} = H_{y0}\,\Delta y + \frac{\Delta x\,\Delta y}{2}\frac{\partial H_y}{\partial x} - H_{x0}\,\Delta x - \frac{\Delta x\,\Delta y}{2}\frac{\partial H_x}{\partial y}$$

$$-H_{y0}\,\Delta y + \frac{\Delta x\,\Delta y}{2}\frac{\partial H_y}{\partial x} + H_{x0}\,\Delta x - \frac{\Delta x\,\Delta y}{2}\frac{\partial H_x}{\partial y}$$

$$\therefore \quad \oint \bar{H} \cdot d\bar{L} = \Delta x\,\Delta y\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] \quad \text{... (11)}$$

According to Ampere's circuital law, this integral must be current enclosed by the differential element.

Current enclosed = Current density normal to closed path × Area of the closed path

$$\therefore \quad I_{enc} = J_z\,\Delta x\,\Delta y \quad \text{... (12)}$$

where J_z = Current density in $\bar{a}_z$ direction as the current enclosed is in $\bar{a}_z$ direction.

From equations (11) and (12) we can write,

$$\frac{\oint \bar{H} \cdot d\bar{L}}{\Delta x \ \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \qquad \text{... (13)}$$

This gives accurate result as the closed path shrinks to a point i.e. $\Delta x \ \Delta y$ area tends to zero.

$$\therefore \quad \lim_{\Delta x \Delta y \to 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \qquad \text{... (14)}$$

This equation gives relation between closed line integral of $\bar{H}$ per unit area and the current per unit area i.e. current density. To have equality sign between the two, the surface area of closed path must shrink to zero.

Considering incremental closed path in yz plane we get the current density normal to it i.e. in x direction considering incremental closed path in zx plane we get the current density normal to it i.e. in y direction. So we can write,

$$\lim_{\Delta y \Delta z \to 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta y \ \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \qquad \text{... (15)}$$

and $$\lim_{\Delta z \Delta x \to 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta z \ \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \qquad \text{... (16)}$$

In general we can write,

$$\lim_{\Delta S_N \to 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta S_N} = J_N \qquad \text{... (17)}$$

where J_N = Current density normal to the surface ΔS.

The term on left hand side of the equation is called curl $\bar{H}$. The ΔS_N is area enclosed by the closed line integral.

The total $\bar{J}$ now can be obtained by adding (14), (15) and (16).

$$\therefore \quad \bar{J} = J_x \bar{a}_x + J_y \bar{a}_y + J_z \bar{a}_z$$

$$\therefore \quad \bar{J} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]\bar{a}_z$$

$$\therefore \quad \boxed{\bar{J} = \text{curl } \bar{H} = \nabla \times \bar{H}} \qquad \text{... (18)}$$

The curl $\bar{H}$ is indicated by $\nabla \times \bar{H}$ which is cross product of operator 'del' and $\bar{H}$.

The equation (18) is called the **point form of Ampere's circuital law.**

$$\therefore \quad \boxed{\text{curl } \bar{H} = \nabla \times \bar{H} = \bar{J}}$$

This is one of the Maxwell's equations.

Key Point: *The curl is not refering to any co-ordinate system though it is developed using cartesian co-ordinate system.*

7.10.1 Curl in Various Co-ordinate Systems

As the curl of $\bar{H}$ i.e. $\nabla \times \bar{H}$ is a cross product it can be expressed in determinant form in various co-ordinate systems.

1. Cartesian co-ordinate system :

$$\nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \begin{matrix} \\ \leftarrow \nabla \\ \leftarrow \bar{H} \end{matrix}$$

$$= \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]\bar{a}_z$$

2. Cylindrical co-ordinate system :

$$\nabla \times \bar{H} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

$$= \left[\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial (rH_\phi)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial \phi}\right]\bar{a}_z$$

Key Point: *Note that in $\frac{\partial (rH_\phi)}{\partial z}$, r is constant as differentiation is with respect to z hence it becomes r $\frac{\partial H_\phi}{\partial z}$. But in $\frac{\partial (rH_\phi)}{\partial r}$, the r can not be taken out of differentiation, as differentiation is with respect to r.*

3. Spherical co-ordinate system :

$$\nabla \times \bar{H} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta\,H_\phi \end{vmatrix}$$

$$\nabla \times \bar{H} = \frac{1}{r\sin\theta}\left[\frac{\partial H_\phi \sin\theta}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial (rH_\phi)}{\partial r}\right]\bar{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial (rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right]\bar{a}_\phi$$

7.10.2 Properties of Curl

The various properties of curl are,

1. The curl of a vector is a vector quantity.

2. $\nabla\times(\overline{A}+\overline{B}) = \nabla\times\overline{A} + \nabla\times\overline{B}$

3. $\nabla\times\nabla\times\overline{A} = \nabla(\nabla\bullet\overline{A}) - \nabla^2\overline{A}$

4. The divergence of a curl is zero.

$\nabla\bullet(\nabla\times\overline{A}) = 0$

5. The curl of a scalar makes no sense

$\nabla\times\alpha$ = No sense if α is scalar.

6. The curl of gradient of a vector is zero.

$\nabla\times\nabla V = 0$

7. $\nabla\times\overline{A}\times\overline{B} = \overline{A}(\nabla\bullet\overline{B}) - \overline{B}(\nabla\bullet\overline{A}) + (\overline{B}\bullet\nabla)\overline{A} - (\overline{A}\bullet\nabla)\overline{B}$

7.10.3 Physical Significance of a Curl

The curl is a closed line integral per unit area as the area shrinks to a point. It gives the circulation per unit area i.e. circulation density of a vector about a point at which the area is going to shrink. Thus curl of a vector at a point quantifies the circulation of a vector around that point. In general if there is no rotation, there is no curl while large angular velocities means greater values of curl. The curl also gives the direction, which is along the axis through a point at which curl is defined.

The magnetic field lines produced by the current carrying conductor are rotating in the form of concentric circles around the conductor. Thus there exists a curl of magnetic field intensity which we have defined as $\nabla\times\overline{H}$. The direction of curl is along the axis about which rotation of a vector field exists and the proper direction is to be obtained by right handed screw rule. If the direction of rotation of vector field about a point reverses, the sign of the curl also reverses.

The water velocity in a river which increases linearily towards the surface, the magnetic field lines due to current carrying conductor, the body rotating about a fixed axis are few examples of a curl.

Key Point: ***Thus if curl of a vector field exists then the field is called rotational. For irrotational vector field, the curl vanishes i.e. curl is zero.***

Another physical interpretation of a curl is about a rigid body rotating about a fixed axis with uniform angular velocity. Thus if v is its linear velocity then its angular velocity (ω) is half the curl of its linear velocity. The curl v represents the net rotation of a body about the axis.

Example 7.9 : *A $\overline{H}$ due to a current source is given by,*
$\overline{H} = [y\cos(\alpha x)]\,\overline{a}_x + (y+e^x)\,\overline{a}_z$. *Describe the current density over the yz plane.*

Solution : From the point form of Ampere's circuital law,

$$\nabla\times\overline{H} = \overline{J}$$

In the cartesian system,

$$\nabla\times\bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos(\alpha x) & 0 & y+e^x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(y+e^x)\right]\bar{a}_x + \left[\frac{\partial y\cos(\alpha x)}{\partial z} - \frac{\partial(y+e^x)}{\partial x}\right]\bar{a}_y + \left[-\frac{\partial}{\partial y}y\cos(\alpha x)\right]\bar{a}_z$$

$$= (1)\bar{a}_x + (0-e^x)\bar{a}_y + (-\cos\alpha x)\,\bar{a}_z$$

On yz plane, $x = 0$

$$\therefore \quad \bar{J} \text{ on yz plane} = \bar{a}_x - e^0\bar{a}_y - \cos 0\,\bar{a}_z$$

$$= \bar{a}_x - \bar{a}_y - \bar{a}_z \ \text{A/m}^2$$

Example 7.10 : *Given the general vector, $\bar{A} = (\sin 2\phi)\bar{a}_\phi$ in cylindrical co-ordinates. Find curl of $\bar{A}$ at $(2, \pi/4, 0)$.*

Solution : In cylindrical co-ordinates $\nabla\times\bar{A}$ is given by,

$$\nabla\times\bar{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right]\bar{a}_z$$

Now $A_r = 0$, $A_\phi = \sin 2\phi$ and $A_z = 0$

$$\therefore \quad \nabla\times\bar{A} = \left[0 - \frac{\partial\sin 2\phi}{\partial z}\right]\bar{a}_r + [0-0]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial(r\sin 2\phi)}{\partial r} - 0\right]\bar{a}_z$$

$$= [0-0]\bar{a}_r + 0\,\bar{a}_\phi + \frac{\sin 2\phi}{r}\bar{a}_z = \frac{\sin 2\phi}{r}\bar{a}_z$$

At $\left(2, \frac{\pi}{4}, 0\right)$, $r = 2$, $\phi = \frac{\pi}{4}$, $z = 0$

$$\therefore \quad \nabla\times\bar{A} = \frac{\sin\left(2\times\frac{\pi}{4}\right)}{2}\bar{a}_z$$

$$= \frac{\sin\left(\frac{\pi}{2}\right)}{2}\bar{a}_z$$

$$= 0.5\ \bar{a}_z$$

Example 7.11 : *Given that the general vector $\bar{A}$ is, $\bar{H} = 2.5\,\bar{a}_\theta + 5\,a_\phi$ in spherical co-ordinates. Find the curl of $\bar{H}$ at $(2, \pi/6, 0)$.*

Solution : In the spherical co-ordinates, curl $\bar{H}$ is given by,

$$\nabla\times\bar{H} = \frac{1}{r\sin\theta}\left[\frac{\partial \sin\theta H_\phi}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(r H_\phi)}{\partial r}\right]\bar{a}_\theta + \frac{1}{r}\left[\frac{\partial(r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\bar{a}_\phi$$

Now $H_r = 0$, $H_\theta = 2.5$, $H_\phi = 5$

$$\therefore\ \nabla\times\bar{H} = \frac{1}{r\sin\theta}\left[\frac{\partial 5\sin\theta}{\partial\theta} - \frac{\partial 2.5}{\partial\phi}\right]\bar{a}_r + \frac{1}{r}\left[0 - \frac{\partial(5r)}{\partial r}\right]\bar{a}_\theta + \frac{1}{r}\left[\frac{\partial(r\,2.5)}{\partial r} - 0\right]\bar{a}_\phi$$

$$= \frac{1}{r\sin\theta}[5\cos\theta - 0]\,\bar{a}_r + \frac{1}{r}[-5]\,\bar{a}_\theta + \frac{1}{r}[2.5]\,\bar{a}_\phi$$

$$= \frac{5}{r}\cot\theta\,\bar{a}_r - \frac{5}{r}\,\bar{a}_\theta + \frac{2.5}{r}\,\bar{a}_\phi$$

At $\left(2, \frac{\pi}{6}, 0\right)$, $r = 2$, $\theta = \frac{\pi}{6}$, $\phi = 0°$

$$\therefore\quad \nabla\times\bar{H} = \frac{5}{2}\cot\frac{\pi}{6}\,\bar{a}_r - \frac{5}{2}\,\bar{a}_\theta + \frac{2.5}{2}\,\bar{a}_\phi$$

$$= \mathbf{4.33\,\bar{a}_r - 2.5\,\bar{a}_\theta + 1.25\,\bar{a}_\phi}$$

7.11 Stoke's Theorem

Analogous to the divergence theorem in electrostatics, there exists Stoke's theorem in magnetostatics. The Stoke's theorem relates the line integral to a surface integral. Basically it is a mathematical theorem which is to be applied in magnetostatics.

The Stoke's theorem states that,

"The line integral of a vector $\bar{A}$ around a closed path L is equal to the integral of curl of $\bar{A}$ over the open surface S enclosed by the closed path L".

The theorem is applicable only when $\bar{A}$ and $\nabla\times\bar{A}$ are continuous on the surface S.

7.11.1 Proof of Stoke's Theorem

Consider a surface S which is splitted into number of incremental surfaces. Each incremental surface is having area ΔS as shown in the Fig. 7.38 (a).

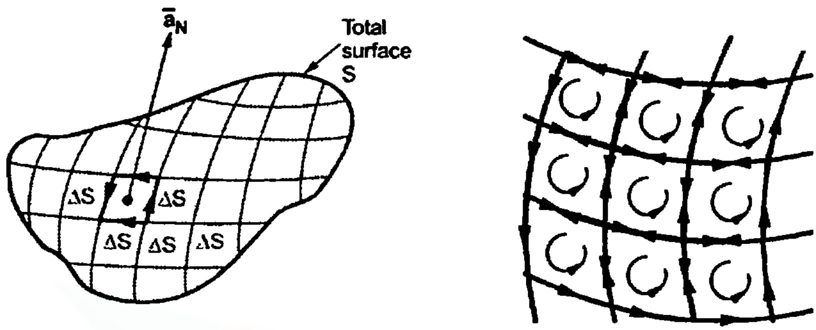


Fig. 7.38 Stoke's theorem

Applying definition of the curl to any of these incremental surfaces we can write,

$$(\nabla \times \bar{H})_N = \frac{\oint \bar{H} \cdot d\bar{L}_{\Delta S}}{\Delta S} \qquad \text{... (1)}$$

where N = Normal to ΔS according to right hand rule

$dL_{\Delta S}$ = Perimeter of the incremental surface ΔS

Now the curl of $\bar{H}$ in the normal direction is the dot product of curl of $\bar{H}$ with $\bar{a}_N$ where $\bar{a}_N$ is unit vector, normal to the surface ΔS, according to right hand rule.

$\therefore \quad (\nabla \times H)_N = (\nabla \times \bar{H}) \cdot \bar{a}_N$ ... Using in (1) we get,

$\therefore \quad \oint H \cdot dL_{\Delta S} = (\nabla \times \bar{H}) \cdot \bar{a}_N \, \Delta S$

$\therefore \quad \oint H \cdot dL_{\Delta S} = (\nabla \times \bar{H}) \cdot \Delta \bar{S}$ ... (2)

To obtain total curl for every incremental surface, add the closed line integrals for each ΔS. From the Fig. 7.38 (b), it can be seen that at a common boundary between the two incremental surfaces, the line integral is getting cancelled as the boundary is getting traced in two opposite directions.

This happens for all the interior boundaries. Only at the outside boundary cancellation does not exist. Hence summation of all closed line integrals for each and every ΔS ends up in a single closed line integral to be obtained for the outer boundary of the total surface S. Hence the equation (3) becomes,

$$\therefore \quad \boxed{\oint \bar{H} \cdot d\bar{L} = \int_S (\nabla \times \bar{H}) \cdot d\bar{S}} \qquad \text{... (3)}$$

where dL = Perimeter of the total surface S

Thus line integral can be expressed as a surface integral which proves the Stoke's theorem.

Key Point: *The Stoke's theorem is applicable for the open surface enclosed by the given closed path. Any volume is a closed surface and hence* ***application of Stoke's theorem to a closed surface which encloses certain volume, produces zero answer.***

Example 7.12 : *Prove that divergence of a curl of a vector is zero, using Stoke's theorem.*

Solution : Consider a vector $\bar{A}$.

The curl of $\bar{A}$ is $\nabla \times \bar{A}$ and its divergence is $\nabla \cdot (\nabla \times \bar{A})$. Now $\nabla \times \bar{A}$ is a vector while divergence of a vector is a scalar say α.

$$\therefore \quad \nabla \cdot (\nabla \times \bar{A}) = \alpha \qquad \text{... (1)}$$

Let us evaluate integral of both sides over a volume

$$\therefore \quad \int_{vol} \nabla \cdot (\nabla \times \bar{A})\, dv = \int_{vol} \alpha \, dv \qquad \text{... (2)}$$

Applying divergence theorem, the left hand side can be converted to a surface integral.

$$\therefore \quad \int_{vol} \nabla \cdot (\nabla \times \bar{A})\, dv = \int_{S} (\nabla \times \bar{A}) \cdot d\bar{S} \qquad \text{... (3)}$$

where the S is closed surface enclosing the given volume.

$$\therefore \quad \int_{S} (\nabla \times \bar{A}) \cdot d\bar{S} = \int_{vol} \alpha \, dv \qquad \text{... (4)}$$

Now if Stoke's theorem is applied, it can be seen that surface S on left hand side of equation (4) is enclosing given volume and **is not the open surface. The Stoke's theorem applied to closed surface produces zero answer.**

$$\therefore \quad \int_{vol} \alpha \, dv = 0 \qquad \text{... (5)}$$

This is true for differential volume also.

$$\therefore \quad \alpha \, dv = 0 \qquad \text{... (6)}$$

But $\quad dv \neq 0 \quad$ as it is a differential volume.

$$\therefore \quad \alpha = 0 \qquad \text{... (7)}$$

From equation (1),

$$\nabla \cdot (\nabla \times \bar{A}) = 0 \qquad \text{... (8)}$$

This proves that the divergence of curl of a vector is zero.

Example 7.13 : *Evaluate both sides of the Stoke's theorem for the field* $\bar{H} = 6xy\bar{a}_x - 3y^2\bar{a}_y$ *A/m and the rectangular path around the region,* $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. *Let the positive direction of* $d\bar{S}$ *be* $\bar{a}_z$.

Solution : According to Stoke's theorem,

$$\oint_{L} \bar{H} \cdot d\bar{L} = \int_{S} (\nabla \times \bar{H}) \cdot d\bar{S}$$

Let us evaluate left hand side. The integral to be evaluated on a perimeter of a closed path shown in the Fig. 7.39. The direction is a-b-c-d-a such that normal to it is positive $\bar{a}_z$ according to right hand rule.

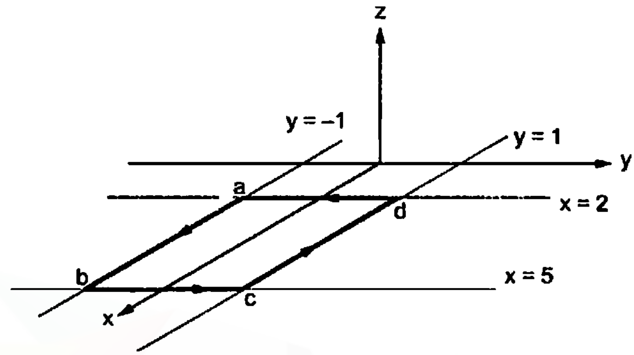


Fig. 7.39

$$\oint \bar{H} \cdot d\bar{L} = \int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \bar{H} \cdot d\bar{L}$$

$$\int_{ab} \bar{H} \cdot d\bar{L} = \int_{x=2}^{5} \left(6xy\,\bar{a}_x - 3y^2\,\bar{a}_y\right) \cdot dx\,\bar{a}_x$$

$$= \int_{x=2}^{5} 6\,xy\ dx = 6y \cdot \left[\frac{x^2}{2}\right]_2^5$$

$$= \frac{6y}{2}[25-4] = 63\ y$$

Now $y = -1$ for path ab, $\int_{ab} \bar{H} \cdot d\bar{L} = 63(-1) = -63$

Similarly $\int_{bc} \bar{H} \cdot d\bar{L} = \int_{y=-1}^{1} -3y^2\ dy = \frac{-3y^3}{3} = -\left[y^3\right]_{-1}^{1} = -[1-(-1)] = -2$

$$\int_{cd} \bar{H} \cdot d\bar{L} = \int_{x=5}^{2} 6\,xy\ dx = 6\left[\frac{x^2}{2}\right]_5^2 (y) = \frac{6y}{2}[4-25] = -63y$$

But $y = 1$ for path cd hence $\int_{cd} \bar{H} \cdot d\bar{L} = -63$

$$\int_{da} \bar{H} \cdot d\bar{L} = \int_{y-1}^{-1} -3y^2\ dy = -[y^3]_1^{-1} = -[(-1)^3 - (1)^3] = -[-1-1] = +2$$

$$\therefore \quad \oint \bar{H} \cdot d\bar{L} = -63-2-63+2 = -126\ \text{A}$$

Now evaluate right hand side.

$$\nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy & -3y^2 & 0 \end{vmatrix}$$

$$= \bar{a}_x\,[0-0]+\bar{a}_y\,[0-0]+\bar{a}_z\,[0-6x]\,\bar{a}_z = -6x\bar{a}_z$$

$$\therefore \quad \int_S (\nabla \times \bar{H})\cdot d\bar{S} = \int_S (-6x\,\bar{a}_z)\cdot(dx\,dy\,\bar{a}_z)$$

$$d\bar{S} = dx\,dy\,\bar{a}_z \quad \text{normal to direction } \bar{a}_z$$

$$\therefore \quad \int_S (\nabla \times \bar{H})\cdot d\bar{S} = \int_{y=-1}^{1}\int_{x=2}^{5} -6x\,dx\,dy = -6\left[\frac{x^2}{2}\right]_2^5 [y]_{-1}^{1}$$

$$= -\frac{6}{2}[25-4][1-(-1)] = -3\times 21\times 2 = -126 \text{ A}$$

Thus both the sides are same, hence Stoke's theorem is verified.

7.12 Magnetic Flux and Flux Density

The magnetic flux density $\bar{B}$ is analogous to the electric flux density $\bar{D}$. The relation between $\bar{B}$ and $\bar{H}$ is already mentioned, which is through the property of medium called permeability μ. The relation is given by,

$$\boxed{\bar{B} = \mu\,\bar{H}} \quad \ldots (1)$$

For the free space, $\mu = \mu_0 = 4\pi\times 10^{-7}$ H/m hence,

$$\boxed{\bar{B} = \mu_0\,\bar{H} \quad \text{for free space}} \quad \ldots (2)$$

The magnetic flux density has units Wb/m^2 and hence it can be defined as the flux in webers passing through unit area in a plane at right angles to the direction flux.

If the flux passing through the unit area is not exactly at right angles to the plane consisting the area but making some angle with the plane then the flux ϕ crossing the area is given by,

$$\boxed{\phi = \int_S \bar{B}\cdot d\bar{S} \quad \text{webers (Wb)}} \quad \ldots (3)$$

where ϕ = Magnetic flux in webers

$\bar{B}$ = Magnetic flux density in Wb/m^2 or Tesla (T)

$d\bar{S}$ = Open surface through which flux is passing.

Now consider a closed surface which is defining a certain volume. The magnetic flux lines are always exist in the form of closed loop. Thus for a closed surface the number of magnetic flux lines entering must be equal to the number of magnetic flux lines leaving.

The single magnetic pole can not exists like a single isolated electric charge. No magnetic flux can reside in a closed surface. Hence the integral $\bar{B} \cdot d\bar{S}$ evaluated over a closed surface is always zero.

$$\therefore \quad \boxed{\oint_S \bar{B} \cdot d\bar{S} = 0} \quad \ldots (4)$$

This is called law of conservation of magnetic flux or Gauss's law in integral form for magnetic fields.

Applying divergence theorem to equation (4),

$$\oint_S \bar{B} \cdot d\bar{S} = \int_{vol} \nabla \cdot \bar{B} \, dv = 0 \quad \ldots (5)$$

where dv = Volume enclosed by the closed surface.

But as dv is not zero, we can write,

$$\boxed{\nabla \cdot \bar{B} = 0} \quad \ldots (6)$$

The divergence of magnetic flux density is always zero. This is called Gauss's law in differential form for magnetic fields. This is another Maxwell's equation.

7.12.1 Maxwell's Equations for Static Electromagnetic Fields

Let us summarize the Maxwell's equations for static electric and magnetic fields.

Maxwell's equations in differential or point form		
1.	$\nabla \cdot \bar{D} = \rho_v$	Gauss's law
2.	$\nabla \times \bar{E} = 0$	Conservation of electric field
3.	$\nabla \times \bar{H} = \bar{J} = \bar{J}$	Ampere's circuital law
4.	$\nabla \cdot \bar{B} = 0$	Single magnetic pole can not exist i.e. conservation of magnetic flux

Table 7.1

The Maxwell's equations in integral form can be summarized as,

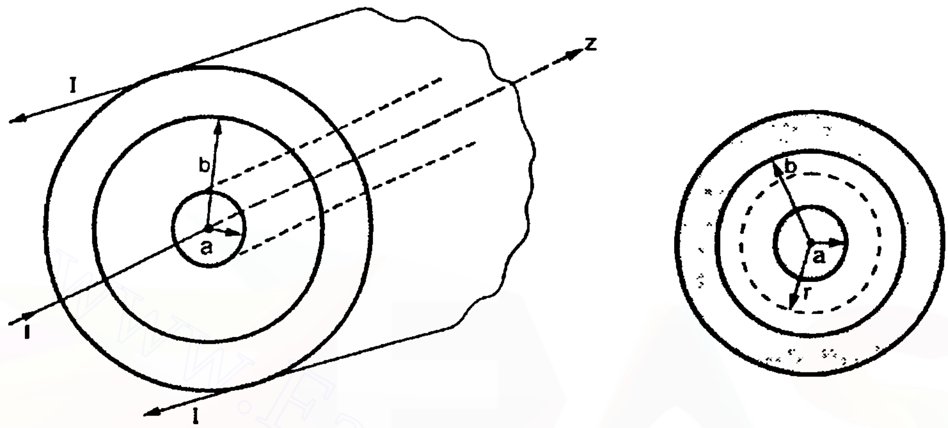
Maxwell's equations in integral form
1. $\oint_S \bar{D} \cdot d\bar{S} = \int_{vol} \rho_v \, dv = Q$
2. $\oint \bar{E} \cdot d\bar{L} = 0$
3. $\oint \bar{H} \cdot d\bar{L} = \int_S \bar{J} \cdot d\bar{S} = I$
4. $\oint \bar{B} \cdot d\bar{S} = 0$

Table 7.2

7.12.2 Application of Flux Density and Flux to Co-axial Cable

Let us obtain the flux between the conductors of a co-axial cable using the concepts of flux density and the flux.

The co-axial cable is shown in the Fig. 7.40, such that its axis is along the z-axis.



(a) Co-axial cable (b) Cross-sectional view

Fig. 7.40

The radius of the inner conductor is 'a' while the inner radius of the outer conductor is 'b'. It carries a direct current I which is uniformly distributed in the inner conductor. The outer conductor carries same current I in opposite direction to that carried by the inner conductor.

As derived in the section 7.9.2, $\bar{H}$ in the region a < r < b is given by,

$$\bar{H} = \frac{I}{2\pi r}\bar{a}_\phi \text{ A/m} \qquad \ldots a < r < b$$

We are interested in the flux in the region a < r < b. The cable is filled with the air as dielectric with $\mu = \mu_0$.

$$\therefore \qquad \bar{B} = \mu_0 \bar{H} = \frac{\mu_0 I}{2\pi r}\bar{a}_\phi \text{ Wb/m}^2 \qquad \ldots a < r < b$$

Let d be the length of the conductors. The magnetic flux contained between the conductors in a length d is the magnetic flux crossing the radial plane from r = a to r = b and for z = 0 to z = d.

The magnetic is given by,

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$

The $d\bar{S}$ normal to the $\bar{a}_\phi$ direction is dr dz.

$$\therefore \quad d\bar{S} = dr\, dz\, \bar{a}_\phi$$

$$\therefore \quad \phi = \int_S \bar{B} \cdot d\bar{S} = \int_S \frac{\mu_0 I}{2\pi r} \bar{a}_\phi \cdot dr\, dz\, \bar{a}_\phi$$

$$= \int_{z=0}^{d} \int_{r=a}^{b} \frac{\mu_0 I}{2\pi r} dr\, dz = \frac{\mu_0 I}{2\pi} [z]_0^d \, [ln\, r]_a^b \qquad \ldots \int \frac{dr}{r} = ln[r]$$

$$= \frac{\mu_0 I d}{2\pi} [ln\, b - ln\, a]$$

$$\therefore \quad \boxed{\phi = \frac{\mu_0 I d}{2\pi} ln\left[\frac{b}{a}\right] \text{ Wb}}$$

Example 7.14 : *A radial field,* $\bar{H} = \frac{2.39 \times 10^6}{r} \cos\phi\, \bar{a}_r$ *A/m exists in free space. Find the magnetic flux crossing the surface defined by* $0 \le \phi \le \pi/4$ *and* $0 \le z \le 1$ *m.*

Solution : The portion of the cylinder is shown in the Fig. 7.41. The flux crossing the given surface is given by,

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$

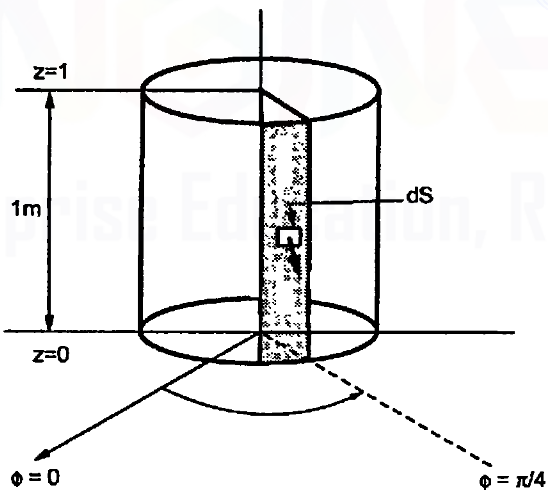


Fig. 7.41

$d\bar{S}$ normal to $\bar{a}_r$ direction is,

$$d\bar{S} = r\, d\phi\, dz\, \bar{a}_r$$

$$\therefore \phi = \int_S \mu_0 \bar{H} \cdot d\bar{S} \qquad \ldots \bar{B} = \mu_0 \bar{H}$$

$$= \mu_0 \int_S \frac{2.39 \times 10^6}{r} \cos\phi\, \bar{a}_r \cdot r\, d\phi\, dz\, \bar{a}_r$$

$$= \mu_0 \int_{z=0}^{1} \int_{\phi=0}^{\pi/4} 2.39 \times 10^6 \cos\phi\, d\phi\, dz$$

$$= 2.39 \times 10^6 \mu_0 [\sin\phi]_0^{\pi/4} \, [z]_0^1$$

$$\therefore \quad \phi = 2.39 \times 10^6 \times 4\pi \times 10^{-7} \times \left[\sin\frac{\pi}{4} - \sin 0\right] [1 - 0]$$

$$= \mathbf{2.1236 \text{ Wb}}$$

Example 7.15 : *In cylindrical co-ordinates $\bar{B} = (2.0/r)\bar{a}_\phi$ Tesla. Determine the magnetic flux ϕ crossing the plane surface defined by $0.5 \le r \le 2.5$ m and $0 \le z \le 2$ m.*

Solution : The surface is shown in the Fig. 7.42.

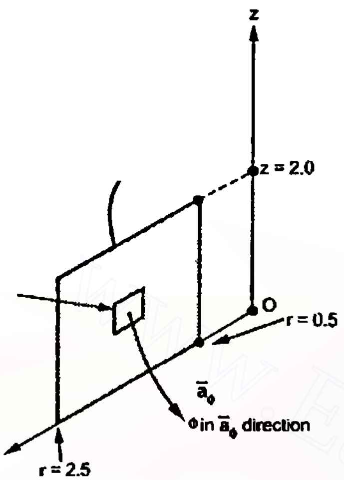


Fig. 7.42

The flux crossing the surface is given by,

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$

The $d\bar{S}$ normal to $\bar{a}_\phi$ direction is dr dz.

$$\therefore \quad d\bar{S} = dr\, dz\, \bar{a}_\phi$$

$$\therefore \quad \phi = \int_S \frac{2.0}{r} \bar{a}_\phi \cdot dr\, dz\, \bar{a}_\phi$$

$$= \int_{z=0}^{2} \int_{r=0.5}^{2.5} \frac{2.0}{r}\, dr\, dz$$

$$= 2.0\, [\ln r]_{0.5}^{2.5}\, [z]_0^2$$

$$= 2.0\, [\ln 2.5 - \ln 0.5]\, [2 - 0]$$

$$= 6.4377 \text{ Wb}$$

Example 7.16 : *Find the flux passing the portion of the plane $\phi = \pi/4$ defined by $0.01 < r < 0.05$m and $0 < z < 2$m. A current filament of 2.5 A is along the z-axis in the $\bar{a}_z$ direction, in free space.*

Solution : The arrangement is shown in the Fig. 7.43.

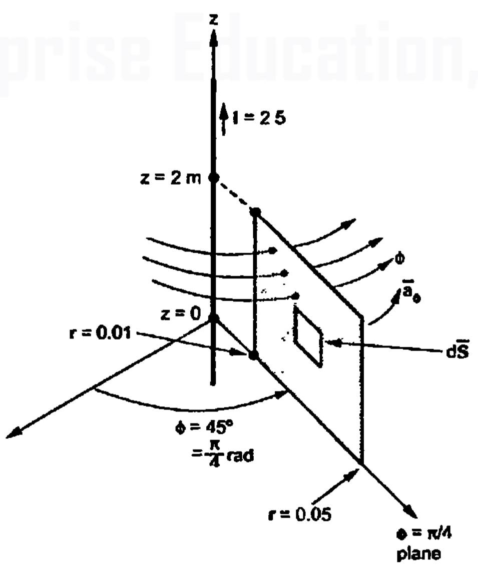


Fig. 7.43

Due to current carrying conductor in free space along z-axis, $\bar{H}$ is given by,

$$\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi$$

$$= \frac{2.5}{2\pi r} \bar{a}_\phi$$

$$= \frac{0.3978}{r} \bar{a}_\phi \quad \text{A/m}$$

$$\bar{B} = \mu_0 \bar{H} = \frac{4\pi \times 10^{-7} \times 0.3978}{r} \bar{a}_\phi$$

$$= \frac{5 \times 10^{-7}}{r} \bar{a}_\phi \quad \text{Wb/m}^2$$

The flux crossing the surface is given by,

$$\phi = \int_S \bar{B} \bullet d\bar{S}$$

Now $\qquad d\bar{S} = dr\, dz\, \bar{a}_\phi \quad$ normal to $\bar{a}_\phi$ direction

$$\therefore \quad \phi = \int_{z=0}^{2} \int_{r=0.01}^{0.05} \frac{5\times10^{-7}}{r} \bar{a}_\phi \bullet dr\, dz\, \bar{a}_\phi$$

$$= \int_{z=0}^{2} \int_{r=0.01}^{0.05} \frac{5\times10^{-7}}{r}\, dr\, dz = 5\times10^{-7} [ln\, r]_{0.01}^{0.05}\, [z]_0^2$$

$$= 5\times10^{-7}\, ln \left[\frac{0.05}{0.01}\right] [2] = \mathbf{1.6094\ \mu Wb}$$

7.13 Magnetic Scalar and Vector Potentials

In electrostatics, it is seen that there exists a scalar electric potential V which is related to the electric field intensity $\bar{E}$ as $\bar{E} = -\nabla V$.

Is there any scalar potential in magnetostatics related to magnetic field intensity $\bar{H}$?

In case of magnetic fields there are two types of potentials which can be defined :

1. The scalar magnetic potential denoted as V_m.
2. The vector magnetic potential denoted as $\bar{A}$.

To define scalar and vector magnetic potentials, let us use two vector identities which are listed as the properties of curl, earlier.

$$\nabla \times \nabla V = 0, \qquad V = \text{Scalar} \qquad \text{... (1)}$$

$$\nabla \bullet (\nabla \times \bar{A}) = 0, \qquad \bar{A} = \text{Vector} \qquad \text{... (2)}$$

Every Scalar V and Vector $\bar{A}$ must satisfy these identities.

7.13.1 Scalar Magnetic Potential

If V_m is the scalar magnetic potential then it must satisfy the equation (1),

$$\therefore \quad \nabla \times \nabla V_m = 0 \qquad \text{... (3)}$$

But the scalar magnetic potential is related to the magnetic field intensity $\bar{H}$ as,

$$\bar{H} = -\nabla V_m \qquad \text{... (4)}$$

Using in equation (3),

$$\therefore \quad \nabla \times (-\bar{H}) = 0 \qquad \text{i.e.} \quad \nabla \times \bar{H} = 0 \qquad \text{... (5)}$$

But $\qquad \nabla \times \bar{H} = \bar{J} \qquad$ i.e. $\qquad \bar{J} = 0 \qquad$ ... (6)

Thus scalar magnetic potential V_m can be defined for source free region where $\bar{J}$ i.e. current density is zero.

$$\therefore \quad \bar{H} = -\nabla V_m \quad \text{only for } \bar{J} = 0 \quad \text{... (7)}$$

Similar to the relation between $\bar{E}$ and electric scalar potential, magnetic scalar potential can be expressed interms of $\bar{H}$ as,

$$\boxed{V_{m\,a,b} = -\int_b^a \bar{H} \cdot d\bar{L} \quad \text{... specified path}}$$

7.13.2 Laplace's Equation for Scalar Magnetic Potential

It is known that as monopole of magnetic field is non existing,

$$\oint \bar{B} \cdot d\bar{S} = 0 \quad \text{... (8)}$$

Using Divergence theorem,

$$\oint \bar{B} \cdot d\bar{S} = \int_{vol} (\nabla \cdot \bar{B})\, dv = 0 \quad \text{... (9)}$$

$$\therefore \quad \nabla \cdot \bar{B} = 0 \quad \text{... (10)}$$

$$\therefore \quad \nabla \cdot (\mu_0 \bar{H}) = 0 \quad \text{but } \mu_0 \neq 0 \quad \text{... (11)}$$

$$\therefore \quad \nabla \cdot \bar{H} = 0 \quad \text{... (12)}$$

$$\therefore \quad \nabla \cdot (-\nabla V_m) = 0 \quad \text{... using } \bar{H} = -\nabla V_m$$

$$\therefore \quad \boxed{\nabla^2 V_m = 0 \quad \text{for} \quad \bar{J} = 0} \quad \text{... (13)}$$

This is Laplace's equation for scalar magnetic potential. This is similar to the Laplace's equation for scalar electric potential $\nabla^2 V = 0$.

7.13.3 Vector Magnetic Potential

The vector magnetic potential is denoted as $\bar{A}$ and measured in Wb/m. It has to satisfy equation (2) that divergence of a curl of a vector is always zero.

$$\therefore \quad \nabla \cdot (\nabla \times \bar{A}) = 0 \quad \text{... } \bar{A} = \text{Vector magnetic potential}$$

$$\text{But} \quad \nabla \cdot \bar{B} = 0 \quad \text{... From equation (10)}$$

$$\therefore \quad \boxed{\bar{B} = \nabla \times \bar{A}} \quad \text{... (14)}$$

Thus curl of vector magnetic potential is the flux density.

$$\text{Now} \quad \nabla \times \bar{H} = \bar{J}$$

$$\therefore \quad \nabla \times \frac{\bar{B}}{\mu_0} = \bar{J} \quad \text{... } \bar{B} = \mu_0 \bar{H}$$

$$\therefore \quad \nabla \times \bar{B} = \mu_0 \bar{J} \qquad \ldots \bar{B} = \nabla \times \bar{A}$$

$$\therefore \quad \nabla \times \nabla \times A = \mu_0 J \qquad \ldots (15)$$

Using vector identity to express left hand side we can write,

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$

$$\therefore \quad \bar{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \bar{A}\right] = \frac{1}{\mu_0}\left[\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}\right] \qquad \ldots (16)$$

Thus if vector magnetic potential is known then current density $\bar{J}$ can be obtained. For defining $\bar{A}$ the current density need not be zero.

7.13.4 Poisson's Equation for Magnetic Field

In a vector algebra, a vector can be fully defined if its curl and divergence are defined.

For a vector magnetic potential $\bar{A}$, its curl is defined as $\nabla \times \bar{A} = \bar{B}$ which is known.

But to completely define $\bar{A}$ its divergence must be known. Assume that $\nabla \cdot \bar{A}$, the divergence of $\bar{A}$ is zero. This is consistent with some other conditions to be studied later in time varying magnetic fields. Using in equation (16),

$$\bar{J} = \frac{1}{\mu_0}\left[-\nabla^2 \bar{A}\right]$$

$$\therefore \quad \boxed{\nabla^2 \bar{A} = -\mu_0 \bar{J}} \qquad \ldots (17)$$

This is the Poisson's equation for magnetostatic fields.

7.13.5 $\bar{A}$ due to Differential Current Element

Consider the differential element $d\bar{L}$ carrying current I. Then according to Biot-Savart law the vector magnetic potential $\bar{A}$ at a distance R from the differential current element is given by,

$$\boxed{\bar{A} = \oint \frac{\mu_0 I \, d\bar{L}}{4\pi R} \text{ Wb/m}} \qquad \ldots (18)$$

For the distributed current sources, $I d\bar{L}$ can be replaced by $\bar{K}\, dS$ where $\bar{K}$ is surface current density.

$$\therefore \quad \boxed{\bar{A} = \oint_S \frac{\mu_0 \bar{K}\, dS}{4\pi R} \text{ Wb/m}} \qquad \ldots (19)$$

The line integral becomes a surface integral. If the volume current density $\bar{J}$ is given in A/m^2 then $I\, d\bar{L}$ can be replaced by $\bar{J}\, dv$ where dv is differential volume element.

$$\therefore \quad \boxed{A = \oint_{vol} \frac{\mu_0 \bar{J}\, dv}{4\pi R} \text{ Wb/m}} \qquad \ldots (20)$$

It can be noted that,

1. The zero reference for $\bar{A}$ is at infinity.
2. No finite current can produce the contributions as $R \to \infty$.

Example 7.17 : *In cylindrical co-ordinates $\bar{A} = 50 r^2 \bar{a}_z$ Wb/m is a vector magnetic potential, in a certain region of free space. Find, $\bar{H}, \bar{B}, \bar{J}$ and using $\bar{J}$ find total current I crossing the surface $0 \le r \le 1$, $0 \le \phi \le 2\pi$ and z = 0.*

Solution : Vector magnetic potential, $\bar{A} = 50\, r^2\, \bar{a}_z$ Wb/m

Now, $$\bar{B} = \nabla \times \bar{A}$$

$$= \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\bar{a}_\phi + \frac{1}{r}\left[\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\bar{a}_z$$

Now $A_r = 0$, $A_\phi = 0$, $A_z = 50\, r^2$

$$\therefore \quad \bar{B} = \left[\frac{1}{r}\frac{\partial (50 r^2)}{\partial \phi} - 0\right]\bar{a}_r + \left[0 - \frac{\partial (50 r^2)}{\partial r}\right]\bar{a}_\phi + \frac{1}{r}[0-0]\bar{a}_z$$

$$\bar{B} = -100\, r\, \bar{a}_\phi \quad \text{Wb/m}^2$$

$$\therefore \quad \bar{H} = \frac{\bar{B}}{\mu_0} = \frac{-100}{\mu_0} r\, a_\phi \quad \text{A/m}$$

Now $$\bar{J} = \nabla \times \bar{H}$$

$$H_r = 0, \quad H_\phi = -\frac{100\, r}{\mu_0}, \quad H_z = 0$$

$$\therefore \quad \nabla \times \bar{H} = \left[0 - \frac{\partial\left(-\frac{100\, r}{\mu_0}\right)}{\partial z}\right]\bar{a}_r + [0-0]\bar{a}_\phi + \frac{1}{r}\left[\frac{\partial\left(\frac{-100\, r^2}{\mu_0}\right)}{\partial r} - 0\right]\bar{a}_z$$

$$= [0-0]\bar{a}_r + 0\, \bar{a}_\phi + \frac{1}{r}\left[-\frac{100}{\mu_0}\right][2r]\, \bar{a}_z = -\frac{200}{\mu_0}\bar{a}_z \quad \text{A/m}^2$$

$$\therefore \quad \bar{J} = -\frac{200}{\mu_0}\, \bar{a}_z$$

Now $$I = \int_S \bar{J} \cdot d\bar{S} \qquad \text{where } d\bar{S} = r\, dr\, d\phi\, \bar{a}_z$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{1} -\frac{200}{\mu_0}\bar{a}_z \cdot r\, dr\, d\phi\, \bar{a}_z = \int_{\phi=0}^{2\pi}\int_{r=0}^{1} -\frac{200}{\mu_0} r\, dr\, d\phi$$

$$= -\frac{200}{\mu_0}\left[\frac{r^2}{2}\right]_0^1 [\phi]_0^{2\pi} = \frac{-200}{\mu_0}\left[\frac{1}{2}\right][2\pi]$$

$$= -500\times 10^6 \text{ A}$$

So current is 500 MA and negative sign indicates the direction of current.

Examples with Solutions

Example 7.18 : *Find out the magnetic vector potential in the vicinity of a very long straight wire carrying a current I. Hence find magnetic field density and magnetic field strength.* **[UPTU : 2002-03, 10 Marks]**

Solution : Consider an infinitely long filament carrying direct current I placed along z-axis as shown in the Fig. 7.44.

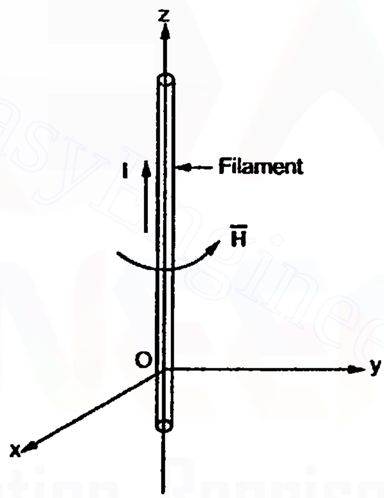


Fig. 7.44

The magnetic field intensity due to such filament is given by,

$$H = \frac{I}{2\pi r}\bar{a}_\phi$$

If it is placed in free space, $\bar{B} = \mu_0 \overline{H}$

$$\therefore \qquad \overline{B} = \frac{\mu_0 I}{2\pi r}\bar{a}_\phi$$

Assuming cylindrical co-ordinate system,

$$\overline{B} = \nabla\times\bar{A}$$

$$\therefore \frac{\mu_0 I}{2\pi r}\bar{a}_\phi = \left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]a_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\bar{a}_\phi + \frac{1}{r}\left[\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right]\bar{a}_z$$

$$= -\frac{\mu_0 I}{2\pi}\frac{\partial}{\partial r}[\ln(r)] = -\frac{\mu_0 I}{2\pi}\times\frac{1}{r}$$

$$\therefore \quad \bar{B} = \nabla\times\bar{A}$$

$$= 0\bar{a}_r + \left[0-\left(-\frac{\mu_0 I}{2\pi r}\right)\right]\bar{a}_\phi + \frac{1}{r}[0-0]\bar{a}_z$$

$$= \frac{\mu_0 I}{2\pi r}\bar{a}_\phi$$

$$\therefore \quad \bar{H} = \frac{\bar{B}}{\mu_0} = \frac{\mu_0 I}{\mu_0 2\pi r}\bar{a}_\phi$$

$$= \frac{I}{2\pi r}\bar{a}_\phi \qquad \text{... Magnetic field strength}$$

Example 7.19 : *A flat perfectly conducting surface in xy plane is situated in a magnetic field,*

$$\bar{H} = 3\cos x\bar{a}_x + z\cos x\bar{a}_y \text{ A/m for } z \geq 0$$

$$= 0 \text{ for } z < 0$$

Find the current density on the conductor surface.

Solution : From the point form of Ampere's circuit law,

$$\nabla\times\bar{H} = \bar{J} = \text{current density}$$

$$\therefore \quad \bar{J} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \text{ in cartesian form}$$

$$= \left[\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right]\bar{a}_z$$

From $\bar{H}$, $\quad H_x = 3\cos x, \quad H_y = z\cos x, \quad H_z = 0$

$$\therefore \quad \bar{J} = \left[0-\frac{\partial z\cos x}{\partial z}\right]\bar{a}_x + \left[\frac{\partial 3\cos x}{\partial z}-0\right]\bar{a}_y + \left[\frac{\partial z\cos x}{\partial x}-\frac{\partial 3\cos x}{\partial y}\right]\bar{a}_z$$

$$= -\cos x\bar{a}_x + 0\bar{a}_y + 0\bar{a}_z = -\cos x\bar{a}_x \text{ A/m}^2$$

Thus, $\quad \bar{J} = -\cos x\bar{a}_x \text{ A/m}^2 \qquad$ ... For $z \geq 0$

$$= 0 \text{ A/m}^2 \qquad \text{... For } z < 0$$

Example 7.20 : *A current sheet $\bar{K} = 10\,\bar{a}_z$ A/m lies in the x = 4 m plane and a second sheet $\bar{K} = -8\bar{a}_z$ A/m is at x = − 5 m plane. Find $\bar{H}$ in all the regions.*

Example 7.21 : *A circular conductor of 1 cm radius has an internal magnetic field given by*

$$\bar{H} = \frac{1}{r}\left[\frac{1}{a^2}\sin ar - \frac{r}{a}\cos ar\right]\bar{a}_\phi \quad A/m$$

Where $a = \dfrac{\pi}{2r_0}$ *and* r_0 *= radius of conductor.*

Calculate the total current through the conductor.

Solution : Consider the conductor as shown in the Fig. 7.46 along z-axis. Consider a closed path of radius r. The current enclosed by the path is part of the total current. The total current I is uniformly distributed in area πr_0^2 while the closed path encloses the area πr^2.

$$\therefore \quad I_{enc} = I\frac{\pi r^2}{\pi r_0^2} = \frac{Ir^2}{r_0^2}$$

Now $$H_\phi = \frac{1}{r}\left[\frac{1}{a^2}\sin ar - \frac{r}{a}\cos ar\right]$$

$$dL = r\,d\phi \quad \text{in } \bar{a}_\phi \text{ direction}$$

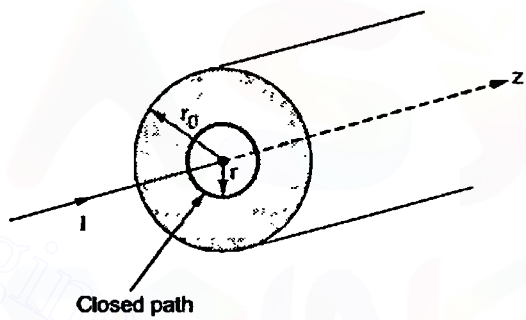


Fig. 7.46

According to Ampere's circuit law,

$$\oint \bar{H}\cdot d\bar{L} = I_{enc}$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\, r\, d\phi = \frac{I r^2}{r_0^2}$$

$$\therefore \quad 2\pi H_\phi\, r = \frac{I r^2}{r_0^2}$$

$$\therefore \quad H_\phi = \frac{I r^2}{2\pi r\, r_0^2} = \frac{I r}{2\pi r_0^2}$$

$$\therefore \quad \frac{1}{r}\left[\frac{1}{a^2}\sin ar - \frac{r}{a}\cos ar\right] = \frac{I r}{2\pi r_0^2}$$

But $a = \dfrac{\pi}{2r_0}$ ... given

$$\therefore \quad \frac{1}{r}\left[\frac{1}{\left(\frac{\pi}{2r_0}\right)^2}\sin\left[\frac{\pi}{2}\frac{r}{r_0}\right] - \frac{r}{\left(\frac{\pi}{2r_0}\right)}\cos\left[\frac{\pi}{2}\frac{r}{r_0}\right]\right] = \frac{I r}{2\pi r_0^2}$$

But if the closed path selected, has to enclose the total current I, then $r = r_0$.

$$\therefore \qquad \frac{1}{r_0}\left[\frac{1}{\left(\frac{\pi}{2r_0}\right)^2}\sin\frac{\pi}{2}-\frac{r_0}{\left(\frac{\pi}{2r_0}\right)}\cos\frac{\pi}{2}\right] = \frac{I}{2\pi r_0}$$

Now $\cos\frac{\pi}{2}=0$ and $r_0=1\text{ cm}=1\times10^{-2}$ m

$$\therefore \qquad \frac{1}{0.01}\left[4.052\times10^{-5}\right] = \frac{I}{2\pi\times0.01}$$

$$\therefore \qquad I = 2\pi\times4.052\times10^{-5}=2.5464\times10^{-4}\text{ A}$$

Example 7.22 : *If a particular field is given by,*
$\bar{F}=(x+2y+az)\bar{a}_x+(bx-3y-z)\bar{a}_y+(4x+cy+2z)\bar{a}_z$
then find the constants a, b and c such that the field is irrotational.

Solution :

Key Point: *The vector field is irrotational if its curl is zero.*

$$\therefore \qquad \nabla\times\bar{F} = 0 \qquad \text{... For } \bar{F} \text{ to be irrotational}$$

$$\text{Now} \qquad \nabla\times\bar{F} = \begin{vmatrix}\bar{a}_x & \bar{a}_y & \bar{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}$$

$$= \left[\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right]\bar{a}_x+\left[\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right]\bar{a}_y+\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right]\bar{a}_z = 0$$

$$\therefore \qquad \frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z} = \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial z}=\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}=0$$

And $F_x=x+2y+az,\quad F_y=bx-3y-z,\quad F_z=4x+cy+2z$

$$\therefore \qquad \frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z} = c-1=0 \qquad \text{i.e. } c=1$$

$$\therefore \qquad \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x} = a-4=0 \qquad \text{i.e. } a=4$$

$$\therefore \qquad \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y} = b-2=0 \qquad \text{i.e. } b=2$$

Thus a = 4, b= 2 and c = 1 to have $\bar{F}$ irrotational.

$$\therefore \quad \bar{H}_2 = \frac{4\times10^{-3}}{r}\bar{a}_\phi \text{ A/m}$$

while $$\bar{H}_1 = \frac{10\times10^{-3}}{r}\bar{a}_\phi \text{ A/m}$$

So at r = 1.5 cm,

$$\bar{H} = \bar{H}_1 + \bar{H}_2 = \frac{14\times10^{-3}}{1.5\times10^{-2}}\bar{a}_\phi = 0.933\,\bar{a}_\phi \text{ A/m}$$

At r = 2.5 cm, second sheet also gets enclosed for which,

$$\bar{K}_2 = -250\times10^{-3}\,\bar{a}_z \text{ A/m}$$

$$\therefore \quad I_{enc} = K_2\times2\pi r_2 = -250\times10^{-3}\times2\pi\times2\times10^{-2} \qquad \ldots r_2 = 2 \text{ cm for sheet}$$

$$= -0.03141 \text{ A}$$

According to Ampere's circuital law,

$$\oint \bar{H}\cdot d\bar{L} = I_{enc}$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\, r\, d\phi = I_{enc}$$

$$\therefore \quad H_\phi = \frac{I_{enc}}{2\pi r} = \frac{-0.03141}{2\pi r}$$

$$\therefore \quad \bar{H}_3 = \frac{-5\times10^{-3}}{r}\bar{a}_\phi \text{ A/m}$$

So at r = 2.5 cm,

$$\bar{H} = \bar{H}_1 + \bar{H}_2 + \bar{H}_3 = \left(\frac{4}{r}+\frac{10}{r}-\frac{5}{r}\right)\times10^{-3}\,\bar{a}_\phi$$

$$= \frac{9}{r}\times10^{-3}\,\bar{a}_\phi = \frac{9\times10^{-3}}{2.5\times10^{-2}}\bar{a}_\phi$$

$$= 0.36\,\bar{a}_\phi \text{ A/m}$$

Example 7.26 : *Evaluate both sides of Stoke's theorem for the field $\bar{H} = 10\sin\theta\,\bar{a}_\phi$ and the surface $r = 3$, $0 \le \theta \le 90°$, $0 \le \phi \le 90°$. Let the surface have the $\bar{a}_r$ direction.*

Solution : According to Stoke's theorem,

$$\oint_L \bar{H}\cdot d\bar{L} = \oint_S (\nabla\times\bar{H})\cdot d\bar{S}$$

In spherical system,

$d\bar{L} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi$

The closed path forming its perimeter is composed of three circular arcs. The first path 1 is r = 3, $\phi = 0$, $0 \le \theta \le 90°$ as shown in the Fig. 7.47. The second path 2 is r = 3, $\theta = 90°$, $0 \le \phi \le 90°$ while the path 3 is r = 3, $\phi = 90°$, $0 \le \theta \le 90°$. For all the three arcs r = 3 m.

Let us evaluate $\oint \bar{H} \cdot d\bar{L}$ over these three paths.

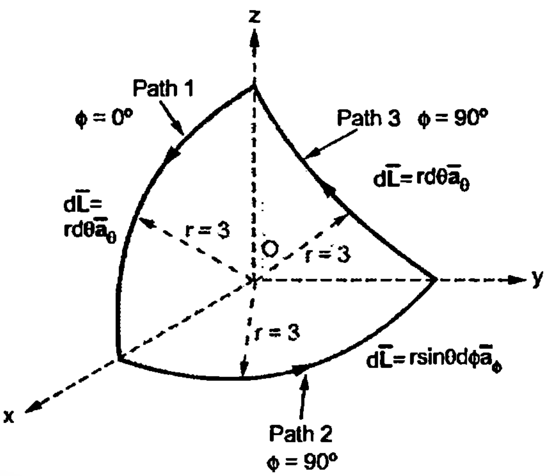


Fig. 7.47

$$\therefore \oint \bar{H} \cdot d\bar{L} = \int_{\text{Path1}} H_\theta\, r\, d\theta + \int_{\text{Path2}} H_\phi\, r \sin\theta\, d\phi + \int_{\text{Path 3}} H_\theta\, r\, d\theta$$

Now, $H_r = 0,\quad H_\theta = 0,\quad H_\phi = 10\sin\theta$ ... Given $\bar{H}$

Thus only second line integral exists.

$$\therefore \quad \oint \bar{H} \cdot d\bar{L} = \int_{\phi=0}^{\pi/2} 10\sin\theta\, r\sin\theta\, d\phi = 10\, r\sin^2\theta\, [\phi]_0^{\pi/2} \quad \text{... Path 2}$$

$$= 10 \times 3 \times [\sin 90°]^2 \times \frac{\pi}{2} \quad \text{... r = 3 m, } \theta = 90° \text{ for path 2}$$

$$= \mathbf{47.1238\ A}$$

Now evaluate second side of Stoke's theorem.

$\nabla \times \bar{H}$ in spherical co-ordinates is,

$$= \frac{1}{r\sin\theta}\left[\frac{\partial H_\phi \sin\theta}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right]\bar{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\bar{a}_\phi$$

As $H_r = 0,\ H_\theta = 0,\ H_\phi = 10\sin\theta$

$$\nabla \times \bar{H} = \frac{1}{r\sin\theta}\left[10\frac{\partial \sin^2\theta}{\partial\theta} - 0\right]\bar{a}_r + \frac{1}{r}\left[0 - \frac{\partial(r10\sin\theta)}{\partial r}\right]\bar{a}_\theta + \frac{1}{r}[0-0]\bar{a}_\phi$$

$$= \frac{1}{r\sin\theta}[10 \times 2\sin\theta\cos\theta]\bar{a}_r + \frac{1}{r}[-10\sin\theta]\bar{a}_\theta$$

$$= \frac{10}{r\sin\theta}\sin 2\theta\,\bar{a}_r - \frac{10}{r}\sin\theta\,\bar{a}_\theta$$

while $\quad d\bar{S} = r^2\sin\theta\, d\theta\, d\phi\, \bar{a}_r \quad$ ... as given in $\bar{a}_r$ direction

$$\therefore \quad (\nabla \times \bar{H}) \cdot d\bar{S} = \frac{10}{r \sin\theta} \sin 2\theta \; r^2 \sin\theta \, d\theta \, d\phi \qquad \ldots \bar{a}_r \cdot \bar{a}_r = 1$$

$$\therefore \quad \int_S (\nabla \times \bar{H}) \cdot d\bar{S} = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} 10 \; r \sin 2\theta \, d\theta \, d\phi$$

$$= 10\,r \left[\frac{-\cos 2\theta}{2} \right]_0^{\pi/2} [\phi]_0^{\pi/2}$$

$$= 10 \times r \times \left[\frac{-\cos\pi}{2} - \frac{(-\cos 0)}{2} \right] \left[\frac{\pi}{2} \right] \qquad \ldots r = 3 \text{ m}$$

$$= 10 \times 3 \times \left[\frac{1}{2} + \frac{1}{2} \right] \times \frac{\pi}{2} = 47.1238 \text{ A}$$

... Thus Stoke's theorem is verified.

Example 7.27 : *A conductor in the form of regular polygon of 'n' sides inscribed in a circle of radius R. Show that the expression for magnetic flux density is* $B = \frac{\mu_0 n I}{2\pi R} \tan\left(\frac{\pi}{n}\right)$ *at the centre, where I is the current. Show also when 'n' is indefinitely increased then the expression reduces to* $B = \frac{\mu_0 I}{2R}$.

Solution : Consider a polygon of n sides inscribed in a circle of radius R, as shown in the Fig. 7.48. It carries a current I.

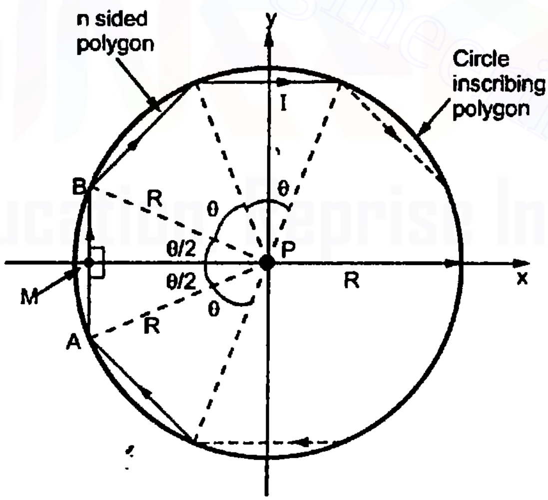


Fig. 7.48

Let the polygon is placed in the xy plane. Consider the side AB of polygon. The angle subtended by each side at the centre is say θ. The PM is perpendicular to AB. Thus $\angle$ BPM = $\angle$ AMP = $\frac{\theta}{2}$.

Now as the sides are n, the angle θ can be written as,

$$\theta = \frac{2\pi \text{ radians}}{n} = \frac{360°}{n} \quad ... (1)$$

$$\therefore \quad \frac{\theta}{2} = \frac{360°}{2n} = \frac{180°}{n} = \frac{\pi}{n} \text{ rad} \quad ... (2)$$

Using the formula for $\bar{H}$ due to finite length conductor, $|\bar{H}_{AB}|$ at centre P is given by,

$$|\bar{H}_{AB}| = \frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1] \quad ... (3)$$

where $\alpha_1 = -\frac{\theta}{2}$ as point A is below P

$\alpha_2 = \frac{\theta}{2}$ as point B is above P

$$\therefore \quad |\bar{H}_{AB}| = \frac{I}{4\pi r}\left[\sin\frac{180°}{n} - \sin\left(\frac{-180°}{n}\right)\right] \quad ... \sin(-\theta) = -\sin\theta$$

$$\therefore \quad |\bar{H}_{AB}| = \frac{I}{4\pi r}\left[\sin\frac{\pi}{n} - \left(-\sin\frac{\pi}{n}\right)\right] = \frac{I}{4\pi r} 2\sin\frac{\pi}{n}$$

$$= \frac{I}{2\pi r}\sin\frac{\pi}{n} \quad ... (4)$$

Now $r = l$ (MP) = Perpendicular distance of P from AB

$$= R\cos\frac{\theta}{2} = R\cos\left(\frac{\pi}{n}\right) \quad ... (5)$$

$$\therefore \quad |\bar{H}_{AB}| = \frac{I}{2\pi R\cos\frac{\pi}{n}}\sin\frac{\pi}{n}$$

$$= \frac{I}{2\pi R}\tan\frac{\pi}{n} \quad ... (6)$$

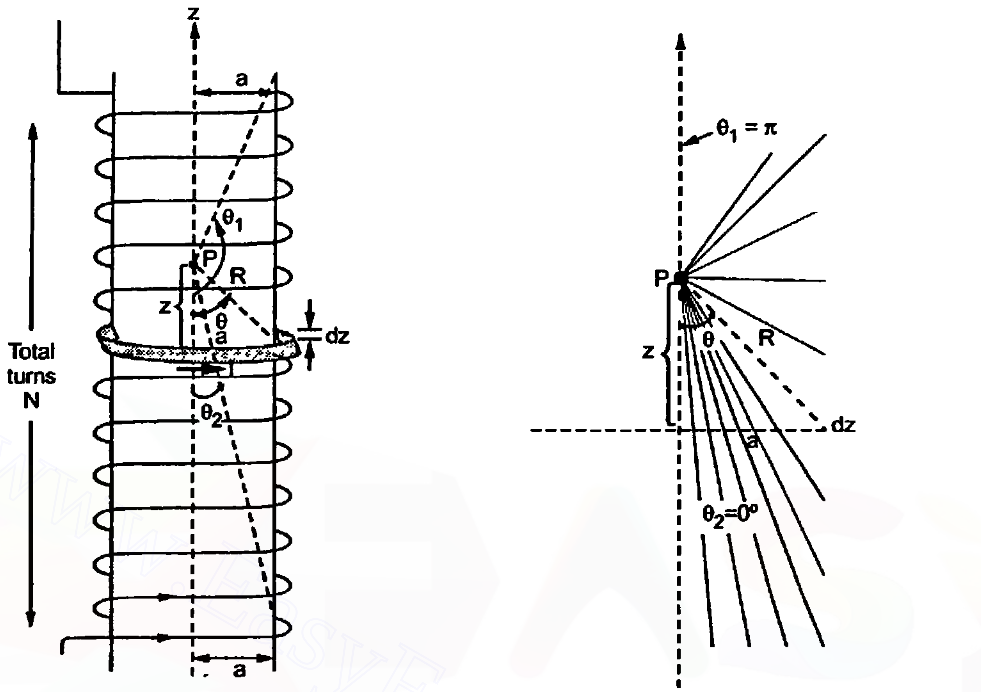
But there are n such sides hence,

$$|\bar{H}_{total}| = n \times \frac{I}{2\pi R}\tan\frac{\pi}{n} = \frac{nI}{2\pi R}\tan\frac{\pi}{n} \quad ... (7)$$

As the polygon is in free space, $|\bar{B}| = \mu_0 |\bar{H}|$,

$$\therefore \quad |\bar{B}| = \frac{\mu_0 n I}{2\pi R}\tan\frac{\pi}{n} \text{ Wb/m}^2 \quad ... \text{Proved}$$

As $n \to \infty$, $\frac{\pi}{n} \to 0$



(a) Finite length, θ varies from θ_2 to θ_1 **(b) Very long solenoid, θ varies from 0 to π**

Fig. 7.50

Now $|\overline{dH}|$ due to circular conductor, at a point P which is at a distance z from the conductor on the axis is given by,

$$|\overline{dH}| = \frac{I a^2 (N/l)\, dz}{2(a^2 + z^2)^{3/2}} \qquad \text{... Refer equation (7) of section (7.7)}$$

where a = Radius of the solenoid

$I \frac{N}{l}$ = Current in the turns N/l

Hence total $\overline{H}$ at a point P on axis can be obtained by summing $\overline{H}$ due to all such elementary rings spreaded from one end of solenoid to other. i.e. within the θ from θ_1 to θ_2.

From the geometry of the Fig. 7.50

$$\tan\theta = \frac{a}{z} \qquad \text{i.e. } z = a \cot\theta$$

$$\therefore \quad |\bar{H}| = \int_{\theta_1}^{\theta_2} |d\bar{H}| = \int_{\theta_1}^{\theta_2} \frac{I a^2 \left(\frac{N}{l}\right) dz}{2\left(a^2 + a^2 \cot^2 \theta\right)^{3/2}}$$

Now $\quad dz = a\left(-\text{cosec}^2 \theta\right) d\theta \qquad \ldots \int \cot\theta = -\text{cosec}^2\theta$

$$\therefore \quad |\bar{H}| = \int_{\theta_1}^{\theta_2} \frac{I a^2 \frac{N}{l} a\left(-\text{cosec}^2 \theta\right) d\theta}{2 a^3 \left(1 + \cot^2 \theta\right)^{3/2}} \qquad \ldots 1 + \cot^2 \theta = \text{cosec}^2\theta$$

$$= \frac{N}{l} \frac{I}{2} \int_{\theta_1}^{\theta_2} -\frac{1}{\text{cosec}\,\theta} d\theta = \frac{NI}{2l} \int_{\theta_1}^{\theta_2} -\sin\theta \, d\theta$$

$$= -\frac{NI}{2l} [-\cos\theta]_{\theta_1}^{\theta_2} = \frac{NI}{2l} [\cos\theta_2 - \cos\theta_1]$$

$$\therefore \quad |\bar{H}| = \frac{NI}{2l} [\cos\theta_2 - \cos\theta_1] \text{ AT/m}$$

The direction of $\bar{H}$ is to be obtained using right hand thumb rule.

$$|\bar{B}| = \mu |\bar{H}| = \frac{\mu NI}{l} [\cos\theta_2 - \cos\theta_1] \text{ Wb/m}^2$$

If the solenoid is having large length then $\theta_1 \approx 180°$ and $\theta_2 = 0°$.

$$\therefore \quad |\bar{H}| = \frac{NI}{2l} [\cos 0° - \cos(180°)] = \frac{NI}{2l}(2)$$

$$= \frac{NI}{l} \text{ AT/m}$$

Review Questions

1. *State and explain Biot-Savart law.*
2. *How Biot-Savart law can be applied to the distributed forces ?*
3. *Using Biot-Savart law, find $\bar{H}$ due to infinitely long straight conductor.*
4. *Using Biot-Savart law, find $\bar{H}$ due to conductor of finite length.*
5. *Using Biot-Savart law, find $\bar{H}$ at the centre of a circular conductor.*
6. *Using Biot-Savart law, find $\bar{H}$ on axis of circular loop.*
7. *State and explain Ampere's circuital law.*
8. *Using Ampere's circuital law, find $\bar{H}$ due to infinitely long straight conductor.*
9. *Using Ampere's circuital law, find $\bar{H}$ due to a co-axial cable carrying current I.*
10. *Using Ampere's circuital law, find $\bar{H}$ due to infinite sheet of current.*
11. *Derive the expression for a curl, applying Ampere's circuital law to an incremental surface element.*

12. *State the point form of Ampere's circuital law and explain it.*
13. *Explain the physical significance of a curl.*
14. *State and prove the Stoke's theorem.*
15. *State the relation between magnetic flux and flux density. Also explain Gauss's law in integral and differential form for the magnetic fields.*
16. *Derive the expression for the flux in a co-axial cable.*
17. *Explain the concept of scalar and vector magnetic potentials.*
18. *Derive the Laplace's and Poisson's equations for the magnetic fields.*
19. *Find the field intensity at a point on the axis, 5 m from the centre of a circular loop carrying current of 50 A with area 100 cm^2* **[Ans. $6.36\times10^{-4}\,\bar{a}_z$ A/m]**
20. *The conducting triangular loop is shown in the Fig. 7.51, carries a current of 10 A. Find $\bar{H}$ at (0, 0, 6) due to side 1 of the loop.* **[Ans. : $-49.257\,\bar{a}_y$ mA/m]**

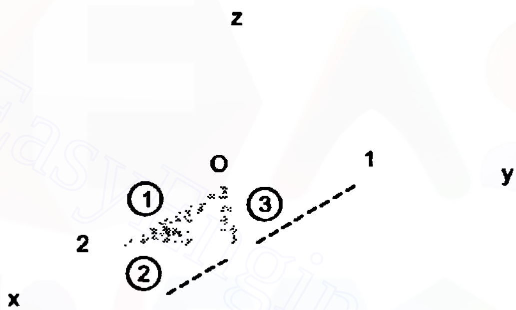


Fig. 7.51

21. *A circuit carrying a direct current of 5 A forms a regular hexagon inscribed in a circle of radius 1 m. Calculate the magnetic flux density at the centre of the hexagon. Assume the medium to be the free space.* **[Ans. : $3.4644\,\bar{a}_N$ μWb/m^2]**
22. *A circular loop located on $x^2+y^2=9$, $z=0$ carries a direct current of 10 A along $\bar{a}_\phi$. Determine $\bar{H}$ at point (0, 0, 5) and (0, 0, – 5).* **[Ans. : $0.227\,\bar{a}_z$ for both]**
23. *The magnetic field intensity $\bar{H}$ is given by,*

$$\bar{H} = -y\left(x^2+y^2\right)\bar{a}_x + x\left(x^2+y^2\right)\bar{a}_y \quad A/m \text{ in } z=0 \text{ plane}$$

for $-5\le x\le5$ and $-5\le y\le5$ m. Calculate the current passing through the z = 0 plane in the $\bar{a}_z$ direction inside the rectangle $-1<x<1$ and $-2<y<2$. **[Ans. : 53.33 A]**

24. *A current carrying conductor is in the form $x^2+y^2=r^2$ at z = 0 plane carrying current I in $\bar{a}_\phi$ direction. Find the expression for $\bar{H}$ at*

a) (0, 0, 0) b) (0, 0, h) c) (0, 0, – h) **[Ans. : $\frac{I}{2r}\bar{a}_z$, $\frac{Ir^2}{2\left(r^2+h^2\right)^{3/2}}\bar{a}_z$]**

25. *Find the flux crossing the plane surface defined by $0.5 \le r \le 2$ m and $0 \le z \le 3$ m if*

$\bar{B} = \left(\frac{4}{r}\bar{a}_\phi\right)$ T.

[Ans. : 16.64 Wb]

26. *The planes z = 0 and z = 6 carry current $\bar{K} = -20\bar{a}_x$ A/m and $\bar{K} = 20\bar{a}_x$ A/m respectively. Determine H at a) (1, 1, 1) b) (0, – 3, 10).* **[Ans. : 20 $\bar{a}_y$, 0 A/m]**

27. *Evaluate both sides of the Stoke's theorem for the field $H = 6\,xy\,\bar{a}_x - 3y^2\bar{a}_y$ A/m and the rectangular path around the strip, $1 \le x \le 3$, $-2 \le y \le z$, $z = 0$. Let the positive direction of $d\bar{S}$ be a_z.* **[Ans. : – 96 A]**

28. *Given the vector magnetic potential as,*

$$\bar{A} = -\frac{r^2}{4}\,\bar{a}_z \text{ Wb/m in cylindrical system.}$$

Calculate the flux crossing the surface $\phi = \frac{\pi}{2}$, $1 \le r \le 2$ m, $0 \le z \le 5$ m. **[Ans. : 3.75 Wb]**

29. *If the vector magnetic potential is given by,*

$$\bar{A} = \frac{10}{(x^2+y^2+z^2)}\,\bar{a}_x \quad \text{Find } \bar{B}.$$

[Ans. : $\frac{20}{x^2+y^2+z^2}(z\,\bar{a}_y - y\,\bar{a}_z)$ Wb/m^2]

30. *If $\bar{A} = r\phi z\,\bar{a}_z$, calculate curl of $\bar{A}$ at the point (2, 30°, 3).* **[Ans. : $3\,\bar{a}_r - 1.571\,\bar{a}_\phi$]**

31. *Find $\bar{J}$ at (3, 2, 1) if $\bar{H} = xyz\bar{a}_x + xyz\,\bar{a}_y$ A/m.* **[Ans. : $-6\,\bar{a}_x + 6\bar{a}_y - \bar{a}_z$ A/m^2]**

32. *The $\bar{H} = 0.1\,y^3\bar{a}_x + 0.4x\bar{a}_z$ A/m in a region. Determine the current flow through the path a-b-c-d when a (5, 4, 1), b (5, 6, 1), c (0, 6, 1) and d (0, 4, 1).* **[Ans. : – 75 A]**

33. *Find $\bar{H}$ at the centre of a square loop of side 'L' in xy plane at the origin as centre, carrying current I. [Hint : Refer Ex. 7.3]* **[Ans. : $\frac{0.9\,I}{L}\,\bar{a}_z$]**

34. *Evaluate both sides of the Stoke's theorem for the portion of a sphere specified by $r = 4$, $0 \le \theta \le 0.1\pi$, $0 \le \phi \le 0.3\pi$. Given that the field $\bar{H}$ is,*

$\bar{H} = 6r\sin\phi\,\bar{a}_r + 18r\sin\theta\cos\phi\,\bar{a}_\phi$ A/m.

Assume $d\bar{S}$ in the direction of $\bar{a}_r$.

[Hint : Refer Ex. 7.26] **[Ans. : 22.2 A]**

University Questions

1. *What is Stokes' theorem ? State and prove it.* **[UPTU : 2002-03, 5 Marks]**
2. *How is magnetic flux density related to the magnetic vector potential ? Find out the magnetic vector potential in the vicinity of a very long straight wire carrying a current I. Hence find the magnetic field strength.* **[UPTU: 2002-03, 10 Marks]**

3. *Verify that within a long conductor carrying a current I, the magnetic field strength at a distance r from the center of the wire is given by*

$$H = \frac{Ir}{2\pi R^2}$$

where R is the radius the wire. The current density is constant across the cross-section of the conductor. **[UPTU: 2003-04(A), 10 Marks]**

4. *A current sheet with surface current density k is given by $\bar{k} = k \cdot \bar{a}_z \ Am^{-1}$ where k is a constant coincides with the xz plane as shown in Fig. 7.52. Find a general relation for flux density.*

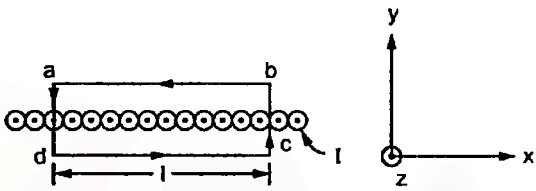


Fig. 7.52

[UPTU : 2005-06, 10 Marks]

5. *Define Biot Savart law and Ampere's law A long, straight conductor cross-section with radius 'a' has a magnetic field strength $\bar{H} = \left(\frac{Ir}{2\pi a^2}\right)\bar{a}_\phi$ with in the conductor ($r < a$) and $\bar{H} = \left(\frac{I}{2\pi r}\right)\bar{a}_\phi$ for ($r < a$). Find $\bar{J}$ in both the region.* **[UPTU : 2006-07, 10 Marks]**

6. *State and explain Ampere's circuital law in integral form.* **[UPTU : 2008-09, 4 Marks]**

7. *Define and explain the terms magnetic force, magnetic flux density and magnetic permeability and the units in which each of these quantities is measured in the MKS unit.*

[UPTU : 2008-09, 10 Marks]

□□□

8 Magnetic Forces, Materials and Inductance

8.1 Introduction

In the last chapter, we have studied various basic concepts in magnetostatics. We have discussed Biot-Savart law, Ampere's circuital law and the concepts of magnetic flux, flux density, scalar and vector potentials.

In this chapter, we shall study the magnetic forces. We shall discuss the concepts of the magnetic torque, magnetic dipole moment. We shall study the behaviour of different magnetic materials on the basis of quantum theory and the magnetization along with the concept of permeability. Similar to the boundary conditions in electrostatic fields, we shall study the boundary conditions for the magnetostatic fields. Under the magnetic circuits, we shall discuss the similarities and dis-similarities between the electric and magnetic circuits.

8.2 Force on a Moving Point Charge

According to the discussion in the previous chapters, a static electric field $\bar{E}$ exerts a force on a static or moving charge Q. Thus according to Coulomb's law, the force $\bar{F}_e$ exerted on an electric charge can be obtained. The force is related to the electric field intensity $\bar{E}$ as,

$$\boxed{\bar{F}_e = Q\,\bar{E}\ \text{N}} \qquad \dots (1)$$

For a positive charge, the force exerted on it is in the direction of $\bar{E}$. This force is also refered as **electric force** $(\bar{F}_e)$.

Now consider that a charge is placed in a steady magnetic field. It experiences a force only if it is moving. Then a magnetic force $(\bar{F}_m)$ exerted on a charge Q, moving with a velocity $\bar{v}$ in a steady magnetic field $\bar{B}$ is given by,

$$\boxed{\bar{F}_m = Q\,\bar{v} \times \bar{B}\ \text{N}} \qquad \dots (2)$$

The magnitude of the magnetic force $\bar{F}_m$ is directly proportional to the magnitudes of Q, $\bar{v}$ and $\bar{B}$ and also the sine of the angle between $\bar{v}$ and $\bar{B}$. The direction of $\bar{F}_m$ is perpendicular to the plane containing $\bar{v}$ and $\bar{B}$ both, as shown in the Fig. 8.1.

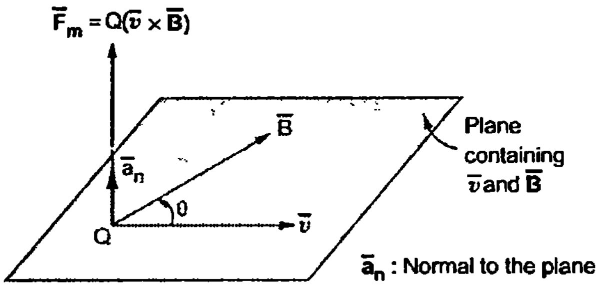


Fig. 8.1 Magnetic force on a moving charge in magnetic field

From equation (1) it is clear that the electric force $\bar{F}_e$ is independent of the velocity of the moving charge. In other words, the electric force exerted on the moving charge by the electric field is independent of the direction in which the charge is moving. Thus the electric force performs work on the charge. On the other hand, the magnetic force $\bar{F}_m$ is dependent on the velocity of the moving charge. But $\bar{F}_m$ cannot perform work on a moving charge as it is at right angle to the direction of motion of charge. ($\bar{F} \bullet d\bar{L} = 0$).

The total force on a moving charge in the presence of both electric and magnetic fields is given by,

$$\boxed{F = \bar{F}_e + \bar{F}_m = Q\left(\bar{E} + \bar{v} \times \bar{B}\right) \quad N} \quad \text{... (3)}$$

Above equation is called **Lorentz Force Equation** which relates mechanical force to the electrical force. If the mass of the charge is m, then we can write,

$$\boxed{\bar{F} = m\,\bar{a} = m\,\frac{d\bar{v}}{dt} = Q\left(\bar{E} + \bar{v} \times \bar{B}\right) \quad N} \quad \text{... (4)}$$

Example 8.1 *A point charge of Q = – 1.2 C has velocity $\bar{v} = \left(5\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z\right)$ m/s. Find the magnitude of the force exerted on the charge if,*

a) $E = -18\bar{a}_x + 5\bar{a}_y - 10\bar{a}_z$ V/m,

b) $\bar{B} = -4\bar{a}_x + 4\bar{a}_y + 3\bar{a}_z$ T,

c) Both are present simultaneous.

Solution : a) The electric force exerted by $\bar{E}$ on charge Q is given by,

$$\bar{F}_e = Q\,\bar{E}$$

$$= -1.2\left[-18\bar{a}_x + 5\bar{a}_y - 10\bar{a}_z\right]$$

$$= 21.6\,\bar{a}_x - 6\,\bar{a}_y + 12\,\bar{a}_z \quad N$$

Thus the magnitude of the electric force is given by

$$\left|\bar{F}_e\right| = \sqrt{(21.6)^2 + (-6)^2 + (12)^2} = 25.4275 \text{ N}$$

b) The magnetic force exerted by $\bar{B}$ on charge Q is given by,

$$\bar{F}_m = Q\,\bar{v}\times\bar{B}$$

$$= -1.2\left[(5\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z)\times(-4\bar{a}_x + 4\,\bar{a}_y + 3\bar{a}_z)\right]$$

$$= (-6\,\bar{a}_x - 2.4\bar{a}_y + 3.6\bar{a}_z)\times(-4\bar{a}_z + 4\,\bar{a}_y + 3\,\bar{a}_z)$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -6 & -2.4 & 3.6 \\ -4 & 4 & 3 \end{vmatrix}$$

$$= [-7.2 - 14.4]\,\bar{a}_x - [-18 + 14.4]\,\bar{a}_y + [-24 - 9.6]\,\bar{a}_z$$

$$= (-21.6\,\bar{a}_x + 3.6\,\bar{a}_y - 33.6\,\bar{a}_z)\text{ N}$$

Thus, the magnitude of the magnetic force is given by

$$|\bar{F}_m| = \sqrt{(-21.6)^2 + (+3.6)^2 + (-33.6)^2} = \mathbf{40.1058\ N}$$

c) The total force exerted by both the fields ($\bar{E}$ and $\bar{B}$) on a charge is given by,

$$\bar{F} = \bar{F}_e + \bar{F}_m = Q\left(\bar{E} + \bar{v}\times\bar{B}\right) = Q\,\bar{E} + Q\,\bar{v}\times\bar{B}$$

$$= \left[(21.6\bar{a}_x - 6\,\bar{a}_y + 12\,\bar{a}_z) + (-21.6\,\bar{a}_x + 3.6\,\bar{a}_y - 33.6\,\bar{a}_z)\right]$$

$$= (0\,\bar{a}_x - 2.4\,\bar{a}_y - 21.6\,\bar{a}_z)\text{ N}$$

Thus, the magnitude of the total force exerted is given by

$$|\bar{F}| = \sqrt{(0)^2 + (-2.4)^2 + (-21.6)^2} = \mathbf{21.7329\ N}$$

8.3 Force on a Differential Current Element

The force exerted on a differential element of charge dQ moving in a steady magnetic field is given by,

$$d\bar{F} = dQ\,\bar{v}\times\bar{B}\text{ N} \quad \text{... (1)}$$

The current density $\bar{J}$ can be expressed interms of velocity of a volume charge density as,

$$\bar{J} = \rho_v\,\bar{v} \quad \text{... (2)}$$

But the differential element of charge can be expressed in terms of the volume charge density as,

$$dQ = \rho_v\ dv \quad \text{... (3)}$$

Substituting value of dQ in equation (1),

$$\therefore \qquad d\bar{F} = \rho_v \, dv \, \bar{v} \times \bar{B}$$

Expressing $d\bar{F}$ interms of $\bar{J}$ using equation (2), we can write,

$$d\bar{F} = \bar{J} \times \bar{B} \; dv \qquad \text{... (4)}$$

But we have already studied in previous chapters, the relationship between current element as,

$$\bar{J} \, dv = \bar{K} \, dS = I \, d\bar{L}$$

Then the force exerted on a surface current density is given by,

$$d\bar{F} = \bar{K} \times \bar{B} \; dS \qquad \text{... (5)}$$

Similarly the force exerted on a differential current element is given by,

$$dF = (I \, dL \times B) \qquad \text{... (6)}$$

Integrating equation (4) over a volume, the force is given by,

$$\bar{F} = \int_{vol} \bar{J} \times \bar{B} \, dv \qquad \text{... (7)}$$

Integrating equation (5) over either open or closed surface, we get,

$$\bar{F} = \int_{S} \bar{K} \times \bar{B} \; dS \qquad \text{... (8)}$$

Similarly integrating equation (6) over a closed path, we get,

$$\boxed{\bar{F} = \oint I \, d\bar{L} \times \bar{B}} \qquad \text{... (9)}$$

If a conductor is straight and the field $\bar{B}$ is uniform along it, then integrating equation (6) we get simple expression for the force as,

$$\boxed{\bar{F} = I \, \bar{L} \times \bar{B}} \qquad \text{... (10)}$$

The magnitude of the force is given by,

$$\boxed{F = I \, L \, B \sin \theta} \qquad \text{... (11)}$$

Actually the magnetic field exerts a magnetic force on the electrons which constitutes the current I. But these electrons are part of the conductor, this magnetic force gets transfered to the conductor lattice. Now this transfered force can perform work on a conductor as a whole.

Example 8.2 : *A conductor 6m long, lies along z-direction with a current of 2A in $\bar{a}_z$ direction. Find the force experienced by conductor if $\bar{B} = 0.08 \, \bar{a}_x$ (T).*

Solution : A force exerted on current carrying conductor in a magnetic field is given by

$$\bar{F} = I \, d\bar{L} \times \bar{B}$$

$\therefore \quad \bar{F} = 2(6\bar{a}_z) \times (0.08\bar{a}_x)$

$\therefore \quad \bar{F} = 12\,\bar{a}_z \times 0.08\,\bar{a}_x$

$\therefore \quad \bar{F} = 0.96\;\bar{a}_y\;\;N \qquad \ldots\; \bar{a}_z \times \bar{a}_x = \bar{a}_y$

8.4 Force between Differential Current Elements

While discussing the electrostatic fields, we have studied that a point charge exerts a force on another point charge, separated by distance R. If these charges are of same type (i.e. both positive or negative), then they repel each other. But when two charges are of different type (i.e. one positive and other negative), then they attract each other.

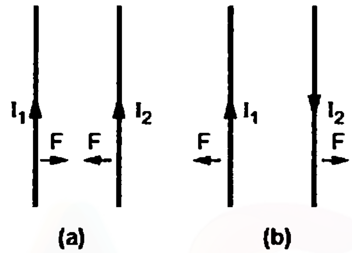


Fig. 8.2 Force between two parallel current carrying conductors

Now consider that two current carrying conductors are placed parallel to each other. Each of this conductor produces its own flux around it. So when such two conductors are placed closed to each other, there exists a force due to the interaction of two fluxes. The force between such parallel current carrying conductors depends on the directions of the two currents. If the directions of both the currents are same, then the conductors experience a force of attraction as shown in the Fig. 8.2 (a). And if the directions of two currents are opposite to each other, then the conductors experience a force of repulsion as shown in the Fig. 8.2 (b).

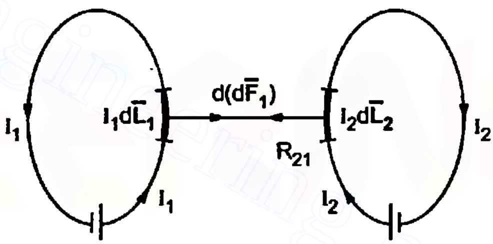


Fig. 8.3 Force between two current elements

Let us now consider two current elements $I_1\,d\bar{L}_1$ and $I_2 d\bar{L}_2$ as shown in the Fig. 8.3. Note that the directions of I_1 and I_2 are same.

Both the current elements produce their own magnetic fields. As the currents are flowing in the same direction through the elements, the force $d(d\bar{F}_1)$ exerted on element $I_1 d\bar{L}_1$ due to the magnetic field $d\bar{B}_2$ produced by other element $I_2 d\bar{L}_2$ is the force of attraction.

From the equation of force the force exerted on a differential current element is given by,

$$d\left(d\bar{F}_1\right) = I_1 d\bar{L}_1 \times d\bar{B}_2 \qquad \ldots (1)$$

According to Biot-Savart's law, the magnetic field produced by current element $I_2 d\bar{L}_2$ is given by, for free space,

$$d\bar{B}_2 = \mu_0 d\bar{H}_2 = \mu_0 \left[\frac{I_2 d\bar{L}_2 \times \bar{a}_{R21}}{4\pi R_{21}^2}\right] \quad ... (2)$$

Substituting value of $d\bar{B}_2$ in equation (1), we can write,

$$d(d\bar{F}_1) = \mu_0 \frac{I_1 d\bar{L}_1 \times (I_2 d\bar{L}_2 \times \bar{a}_{R21})}{4\pi R_{21}^2} \quad ... (3)$$

The equation (3) represents force between two current elements. It is very much similar to Coulomb's law. By integrating $d(d\,F_2)$ twice, the total force $\bar{F}_1$ on current element 1 due to current element 2 is given by,

$$\boxed{\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_1 \times (d\bar{L}_2 \times \bar{a}_{R21})}{R_{21}^2}} \quad ... (4)$$

Exactly following same steps, we can calculate the force $\bar{F}_2$ exerted on the current element 2 due to the magnetic field $\bar{B}_1$ produced by the current element 1. Thus,

$$\boxed{\bar{F}_2 = \frac{\mu_0 I_2 I_1}{4\pi} \oint_{L_2} \oint_{L_1} \frac{d\bar{L}_2 \times (d\bar{L}_1 \times \bar{a}_{R12})}{R_{12}^2}} \quad ... (5)$$

Actually equation (5) is obtained from equation (4) by interchanging the subscripts 1 and 2. By using back-cab rule for expanding vector triple product, we can show that

$$\bar{F}_2 = -\bar{F}_1 \quad ... (6)$$

Thus, above condition indicates that both the forces $\bar{F}_1$ and $\bar{F}_2$ obey Newton's third law that for every action there is equal and opposite reaction.

For the two current carrying conductors of length l each, the force exerted is given by

$$\boxed{F = \frac{\mu I_1 I_2 l}{2\pi d}} \quad ... (7)$$

where I_1 and I_2 are the currents flowing through conductor 1 and conductor 2 and d is the distance of separation between two conductors.

If the two currents flow in same directions, the current carrying conductors attract each other. While if, the two currents flow in opposite direction to each other, the current carrying conductors repel each other.

Example 8.3 : *A current element, $I_1 \Delta\bar{L}_1 = 10^{-5}\,\bar{a}_z$ A.m is located at $P_1(1,0,0)$ while a second element, $I_2 \Delta\bar{L}_2 = 10^{-5}(0.6\,\bar{a}_x - 2\,\bar{a}_y + 3\,\bar{a}_z)$ A.m is at $P_2(-1,0,0)$, both in free space. Find the vector force exerted on $I_2 \Delta\bar{L}_2$ by $I_1 \Delta\bar{L}_1$.*

Solution : The magnetic field intensity at point P_1 due to $I_1 \Delta\bar{L}_1$ can be obtained using Biot-Savart's law as follows.

For the side BC, the lever arm extends from origin to the midpoint of the side BC. Thus the lever arm for side BC is given by,

$$\bar{R}_2 = \frac{1}{2} dx\ \bar{a}_x \quad ...(6)$$

Hence torque on side 2 is given by,

$$d\bar{T}_2 = \bar{R}_2 \times d\ F_2$$

$$= \frac{1}{2} dx\ \bar{a}_x \times I\ dy\ [B_{oz}\ \bar{a}_x - B_{ox}\ \bar{a}_z]$$

$$= +\frac{1}{2} dx\ dy\ I\ B_{ox}\ \bar{a}_y \quad ...(7)$$

For side CD, the torque contribution is exactly same as that by side AB. The torque on side 3 is given by,

$$d\bar{T}_3 = -\frac{1}{2} dx\ dy\ I\ B_{oy}\ \bar{a}_x \quad ...(8)$$

Similarly for DA, the torque contribution is exactly same as that by side BC. The torque on side 4 is thus given by,

$$d\bar{T}_4 = +\frac{1}{2} dx\ dy\ I\ B_{ox}\ \bar{a}_z \quad ...(9)$$

Hence the total torque is given by,

$$d\bar{T} = d\bar{T}_1 + d\bar{T}_2 + d\bar{T}_3 + d\bar{T}_4$$

$$\therefore \quad d\bar{T} = \left(-\frac{1}{2} dx\, dy\, I\, B_{oy}\ \bar{a}_x\right) + \left(+\frac{1}{2} dx\, dy\, I\, B_{ox}\ \bar{a}_y\right) + \left(-\frac{1}{2} dx\, dy\, I\, B_{oy}\ \bar{a}_x\right) + \left(+\frac{1}{2} dx\, dy\, I\, B_{ox}\ \bar{a}_y\right)$$

$$\therefore \quad d\bar{T} = -dx\, dy\, I\, B_{oy}\ \bar{a}_x + dx\, dy\, I\, B_{ox}\ \bar{a}_y$$

$$\therefore \quad d\bar{T} = I\, dx\, dy\, (B_{ox}\ \bar{a}_y - B_{oy}\ \bar{a}_x)$$

$$\therefore \quad d\bar{T} = I\, dx\, dy\, [\bar{a}_z \times (B_{ox}\ \bar{a}_x + B_{oy}\ \bar{a}_y + B_{oz}\ \bar{a}_z)]$$

$$\therefore \quad \boxed{d\bar{T} = I\, dx\, dy\, (\bar{a}_z \times \bar{B}_o)} \quad ...(10)$$

We can modify above equation by replacing the product term ie. dx dy by vector area of the differential current loop i.e. $d\bar{S}$.

$$\therefore \quad \boxed{d\bar{T} = I\, d\bar{S} \times \bar{B}} \quad ...(11)$$

Above equation indicates that even though the total force exerted on the rectangular loop as a whole is zero, the torque exsists along the axis of rotation. i.e. in the z-direction. The expression is valid for all the flat loops of any arbitrary shape.

8.5.2 Magnetic Dipole Moment

The **magnetic dipole moment** of a current loop is defined as the product of current through the loop and the area of the loop, directed normal to the current loop. From the definition it is clear that, the magnetic dipole moment is a vector quantity. It is denoted by $\bar{m}$. The direction of the magnetic dipole moment $\bar{m}$ is given by the right hand thumb rule. The right hand thumb indicates the direction of the unit vector in which $\bar{m}$ is directed and the figures represents the current direction. The **magnetic dipole moment** is given by

$$\boxed{\bar{m} = (IS)\bar{a}_n \quad A \cdot m^2} \qquad \text{... (12)}$$

In the previous section we have obtained the expression for the torque along the axis of rotation of a planar coil as,

$$\bar{T} = BIS\left(-\bar{a}_y\right)$$

Using definition for the magnetic dipole moment, the torque can be expressed as,

$$\boxed{\bar{T} = \bar{m} \times \bar{B} \quad Nm} \qquad \text{... (13)}$$

Above expression is in general applicable in calculating the overall torque on a planar loop of any arbitrary shape. But the basic requirement is that the magnetic field must be uniform. The torque is always in the direction of axis of rotation. When the planar loop or coil is normal to the mangetic field, the sum of the forces on the planar loop as well as the torque will be zero.

Example 8.4 : *A rectangular coil as shown in the Fig. 8.7 is in the magnetic field given by* $\bar{B} = 0.05 \dfrac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}$ *T*

Find the torque about z-axis when the coil is in the position shown and carries a current of 5 A.

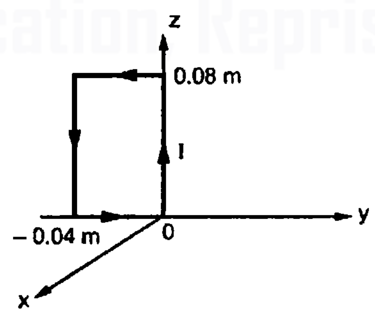


Fig. 8.7

Solution : The magnetic dipole moment is given by,

$$\bar{m} = I\,S\,\bar{a}_n$$

$$\text{where } S = \text{Area of coil} = (0.08) \times (0.04) = 3.2 \times 10^{-3} \text{ m}^2$$

The angular momentum of an electron is called **spin** of the electron. As electron is a charged particle, the spin of the electron produces magnetic dipole moment. In an atom with completely filled orbits the contribution in spin magnetic moment is zero. In other words, the spins of the electrons in incompletely filled shells contribute more in the resultant spin magnetic moment.

Similar to the electro spin, the nuclear spin contributes to the magnetic moment called nuclear spin magnetic moment. The mass of the nucleus is much larger than an electron. Thus the dipole moments due to the nuclear spin are very small. The contribution of nuclear magnetic moment to the magnetic properties of materials is negligible.

The total magnetic dipole moment of an atom can be calculated by summing up all the above mentioned magnetic dipole moments in appropriate manner.

8.6.2 Classification of Magnetic Materials

According to the previous discussion, it is clear that the characteristics of the magnetic materials are decided by the different components of moments and also their summations. On the basis of the magnetic behaviour, the magnentic materials are classified as diamagnetic, paramagnetic, ferromagnetic, antiferomagnetic, ferrimagnetic and supermagnentic.

The magnetic materials in which the orbital magnetic moment and electron spin magnetic moment cancel each other making net permanent magnetic moment of each atom zero are called **diamagnetic** materials. Thus with no external field, in diamagnetic materials the net torque produced on atom is zero with no effective realignment of magnetic moment. But an applied field makes spin moment slightly greater than that of orbital moment. This results in small magnetic moment which opposes the applied field. Hence when a diamagnetic material like bismuth is kept near either pole of a strong magnet gets repelled. Other examples of diamagnetic materials are lead, copper, silicon, diamond, graphite, sulphur, sodium chloride and inert gases.

The magnetic materials in which the orbital and spin magnetic moments do not cancel each other resulting in a net magnetic moment of an atom are called **paramagnetic** materials. In the paramagnetic materials, atoms are oriented randomly. In the absence of an external field, the paramagnetic materials do not show any magnetic effect. But when an external field is applied, each atomic dipole moment experiences a torque. Due to this, all the atomic dipole moments tend to align with the external field. Thus inside material, value of the field increases than the value of the external field if the perfect alignment of the dipole moments is achieved. When the paramagnetic material is kept near the pole of a strong magnets, it gets attracted. The common examples of paramagnetic materials are potassium, tungston, oxygen, rare earth metals.

The materials in which the atoms have large dipole moment due to electron spin magnetic moments are called **ferromagnetic** materials. In the ferromagnetic materials, the adjacent atoms line up their magnetic dipole moments in parallel fashion in the lattice. The regions in which large number of magnetic moments lined in parallel are called **domains**.

When an external field is applied, the domains increase their size increasing internal field to a high value. When the external field is removed, the original random alignment of dipole moments is not achieved. Some of the moments remain in a small region which results in **residual** field or **remanant** field. This effect is called **hysteresis**. Iron, nickel and cobalt are the examples of ferromagnetic materials.

The behaviour of a ferromagnetic material can be represented interms of magnetic domains. A magnetic domain is a small region in which all the magnetic dipoles are perfectly alligned as shown in the Fig. 8.8. But the direction of allignment of the dipoles vary from domain to domain. Hence this virgin material is said to be in non-magnetized state.

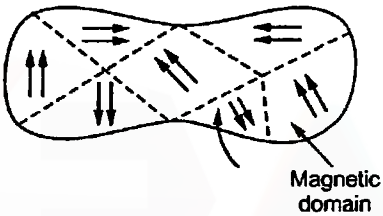


Fig. 8.8 Random orientation of the magnetic dipoles in ferromagnetic material under nonmagnetized state

When a current carrying wire is wound around the magnetic material, a magnetic field is produced. So when the magnetic material is placed in an external magnetic field, all the magnetic dipoles will try to allign in the direction of magnetic field. The domains in the material, which are already in the direction of magnetic field, grow in size at the cost of neighbouring domains. The remaining domains rotate their dipoles in the direction of magnetic field. Thus magnetic flux density within the magnetic material increases.

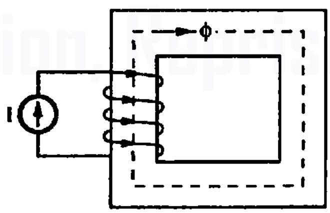
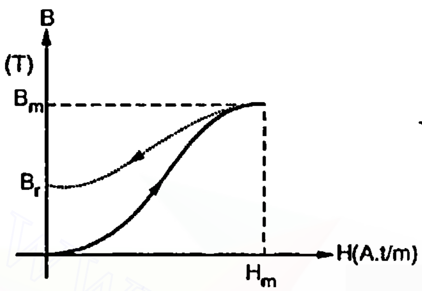


Fig. 8.9 Magnetic flux due to current carrying coil

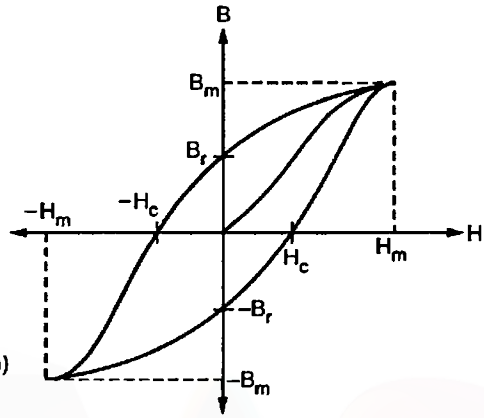
Due to the current in wire, $\bar{H}$ is produced in material. The applied field $\bar{H}$ produces $\bar{B}$ field within the medium. The movement of the domain walls is reversible till the $\bar{B}$ field within medium is weak.

When the current in wire is increased, the $\bar{H}$ field increases and $\bar{B}$ field becomes stronger and stronger. This is due to more number of magnetic dipoles align with the $\bar{B}$ field. Now if we measure $\bar{B}$ field, it is observed that initially $\bar{B}$ field increases slowly, then

it increases rapidly. Then again slowly $\bar{B}$ field increases and then it flattens off finally as shown in the Fig. 8.10 (a).



(a) Magnetization characteristics of a ferromagnetic material



(b) Hysteresis loop

Fig. 8.10 Magnetization characteristic and hysteresis loop of the ferromagnetic material

The changes in $\bar{B}$ are due to the changes in $\bar{M}$ is magnetization. The flattened region indicate that almost all the magnetic dipoles are alligned themselves in the direction of $\bar{B}$ field.

Now if we lower the $\bar{H}$ field by decreasing current in wire, the $\bar{B}$ field does not follow the same path but it slowly decreases as shown by the dashed path in the Fig. 8.10 (a). Thus it is clear that even though the $\bar{H}$ field becomes zero, there exists certain magnetic field density in material. So it is called **residual flux density** or **remanent flux density.** It is denoted by $\overline{B_r}$. The magnetic material which retains high residual flux density is called **hard magnetic material.**

When the direction of the current through wire is reversed, the flux density $\bar{B}$ becomes zero at a certain value of $\bar{H}$ in opposite direction. The value of $\bar{H}$ to make $\bar{B}$ zero is called **coercive force** denoted by $\overline{H_c}$. By increasing and decreasing $\bar{H}$ field in both the directions, we get a loop which is called **hystersis loop.** The area of this hysteresis loop determines the loss in the energy per cycle which is known as **hysteresis loop.** Thus to minimise the hysteresis loss, the area of the hysteresis loop should be small. That means the residual flux density of the material should be as small as possible. The material with small values of the residual flux density is called **soft material.**

The materials in which the dipole moments of adjacent atoms line up in **antiparallel** fashion are called **antiferromagnetic** materials. The net magnetic moment in such materials is zero. Thus when a specimen of antiferromagnetic material is kept near a strong magnet gets neither attracted nor repelled. This property is observed in materials like many of the oxides, chlorides and sulphides at low termperatures.

Let the bound current I_b flows through a closed path. Assume that this closed path encloses a differential area $d\bar{S}$. Then the magnetic dipole moment is given by

$$\bar{m} = I_b d\bar{S} \quad \text{... (1)}$$

Now consider a differential volume ΔV. Assume that there are n magnetic dipoles per unit volume. The total magnetic dipole moment can be obtained summing up all the individual magnetic dipole moment of each magnentic dipole. Note that, we must add all these moment vectorically.

$$\therefore \quad \bar{m}_{total} = \sum_{a=1}^{n\Delta v} \bar{m}_a \quad \text{... (2)}$$

The **magnetization** is defined as the magnetic dipole moment per unit volume. Its unit is A/m.

$$\therefore \quad \boxed{\bar{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{a=1}^{n\Delta v} \bar{m}_a} \quad \text{... (3)}$$

Consider a differential volume ∇v. When external field is not applied to the material, there is random orientation of the magnetic dipole moments as shown in the Fig. 8.12 (a). Thus the total sum of the magnetic dipole moments is zero. Thus the magnetization $\bar{M}$ is also zero. With the application of an external field $\bar{B}$, the magnetic moments of electrons tend to aling with $\bar{B}$ on there own such that net magnetic moment is not equal to zero. Refer Fig. 8.12 (b).

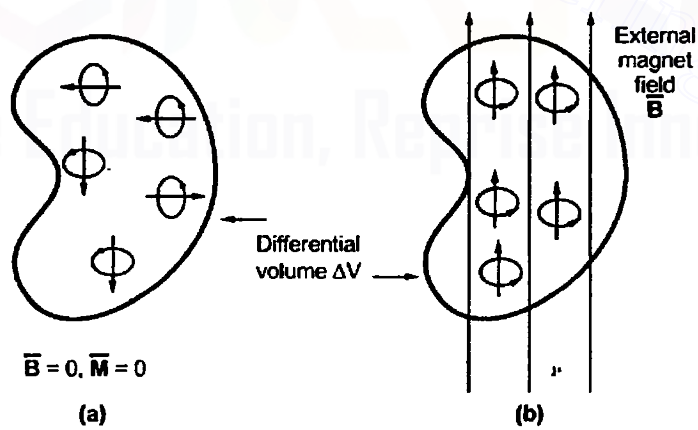


Fig. 8.12 Magnetic dipole moments in differential volume ∇v (a) with no external field ($\bar{B} = 0$) (b) with external field applied.

Let us consider alignment of a magnetic dipole along a closed path as shown in the Fig. 8.13.

$$\therefore \qquad I = \oint \left(\frac{\bar{B}}{\mu_0} - \bar{M}\right) \cdot d\bar{L} \qquad \text{... (8)}$$

Compare this equation with the expression of Ampere's circuital law given by,

$$I = \oint \bar{H} \cdot d\bar{L}$$

We can write the relationship between $\bar{B}$, $\bar{H}$ and $\bar{M}$ as,

$$\bar{H} = \frac{\bar{B}}{\mu_0} - \bar{M} \qquad \text{... (9)}$$

or

$$\boxed{\bar{B} = \mu_0(\bar{H} + \bar{M})} \qquad \text{... (10)}$$

The relationship in equation (10) is true for all the materials irrespective of the nature of material whether it is linear or not.

For linear, isotropic magnetic materials,

$$\boxed{\bar{M} = \chi_m \bar{H}} \qquad \text{... (11)}$$

The quantity χ_m is dimentionless and is called **magnetic susceptibility** of the medium. Thus the magnetic susceptibility measures how susceptible the material is to a magnetic field.

Substituting value of $\bar{M}$, from equation (11), in equation (10),

$$\bar{B} = \mu_0(\bar{H} + \chi_m \bar{H}) = \mu_0(1 + \chi_m)\bar{H} \qquad \text{... (12)}$$

But for any magnetic material we can write

$$\bar{B} = \mu \bar{H} = \mu_0 \mu_r \bar{H} \qquad \text{... (13)}$$

Comparing equations (12) and (13), the relative permeability can be expressed interms of magnetic susceptibility as

$$\boxed{\mu_r = (1 + \chi_m) = \frac{\mu}{\mu_0}} \qquad \text{... (14)}$$

In general $\mu = \mu_0 \mu_r$ is called **permeability** of a material. It is measured in henry/meter (H/m). But the relative permeability is a dimensionless quantity similar to the magnetic susceptibility.

Consider again the expressions for the currents. These currents can be expressed interms of the current densities. Let $\bar{J}_T$ be the total current density, $\bar{J}_b$ be the bound current density and $\bar{J}$ be the free current density. Then we can write the expressions for currents as,

$$I_b = \oint_S \bar{J}_b \cdot d\bar{S},$$

c)
$$n = 8.2\times10^{28} \text{ atoms/m}^3$$
$$m = 5\times10^{-27} \text{ A}\cdot\text{m}^2$$

The magnetization is given by,

$$M = (n)(m) = (8.2\times10^{28})(5\times10^{-27}) = 410 \text{ A/m}$$

The magnetic field intensity is given by,

$$H = \frac{M}{\mu_r - 1} = \frac{410}{30-1} = 14.1379 \text{ A/m}$$

Example 8.6 : *In certain region, the magnetic flux density in a magnetic material with $\chi_m = 6$ is given as $\bar{B} = 0.005\,y^2\,\bar{a}_x$ T. At y = 0.4 m, find the magnitude of ;*
(a) $\bar{J}$ (b) $\bar{J}_b$ and (c) $\bar{J}_T$.

Solution : a)
$$\bar{J} = \nabla\times\bar{H} \quad \text{... (1)}$$

But
$$\bar{H} = \frac{\bar{B}}{\mu} = \frac{\bar{B}}{\mu_0\mu_r} = \frac{\bar{B}}{\mu_0(\chi_m+1)}$$

Putting value of $\bar{H}$ in equation (1)

$$\therefore \quad \bar{J} = \nabla\times\frac{\bar{B}}{\mu_0(\chi_m+1)} = \frac{1}{\mu_0(\chi_m+1)}(\nabla\times\bar{B})$$

$$\therefore \quad \nabla\times\bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0.005\,y^2 & 0 & 0 \end{vmatrix}$$

$$= [0-0]\bar{a}_x - \left[0-\frac{\partial}{\partial z}(0.005\,y^2)\right]\bar{a}_y + \left[0-\frac{\partial}{\partial y}(0.005\,y^2)\right]\bar{a}_z$$

$$= -(0.01\ y)\,\bar{a}_z$$

$$\therefore \quad \bar{J} = \frac{1}{\mu_0(\chi_m+1)}[-0.01\ y\bar{a}_z]$$

$$\therefore \quad \bar{J} = \frac{-0.01\,y}{4\times\pi\times10^{-7}(6+1)}\,\bar{a}_z$$

Calculating value of $\bar{J}$ at y = 0.4 m.

$$\therefore \quad \bar{J} = -454.7284\,\bar{a}_z \ \text{A/m}^2$$

The magnitude of $\bar{J}$ is 454.7284 A/m²

8.8 Magnetic Boundary Conditions

The conditions of the magnetic field existing at the boundary of the two media when the magnetic field passes from one medium to other are called **boundary conditions** for magnetic fields or simply **magnetic boundary conditions.** When we consider magnetic boundary conditions, the conditions of $\bar{B}$ and $\bar{H}$ are studied at the boundary. The boundary between the two different magnetic materials is considered. To study conditions of $\bar{B}$ and $\bar{H}$ at the boundary, both the vectors are resolved into two components ;

a) Tangential to boundary and

b) Normal (perpendicular) to boundary.

Consider a boundary between two isotropic, homogeneous linear materials with different permeabilities μ_1 and μ_2 as shown in the Fig. 8.14. To determine the boundary conditions, let us use the closed path and the Gaussian surface.

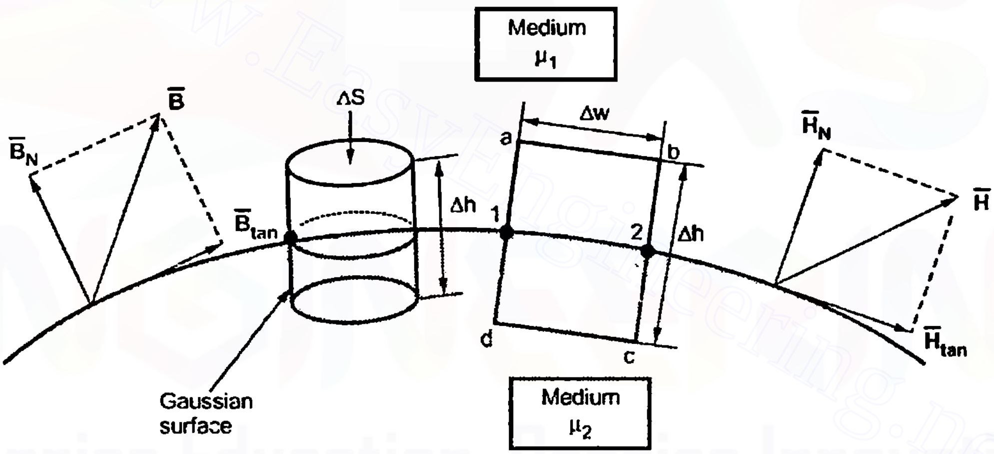


Fig. 8.14 Boundary between two magnetic materials of different permeabilities

8.8.1 Boundary Conditions for Normal Component

To find the normal component of $\bar{B}$, select a closed Gaussian surface in the form of a right circular cylinder as shown in the Fig. 8.14. Let the height of the cylinder be Δh and be placed in such a way that $\Delta h/2$ is in medium 1 and remaining $\Delta h/2$ is in medium 2. Also the axis of the cylinder is in the normal direction to the surface.

According to the Gauss's law for the magnetic field,

$$\oint_S \overline{B} \cdot d\overline{S} = 0 \qquad \text{... (1)}$$

The surface integral must be evaluated over three surfaces, (i) Top, (ii) Bottom and (iii) Lateral.

Let the area of the top and bottom is same, equal to ΔS.

$$\therefore \oint_{top} \overline{B} \cdot d\overline{S} + \oint_{bottom} \overline{B} \cdot d\overline{S} + \oint_{lateral} \overline{B} \cdot d\overline{S} = 0 \qquad \text{... (2)}$$

As we are very much intetrested in the boundary conditions, reduce Δh to zero. As $\Delta h \rightarrow 0$, the cylinder tends to boundary and only top and bottom surfaces contribute in the surface integral. Thus surface integrals are calculated for top and bottom surfaces only. These surfaces are very small. Let the magnitude of normal component of $\overline{B}$ be B_{N_1} and B_{N_2} in medium 1 and medium 2 respectively. As both the surfaces are very small, we can assume B_{N_1} and B_{N_2} constant over their surfaces. Hence we can write,

For top surfaces :

$$\oint_{Top} \overline{B} \cdot d\overline{S} = B_{N_1} \oint_{Top} d\overline{S} = B_{N_1} \Delta S \qquad \text{... (3)}$$

For bottom surface :

$$\oint_{Bottom} \overline{B} \cdot d\overline{S} = B_{N_2} \oint_{Bottom} d\overline{S} = B_{N_2} \Delta S \qquad \text{... (4)}$$

For lateral surface

$$\oint_{Lateral} \overline{B} \cdot d\overline{S} = 0 \qquad \text{... (5)}$$

Putting values of surface integrals in equation (2), we get

$$B_{N_1} \Delta S - B_{N_2} \Delta S = 0 \qquad \text{... (6)}$$

Note that the negative sign is used for one of the surface integrals because normal component in medium 2 is entering the surface while in medium 1 the component is leaving the surface. Hence B_{N_1} and B_{N_2} are in opposite direction.

From equation (6), we can write,

$$B_{N_1} \Delta S = B_{N_2} \Delta S$$

i.e.

$$\boxed{B_{N_1} = B_{N_2}} \qquad \text{... (7)}$$

Thus the normal component of $\overline{B}$ is continuous at the boundary.

As the magnetic flux density and the magnetic field intensity are related by

$$\overline{B} = \mu \overline{H}$$

Thus, equation (7) can be written as,

$$\mu_1 H_{N_1} = \mu_2 H_{N_2}$$

$$\therefore \quad \boxed{\frac{H_{N_1}}{H_{N_2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_{r2}}{\mu_{r1}}} \quad \text{... (8)}$$

Hence the normal component of $\bar{H}$ is not continuous at the boundary. The field strengths in two media are inversely proportional to their relative permeabilities.

8.8.2 Boundary Conditions for Tangential Component

According to Ampere's circuital law,

$$\oint \bar{H} \cdot d\bar{L} = I \quad \text{... (9)}$$

Consider a rectangular closed path abcda as shown in the Fig. 8.13. It is traced in clockwise direction as a-b-c-d-a. This closed path is placed in a plane normal to the boundary surface. Hence $\oint \bar{H} \cdot d\bar{L}$ can be divided into 6 parts.

$$\oint \bar{H} \cdot d\bar{L} = \int_a^b \bar{H} \cdot d\bar{L} + \int_b^1 \bar{H} \cdot d\bar{L} + \int_1^c \bar{H} \cdot d\bar{L} + \int_c^d \bar{H} \cdot d\bar{L} + \int_d^2 \bar{H} \cdot d\bar{L} + \int_2^a \bar{H} \cdot d\bar{L} = I \quad \text{... (10)}$$

From the Fig. 8.14 it is clear that, the closed path is placed in such a way that its two sides a-b and e-d are parallel to the tangential direction to the surface while the other two sides are normal to the surface at the boundary. This closed path is placed in such a way that half of its portion is in medium 1 and the remaining is in medium 2. The rectangular path is an elementary rectangular path with elementary height Δh and elementary width Δw. Thus over small width Δw, $\bar{H}$ can be assume constant say H_{tan1} in medium 1 and H_{tan2} in medium 2. Similarly over a small height $\frac{\Delta h}{2}$, $\bar{H}$ can be assumed constant say H_{N1} in medium 1 and H_{N2} in medium 2. Now assume that $\bar{K}$ is the surface current normal to the path. Also from the Fig. 8.14 it is clear that the normal and tangential components in medium 1 and medium 2 are in opposite direction. Thus equation (10) can be written as,

$$K \cdot dw = H_{tan1}(\Delta w) + H_{N1}\left(\frac{\Delta h}{2}\right) + H_{N2}\left(\frac{\Delta h}{2}\right) - H_{tan2}(\Delta w)$$

$$- H_{N2}\left(\frac{\Delta h}{2}\right) - H_{N1}\left(\frac{\Delta h}{2}\right) \quad \text{... (11)}$$

To get conditions at boundary, $\Delta h \to 0$. Thus,

$$K \cdot dw = H_{tan1}(\Delta w) - H_{tan2}(\Delta w)$$

$$H_{tan1} - H_{tan2} = K \quad \text{... (12)}$$

In vector form, we can express above relation by a cross product as

$$\boxed{\bar{H}_{tan1} - \bar{H}_{tan2} = \bar{a}_{N_{12}} \times \bar{K}} \quad \text{... (13)}$$

where $\bar{a}_{N_{12}}$ is the unit vector in the direction normal at the boundary from medium 1 to medium 2.

For $\bar{B}$, the tangential components can be related with permeabilities of two media using equation (12),

$$\therefore \quad \frac{B_{tan1}}{\mu_1} - \frac{B_{tan2}}{\mu_2} = K \quad \text{... (14)}$$

Consider a special case that the boundary is free of current. In other words, media are not conductors; so K = 0. Then equation (12) becomes

$$H_{tan1} - H_{tan2} = 0$$

$$\text{or} \quad H_{tan1} = H_{tan2} \quad \text{... (15)}$$

For tangential components of $\bar{B}$ we can write,

$$\frac{B_{tan1}}{\mu_1} - \frac{B_{tan2}}{\mu_2} = 0$$

$$\therefore \quad \frac{B_{tan1}}{\mu_1} = \frac{B_{tan2}}{\mu_2}$$

$$\therefore \quad \boxed{\frac{B_{tan1}}{B_{tan2}} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}} \quad \text{... (16)}$$

From equations (15) and (16) it is clear that tangential component of $\bar{H}$ are continuous, while tangential component of $\bar{B}$ are discontinuous at the boundary, with the condition that the boundary is current free.

Let the fields make angles α_1 and α_2 with the normal to the interface as shown in the Fig. 8.15.

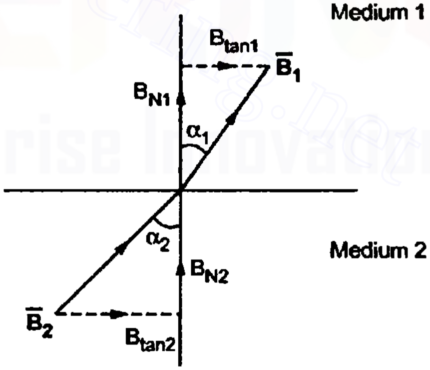


Fig. 8.15 Component of $\bar{B}$ at boundary

Interms of angle α_1 and α_2, we can write relationship between normal components and tangential components of $\bar{B}$.

In medium 1,

$$\tan \alpha_1 = \frac{B_{tan1}}{B_{N1}} \quad \text{... (17)}$$

Similarly in medium 2,

$$\tan \alpha_2 = \frac{B_{tan2}}{B_{N2}} \quad \text{... (18)}$$

Dividing equation (17) by equation (18)

$$\therefore \quad \frac{\tan \alpha_1}{\tan \alpha_2} = \frac{B_{tan1}}{B_{N1}} \cdot \frac{B_{N2}}{B_{tan2}}$$

As we know, $B_{N1} = B_{N2}$,

$$\boxed{\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{B_{tan1}}{B_{tan1}} = \frac{\mu_{r1}}{\mu_{r2}}} \quad \text{... (19)}$$

Consider an interface between air (medium 1) and soft iron (medium 2) For air, $\mu_{r1} = 1$. For soft iron, let $\mu_{r2} = 7000$. Then

$$\frac{B_{tan1}}{B_{tan2}} = \frac{\tan \alpha_1}{\tan \alpha_2} = \frac{1}{7000}$$

If $\alpha_2 = 85°$, then $\alpha_1 = 0.093°$ and $B_{tan1} = 0$.

Thus practically when fields cross a medium of high μ_r to low μ_r, then the magnetic fields $\bar{B}$ and $\bar{H}$ are always **perpendicular to the boundary.**

Example 8.7 : *In region 1, as shown in the Fig. 8.16.*

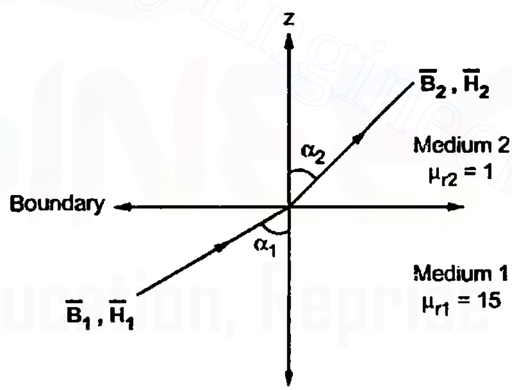


Fig. 8.16

$$\bar{B}_1 = 1.2\bar{a}_x + 0.8\bar{a}_y + 0.4\bar{a}_z \ T$$

Determine $\bar{B}_2$ and $\bar{H}_2$ in other medium and also calculate the angles made by the fields with the normal.

Solution : Assume that the boundary is current free.

In medium 1,

$$\bar{H}_1 = \frac{\bar{B}_1}{\mu_1} = \frac{\bar{B}_1}{\mu_0 \mu_{r1}} = \frac{1}{\mu_0}\left[\frac{1.2\bar{a}_x + 0.8\bar{a}_y + 0.4\bar{a}_z}{15}\right]$$

$$\therefore \quad \bar{H}_1 = \frac{1}{\mu_0}\left[0.08\bar{a}_x + 0.0533\bar{a}_y + 0.0266\bar{a}_z\right] \text{ A/m} \quad \text{... (1)}$$

As per the boundary shown in the Fig. 8.16, x and y are tangential, while z component is normal. So according to the boundary conditions for current free boundary, the tangential component for $\bar{H}_1$ and $\bar{H}_2$ remain same. The normal component can be calculated as

$$\frac{H_{N1}}{H_{N2}} = \frac{\mu_{r2}}{\mu_{r1}}$$

$$\therefore \quad H_{N2} = \frac{\mu_{r1}}{\mu_{r2}} H_{N1}$$

From equation (1), the normal component i.e. component in z-direction is

$$H_{N1} = 0.0266$$

$$\therefore \quad H_{N1} = \frac{15}{1}(0.0266) = 0.399 \approx 0.4 \text{ A/m} \quad \text{... (2)}$$

Hence in medium 2,

$$H_{tan2\,x} = H_{tan1\,x} = 0.08,$$

$$H_{tan2\,y} = H_{tan1\,y} = 0.0533,$$

$$H_{N2} = 0.4$$

$$\therefore \quad \bar{H}_2 = \frac{1}{\mu_0}\left[0.08\,\bar{a}_x + 0.0533\,\bar{a}_y + 0.4\,\bar{a}_z\right] \text{ A/m} \quad \text{... (3)}$$

Then the magnetic flux density in medium 2 is given by

$$\bar{B}_2 = \mu_2\,\bar{H}_2 = (\mu_0\mu_{r2})\bar{H}_2$$

$$\bar{B}_2 = \mu_0\left[\frac{1}{\mu_0}\left(0.08\bar{a}_x + 0.0533\bar{a}_y + 0.4\bar{a}_z\right)\right]$$

$$\therefore \quad \bar{B}_2 = 0.08\,\bar{a}_x + 0.0533\,\bar{a}_y + 0.4\,\bar{a}_z \text{ T} \quad \text{... (4)}$$

As z-direction is perpendicular to boundary, in medium 1, we can write,

$$\bar{B}_1 \cdot \bar{a}_z = |\bar{B}_1|\,|\bar{a}_z|\cos\alpha_1$$

$$\therefore \left(1.2\bar{a}_x + 0.8\bar{a}_y + 0.4\bar{a}_z\right)\cdot\bar{a}_z = \sqrt{(1.2)^2 + (0.8)^2 + (0.4)^2}\,(1)(\cos\alpha_1)$$

$$\therefore \quad 0.4 = (1.4966)\;(\cos\alpha_1) \quad \text{... } \bar{a}_x\cdot\bar{a}_x = \bar{a}_y\cdot\bar{a}_z = 0$$

$$\therefore \quad \alpha_1 \doteq 74.49°$$

Similarly in medium 2,

$$\therefore \qquad \cos\alpha_1 = \frac{24}{(49.6814)} = 0.483$$

$$\therefore \qquad \alpha_1 = 61.12°$$

Thus the angle made by B_1 with tangent to the interface is given by,

$$\theta_1 = 90° - \alpha_1 = 90° - 61.12° = 28.88°$$

The angle made by $\bar{B}_2$ with normal i.e. $\bar{a}_z$ is given by,

$$\bar{B}_2 \cdot \bar{a}_z = |\bar{B}_2||\bar{a}_z| \cos\alpha_2$$

$$\therefore (22\bar{a}_x + 24\bar{a}_z)\cdot\bar{a}_z = \sqrt{(22)^2 + (24)^2}\,(1)\cos\alpha_2$$

$$\therefore \qquad \cos\alpha_2 = \frac{24}{(32.5576)} = 0.7371$$

$$\therefore \qquad \alpha_2 = 42.51°$$

Thus the angle made by $\bar{B}_2$ with tangent to the interface is given by

$$\theta_2 = 90° - \alpha_2 = 90° - 42.51° = 47.49°$$

$$\therefore \qquad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\tan(28.88°)}{\tan(47.49°)} = \frac{0.5515}{1.0909} = 0.5055$$

8.9 Magnetic Circuits

In general, in magnetic circuits, we determine the magnetic fluxes and magnetic field intensities in various parts of the circuits. The magnetic circuits are analogous to the electric circuits. If we study this analogy between the electric and magnetic circuits, we can achieve simple techniques for analysis of the magnetic circuits. Also we can directly use the concepts of the electric circuits in solving the magnetic circuit problems.

The common examples of the magnetic circuits are transformers, toroids motors, generators, relays and magnetic recording devices.

Let us study analogy between the electric and magnetic circuits. An electric circuit forms a circuit (i.e. closed path) through which current can flow. Similar to this magnetic lines of flux are continuous and can form closed paths. So a single magnetic line of flux or all parallel magnetic lines of flux may be considered as magnetic circuit.

Similar to **electromotive force** (e.m.f.) in an electric circuit, we can define a new quantity in case of a magnetic circuit called **magnetomotive force** (m.m.f.). The magnetomotive force (m.m.f.) is defined as,

$$e_m = NI = \oint \bar{H}\cdot d\bar{L} \qquad \text{... (1)}$$

The SI unit of m.m.f. is ampere (A). But generally, in magnetic circuits, the source of m.m.f. is a coil carrying conductors with N number of turns as shown in the Fig. 8.15 (c). Thus m.m.f. is measured in ampere-turn (A-t) very often. Note that m.m.f. is not a force measured in newton.

In an electric circuit, resistance is defined as the ratio of voltage to current given by

$$R = \frac{V}{I}$$

In case of analogous magnetic circuits, we define a new quantity **reluctance** ($\Re$) as the ratio of the magnetomotive force to the total flux.

$$\boxed{\Re = \frac{e_m}{\phi}} \qquad \text{... (2)}$$

The reluctance is measured in $\dfrac{\text{Ampere} \cdot \text{turn}}{\text{Weber}}$.

The resistance in electric circuit can be expressed interms of conductivity σ as

$$R = \frac{l}{\sigma S}$$

where l = Length in m

S = Cross-sectional area in m^2

σ = Conductivity of the linear isotropic homogeneous material.

In case of magnetic circuits, we can define reluctance in very much similar way as,

$$\boxed{\Re = \frac{l}{\mu S}} \qquad \text{... (3)}$$

where μ = Permeability of the isotropic, linear homogeneous material

For electric circuit, Ohm's law can be expressed in point from as,

$$\bar{J} = \sigma \bar{E}$$

Now consider magnetic circuit. The magnetic flux density is analogous to the current density, thus we can write,

$$\bar{B} = \mu \bar{H} \qquad \text{... (4)}$$

The basic equations derived in magnetostatics are very much helpful in the analysis of the magnetic circuits. These basic equations are

$$\nabla \cdot \bar{B} = 0 \text{, and} \qquad \text{... (5)}$$

$$\nabla \times \bar{H} = \bar{J} \qquad \text{... (6)}$$

In other form, the two equations can be expressed in terms of the total current flowing in the magnetic circuit and the total magnetic flux density through cross-section of the magnetic circuit.

The total current in the magnetic circuit is given by,

$$I = \int_S \bar{J} \cdot d\bar{S} \quad \text{... (7)}$$

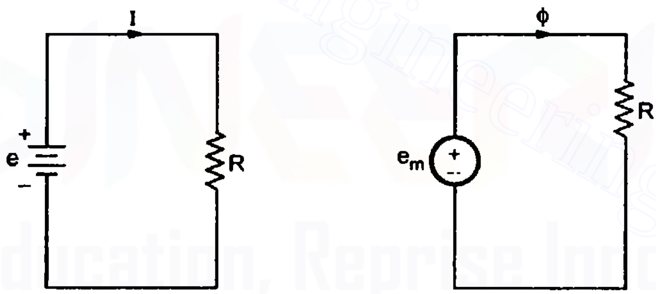
The total magnetic flux density flowing through the cross-section of the magnetic circuit is given by

$$\phi = \int_S \bar{B} \cdot d\bar{S} \quad \text{... (8)}$$

In electric circuit, the reciprocal of the resistance is called conductance. In magnetic circuits, the reciprocal of the reluctance is called permeance denoted by $\mathcal{P}$. The permeance is measured in henries (H).

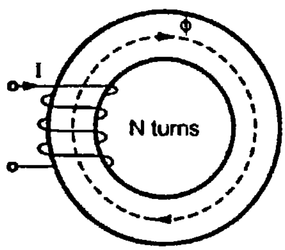
$$\therefore \quad \boxed{\mathcal{P} = \frac{\mu S}{l}} \quad \text{... (9)}$$

The analogous electric and magnetic circuits can be represented in simple way as shown in the Fig. 8.18.



(a) Electric circuit

(b) Magnetic circuit



(c) Coil of N turns carrying current

Fig. 8.18 Analogous electric and magnetic circuits

4.	Conductivity σ	Permeability μ
5.	Field intensity E	Field intensity H
6.	Current density, $J = \frac{I}{S} \sigma E \text{ A/m}^2$	Flux density, $B = \frac{\phi}{S} = \mu H \text{ Wb/m}^2$
7.	Reciprocal of resistance is conductance (G)	Reciprocal of reluctance is permeance ($\mathcal{P}$)
8.	Ohm's law : $e = I R$	Ohm's law : $e_m = \phi R$
9.	Kirchhoff's laws : $\sum I = 0$ $\sum \text{E.M.F.} = 0$	Kirchhoff's laws : $\sum \phi = 0$ $\sum \text{M.M.F.} = \sum \phi S = \sum H . l$

Table 8.1 Similarities between the electric and magnetic circuits

The dis-similarities between the electric and magnetic circuits are given in Table 8.2

Sr.No.	Electric Circuit	Magnetic Circuit
1.	In the electric circuit the current actually flows i.e. there is a movement of electrons.	Due to the magnetomotive force, flux gets established and does not flow in the sense in which current flows.
2.	The energy must be supplied to the electric circuit to maintain the flow of current.	The energy is required to create the magnetic flux, but is not required to maintain it.
3.	The resistance and the conductivity are independent of current density under constant temperature ; but may change due to the temperature.	The reluctance, permeance and the permeability are all dependent on the flux density.
4.	The electric lines of flux are not closed. They start from positive charge and end on negative charge.	The magnetic lines of flux are closed lines.

Table 8.2 Dissimilarities between the electric and magnetic circuits

8.10 Inductance and Mutual Inductance

When a coil with N turns, carrying current I, the flux is produced by it. This flux links with each turn of the coil. Thus total flux linkage of the coil having N turn $N\phi$ Wb-turns (Wb · t).

The flux linked with the coil is proportional to the current I flowing through it. The ratio of total flux linkage to the current producing that flux is called inductance denoted by L. It is measured in henry (H).

Thus the inductance of a solenoid is given by

$$L = \frac{\text{Total flux linkage}}{\text{Total current}}$$

$$\therefore \quad L = \frac{\mu N^2 I A}{l(I)}$$

$$\therefore \quad \boxed{L = \frac{\mu N^2 A}{l} \quad H} \quad \dots (4)$$

Example 8.9 : *Calculate the inductance of a solenoid of 200 turns wound tightly on a cylindrical tube of 6 cm diameter. The length of the tube is 60 cm and the solenoid is in air*

Solution : For a given solenoid in air,

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ Wb/A.m}$$

$$N = 200$$

$$d = 6 \text{ cm} = 6 \times 10^{-2} \text{m} \quad \text{hence } r = \frac{d}{2} = 3 \times 10^{-2} \text{ m}$$

$$l = 60 \text{ cm} = 60 \times 10^{-2} \text{ m}$$

The inductance of a solenoid is given by,

$$L = \frac{\mu N^2 A}{l}$$

$$= \frac{\mu_0 N^2 (\pi r^2)}{l}$$

$$= \frac{4 \times \pi \times 10^{-7} \times (200)^2 \times \pi \times (3 \times 10^{-2})^2}{60 \times 10^{-2}}$$

$$= 2.3687 \times 10^{-4} \text{ H}$$

$$= \mathbf{0.2368 \text{ mH}}$$

8.10.2 Inductance of a Toroid

Consider a toroidal ring with N turns and carrying current I. Let the radius of the toroid be R as shown in the Fig. 8.21.

The magnetic flux density inside a toroidal ring is given by,

$$B = \frac{\mu N I}{2\pi R} \quad \dots (5)$$

$$\boxed{L = \frac{\mu N^2 h}{2\pi} ln\left(\frac{r_2}{r_1}\right) \text{ H}} \quad \text{... (7)}$$

Example 8.10 : *A coil of 500 turns is wound on a closed iron ring of mean radius 10 cm and cross-section area of 3 cm^2. Find the self inductance of the winding if the relative permeability of iron is 800.*

Solution : For a toroidal ring of iron,

$$\mu_r = 800$$

$$A = 3 \text{ cm}^2 = 3\times10^{-4} \text{ m}^2$$

$$R = 10 \text{ cm} = 10\times10^{-2} \text{ m}$$

$$N = 500$$

The inductance of a iron toroid is given by

$$L = \frac{\mu N^2 A}{2\pi R} = \frac{(\mu_0 \mu_r) N^2 A}{2\pi R}$$

$$\therefore \quad L = \frac{\left(4\times\pi\times10^{-7}\times800\right)(500)^2\left(3\times10^{-4}\right)}{\left(2\times\pi\times10\times10^{-2}\right)}$$

$$\therefore \quad L = 0.12 \text{ H} = \mathbf{120 \text{ mH}}$$

8.10.3 Inductance of a Co-axial Cable

Consider a co-axial cable with inner conductor radius a and outer conductor radius b as shown in the Fig. 8.22. Let the current through the coaxial cable be I.

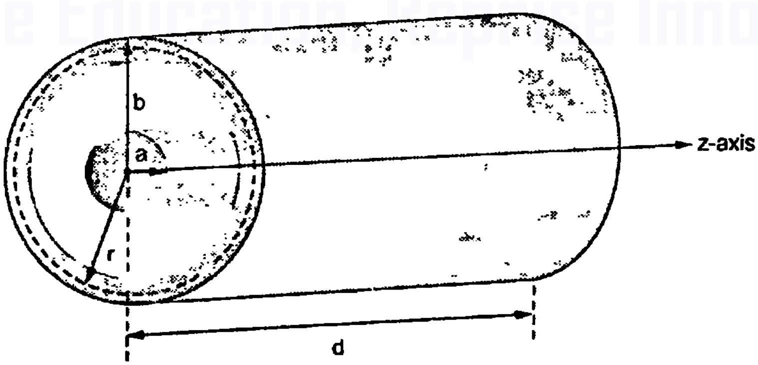


Fig. 8.22 Coaxial cable carrying current I

For the co-axial cable the field intensity at any point between inner and outer conductors is given by,

Examples with Solutions

Example 8.14 : *A point charge, Q = – 60 nC, is moving with a velocity 6×10^6 m/s in the direction specified by unit vector – 0.48 $\bar{a}_x$ – 0.6 $\bar{a}_y$ + 0.64 $\bar{a}_z$. Find the magnitude of the force on a moving charge in the magnetic field, $\bar{B} = 2\,\bar{a}_x - 6\,\bar{a}_y + 5\,\bar{a}_z$ mT.*

Solution : The magnitude of velocity is given as $v = 6\times10^6$ m/s. The direction of this velocity is specified by an unit vector. Thus we can write,

$$\bar{v} = v\bar{a}_v = 6\times10^6\left[-0.48\bar{a}_x - 0.6\,\bar{a}_y + 0.64\,\bar{a}_z\right] \text{ m/s}$$

The force experience by a moving charge in a steady magnetic field $\bar{B}$ is given by

$$\bar{F} = Q\,\bar{v}\times\bar{B}$$

$$= -60\times10^{-9}\left[(6\times10^6)(-0.48\bar{a}_x - 0.6\bar{a}_y + 0.64\bar{a}_z)\times(2\bar{a}_x - 6\bar{a}_y + 5\,\bar{a}_z)(1\times10^{-3})\right]$$

$$= (-3.6\times10^{-4})\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -0.48 & -0.6 & 0.64 \\ 2 & -6 & 5 \end{vmatrix}$$

$$= (-3.6\times10^{-4})\left[0.84\bar{a}_x + 3.68\,\bar{a}_y + 4.08\,\bar{a}_z\right]$$

$$= (-0.3024\,\bar{a}_x - 1.3248\,\bar{a}_y - 1.4688\,\bar{a}_z)\times10^{-3}\text{ N}$$

Thus the magnitude of the force on a moving charge is given by

$$|\bar{F}| = \sqrt{(-0.3024\times10^{-3})^2 + (-1.3248\times10^{-3})^2 + (-1.4688\times10^{-3})^2}$$

$$= \mathbf{2.0009\ mN}$$

Example 8.15 : *A conductor of length 2.5 m in z = 0 and x = 4 m carries a current of 12 A in $-\bar{a}_y$ direction. Calculate the uniform flux density in the region, if the force on the conductor is 12×10^{-2} N in the direction specified by $\left[\frac{-\bar{a}_x + \bar{a}_z}{\sqrt{2}}\right]$.*

Solution : Let $\bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_x$ T

The force exerted on the conductor is given by,

$$\bar{F} = I\,d\bar{L}\times\bar{B}$$

$$\therefore\ 12\times10^{-2}\left[\frac{-\bar{a}_x + \bar{a}_z}{\sqrt{2}}\right] = (-12\bar{a}_y)(2.5)\times(B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z)$$

$$\therefore\ -(0.0848)\,\bar{a}_x + (0.0848)\,\bar{a}_z = -30\left[\bar{a}_y\times(B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z)\right]$$

$$\therefore\ (2.8267\times10^{-3})\bar{a}_x - (2.8267\times10^{-3})\bar{a}_z = -B_x\,\bar{a}_z + B_z\,\bar{a}_x$$

Comparing components on both the sides of the equation,

$$B_x = 2.8267 \times 10^{-3}$$

$$B_y = 0$$

$$B_z = 2.8267 \times 10^{-3}$$

Thus, $$\bar{B} = 2.8267 \times 10^{-3}\, \bar{a}_x + 2.8267 \times 10^{-3}\, \bar{a}_z \text{ T}$$

i.e. $$\bar{B} = (2.8267\, \bar{a}_x + 2.8267\, \bar{a}_z) \text{ mT}$$

Example 8.16 : *A circular loop of radius r and current I lies in z = 0 plane. Find the torque which results if the current is in $\bar{a}_\phi$ and there is a uniform field $\bar{B} = \frac{B_0}{\sqrt{2}}(\bar{a}_x + \bar{a}_z)$ T.*

Solution : Consider a circular loop in z = 0 plane as shown in the Fig. 8.26.

Current is in $\bar{a}_\phi$ as shown in the Fig. 8.26. The given magnetic field is uniform given by

$$\bar{B} = B_0 \left(\frac{\bar{a}_x + \bar{a}_z}{\sqrt{2}} \right) \text{ T}$$

The magnetic dipole moment of a planar circular loop is given by,

$$\bar{m} = (I\, S)\, \bar{a}_n$$

where S is the area of the circular loop.

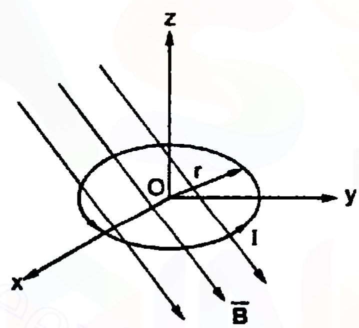


Fig. 8.26

Note that the loop is laying in z = 0 plane. Thus the direction of unit normal $\bar{a}_n$ must be decided by the right hand thumb rule. Let the fingures point in the direction of current (in $\bar{a}_\phi$ direction), then the right thumb gives the direction of $\bar{a}_n$ which is clearly $\bar{a}_z$.

$$\therefore \quad \bar{m} = I(\pi r^2)\, \bar{a}_z = (\pi r^2 I)\, \bar{a}_z \quad \text{... (1)}$$

The total torque is given by

$$\bar{T} = \bar{m} \times \bar{B}$$

$$= (\pi r^2 I)\, \bar{a}_z \times \frac{B_0}{\sqrt{2}}(\bar{a}_x + \bar{a}_z) = \frac{\pi r^2 B_0 I}{\sqrt{2}} [\bar{a}_z \times (\bar{a}_x + \bar{a}_z)]$$

$$= \frac{\pi r^2 B_0 I}{\sqrt{2}} \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \frac{\pi r^2 B_0 I}{\sqrt{2}} [-(-\bar{a}_y)]$$

$$= \left(\frac{\pi r^2 B_0 I}{\sqrt{2}} \right) \bar{a}_y \text{ N-m}$$

Example 8.18 : *Two long parallel conductors are separated by 2 cm in air carrying current of 100 ampere flowing in opposite directions. Find the force per meter length of the conductor.*

Solution : $I_1 = I_2 = 100$ A

d = Distance of separation = 2 cm = 2×10^{-2} m

$\mu = \mu_0\mu_r = \mu_0$ .... for air $\mu_r = 1$

The force per meter length between two long current carrying conductors is given by,

$$\frac{F}{L} = \frac{\mu I_1 I_2}{2\pi d}$$

$$= \frac{4\times\pi\times10^{-7}\times(100)^2}{2\times\pi\times2\times10^{-2}}$$

$$= 0.1 \text{ N/m}$$

As the currents in parallel conductors are flowing in opposite direction, the force will be force of repulsion.

Example 8.19 : *A rectangular loop in the xy plane with sides b_1 and b_2 carrying a current I lies in a uniform magnetic field $B = \bar{a}_x B_x + \bar{a}_y B_y + \bar{a}_z B_z$. Determine the force and torque on the loop.*

Solution : Consider a rectangular loop in the xy plane with sides b_1 and b_2 as shown in the Fig. 8.28.

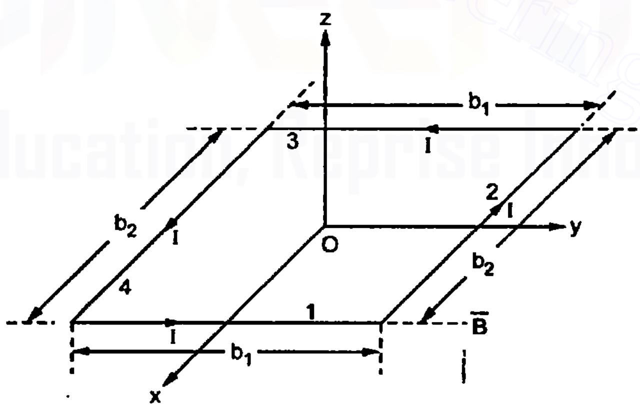


Fig. 8.28

Sides 1 and 3 are of length b_1 and are parallel to y-axis with sides 2 and 4 are of length b_2 and are parallel to x-axis. The origin is at the centre of the loop.

The vector force on side 1 is given by,

$$\bar{F}_1 = I\, d\bar{L}_1 \times \bar{B}$$

$$= \left[I\left(b_1 \bar{a}_y\right) \times \left(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right)\right]$$

$$= I b_1 \left[- B_x \bar{a}_z - B_z \bar{a}_x\right]$$

$$= I b_1 \left[B_z \bar{a}_x - B_x \bar{a}_z\right] \quad ...(1)$$

The vector force on the side 2 is given by,

$$\bar{F}_2 = I d\bar{L}_2 \times B$$

$$= \left[I\left(-b_2 \bar{a}_x\right) \times \left(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right)\right]$$

$$= - I b_2 \left[B_y \bar{a}_z - B_y \bar{a}_y\right]$$

$$= - I b_2 \left[- B_z \bar{a}_y + B_y \bar{a}_z\right] \quad ...(2)$$

The vector force on the side 3 is given by,

$$\bar{F}_3 = I d\bar{L}_3 \times \bar{B}$$

$$= \left[I\left(-b_1 \bar{a}_y\right) \times \left(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right)\right]$$

$$= - I b_1 \left[- B_x \bar{a}_z + B_z \bar{a}_x\right]$$

$$= I b_1 \left[- B_z \bar{a}_x + B_x \bar{a}_z\right] \quad ...(3)$$

Finally the vector force on the side 4 is given by,

$$\bar{F}_4 = I d\bar{L}_4 \times \bar{B}$$

$$= \left[I\left(b_2 \bar{a}_x\right) \times \left(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right)\right]$$

$$= I b_2 \left[B_y \bar{a}_z - B_z \bar{a}_y\right]$$

$$= I b_2 \left[- B_z \bar{a}_y + B_y \bar{a}_z\right] \quad ...(4)$$

Hence total force on the loop of sides b_1 and b_2 is given by,

$$\bar{F} = \bar{F}_1 + F_2 + \bar{F}_3 + \bar{F}_4$$

$$= I b_1 \left[B_z \bar{a}_x - B_x \bar{a}_z\right] - I b_2 \left[-B_z \bar{a}_y + B_y \bar{a}_z\right]$$

$$+ I b_1 \left[-B_z \bar{a}_x + B_x \bar{a}_z\right] + I b_2 \left[- B_z \bar{a}_y + B_y \bar{a}_z\right]$$

$$= I \left\{\left(b_1 B_z \bar{a}_x - b_1 B_x \bar{a}_z - b_1 B_z \bar{a}_x + b_1 B_x \bar{a}_z\right)\right.$$

$$\left. + \left(b_2 B_z \bar{a}_y - b_2 B_y \bar{a}_z - b_2 B_z \bar{a}_y + b_2 B_y \bar{a}_z\right)\right\}$$

$$= 0$$

Total torque on the rectangular loop can be obtained by choosing origin of the torque at the centre of the loop. Hence total torque is given by.

$$\bar{T} = \bar{T}_1 + \bar{T}_2 + \bar{T}_3 + \bar{T}_4$$

But

$$T = \bar{R}_1 \times \bar{F}_1 = \left(\frac{b_2}{2}\,\bar{a}_x\right) \times \left[I\, b_1\, (B_z\,\bar{a}_x - B_x\,\bar{a}_z)\right]$$

$$= I\left(\frac{b_1\, b_2}{2}\right)(B_x\,\bar{a}_y) \qquad ...(5)$$

$$\bar{T}_2 = \bar{R}_2 \times \bar{F}_2 = \left(\frac{b_1}{2}\,\bar{a}_y\right) \times \left[-I\, b_2\, (-B_z\,\bar{a}_y + B_y\,\bar{a}_z)\right]$$

$$= -I\left(\frac{b_1\, b_2}{2}\right)(B_y\,\bar{a}_x)$$

$$= I\left(\frac{b_1\, b_2}{2}\right)(-B_y\,\bar{a}_x) \qquad ...(6)$$

$$\bar{T}_3 = \bar{R}_3 \times \bar{F}_3 = \left(-\frac{b_2}{2}\,\bar{a}_x\right) \times \left[I\, b_1\, (-B_z\,\bar{a}_x + B_x\,\bar{a}_z)\right]$$

$$= -I\left(\frac{b_1\, b_2}{2}\right)(-B_x\,\bar{a}_y)$$

$$= I\left(\frac{b_1\, b_2}{2}\right)(B_x\,\bar{a}_y) \qquad ...(7)$$

$$\bar{T}_4 = \bar{R}_4 \times \bar{F}_4 = \left(-\frac{b_1}{2}\,\bar{a}_y\right) \times \left[I\, b_2\, (-B_z\,\bar{a}_y + B_y\,\bar{a}_z)\right]$$

$$= -I\left(\frac{b_1\, b_2}{2}\right)(B_y\,\bar{a}_x)$$

$$= I\left(\frac{b_1\, b_2}{2}\right)(-B_y\,\bar{a}_x) \qquad ...(8)$$

Hence adding equations (5), (6), (7), and (8), the total torque is given by,

$$\bar{T} = \bar{T}_1 + \bar{T}_2 + \bar{T}_3 + \bar{T}_4$$

$$= I\left(\frac{b_1\, b_2}{2}\right)(B_x\,\bar{a}_y) + I\left(\frac{b_1\, b_2}{2}\right)(-B_y\,\bar{a}_x)$$

$$+ I\left(\frac{-b_1\, b_2}{2}\right)(B_x\,\bar{a}_y) + I\left(\frac{b_1\, b_2}{2}\right)(-B_y\,\bar{a}_x)$$

$$= I\left(\frac{b_1 b_2}{2}\right)\left[-2\,B_y\,\bar{a}_x + 2\,B_x\bar{a}_y\right]$$

$$= I\; b_1\; b_2\; (-B_y\,\bar{a}_x + B_x\,\bar{a}_y)$$

Alternatively we can find total torque by using expression,

$$\bar{T} = I\left(d\bar{S}\right) \times \bar{B}$$

Now loop is in x-y plane with dimensions of sides as b_1 and b_2.

Then $d\bar{S} = (b_1 \ b_2)\,\bar{a}_z$. Let I be the current. Then total torque is given by,

$$\bar{T} = I\left(b_1 \ b_2 \bar{a}_z\right) \times \left(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right)$$

$$= I\, b_1 \ b_2 \left(+ B_x \bar{a}_y - B_y \bar{a}_x\right)$$

$$= I\, b_1 \ b_2 \left(- B_y \bar{a}_x + B_x \bar{a}_y\right)$$

Example 8.20 : *Two wires carrying current in the same direction of 3 A and 6 A are placed with their axes 5 cm apart, free space permeability = $4\pi \times 10^{-7}$ H/m. Calculate the force between them in kg/m length.*

Solution : Force between two parallel conductors is given by,

$$F = \frac{\mu I_1 \ I_2 \, l}{2\pi d}$$

d = distance of separation = 5 cm = 5×10^{-2} m

$$I_1 = 3 \text{ A}$$

$$I_2 = 6 \text{ A}$$

$$l = \text{Length of conductors}$$

Hence force per unit meter length is given by,

$$\frac{F}{l} = \frac{\mu I_1 \ I_2}{2\pi d} = \frac{\mu_0 \mu_r \ I_1 I_2}{2\pi d}$$

For free space $\mu_r = 1$

$$\therefore \quad \frac{F}{l} = \frac{4 \times \pi \times 10^{-7} \times 3 \times 6}{2 \times \pi \times 5 \times 10^{-2}} = 7.2 \times 10^{-5} \text{ N/m}$$

$$\therefore \quad \frac{F}{l} = 72 \, \mu \text{N/m}$$

The force expressed in kg/m is given by,

$$\frac{F}{l} = \frac{72 \times 10^{-6} \text{ N/m}}{9.8} = 7.3469 \times 10^{-6} \text{ kg/m}$$

Example 8.21 : *A Point charge Q = – 0.3 μC and m = 3×10^{-16} kg is moving through the field $\bar{E}$ = 30 $\bar{a}_z$ v/m. Using equation for force on a charge partical by $\bar{E}$ and Newton's*

laws, develop a differential equations and solve them, subject to the initial conditions at $t = 0$, $\bar{v} = 3 \times 10^5\ \bar{a}_x$ m/s at origin. At $t = 3\ \mu s$, find
(i) Position P (x, y, z) of charge,
(ii) Velocity $\bar{v}$,
(iii) The kinetic energy of the charge.

Solution : (i) Let the position of the charge is given by P(x, y, z).

The force exerted on charge by $\bar{E}$ is given by,

$$\bar{F} = Q\bar{E} \quad ...(1)$$

According to Newton's second law,

$$\bar{F} = m\,\bar{a} = m\frac{d\bar{v}}{dt} = \frac{d^2\bar{z}}{dt^2} \quad ...(2)$$

Equating equations (1) and (2) we can write,

$$m\frac{d^2\bar{z}}{dt^2} = Q.\bar{E} = (-0.3\times10^{-6}\times 30\bar{a}_z) \quad ...(3)$$

The initial velocity is constant and it is in x-direction so no force is applied in that direction. Rewritting equation (3), we get,

$$\frac{d^2\bar{z}}{dt^2} = \frac{Q\bar{E}}{m} \quad ...(4)$$

Integrating once equation (4) by separating variables, we get

$$\frac{d\bar{z}}{dt} = \bar{v}_z = \frac{Q\bar{E}}{m}t + k_1 \quad ...(5)$$

where k_1 is constant of integration.

To find k_1 : At t = 0, initial velocity in z-direction is zero Substituting values in equation (5), we get

$$0 = 0 + k_1 \quad \text{i.e. } k_1 = 0$$

Thus equation (5) becomes,

$$\bar{v} = \frac{d\bar{z}}{dt} = \frac{Q\bar{E}}{m}t \quad ...(6)$$

Integrating equation (6) with respect to corresponding variables we get,

$$\bar{z} = \frac{Q\bar{E}}{m}\left(\frac{t^2}{2}\right) + k_2 \quad ...(7)$$

where k_2 constant of integration.

To find k_2 : At t = 0, charge is at origin. Substituting values in equation (7) we get,

$$0 = \frac{Q\bar{E}}{m}\left(\frac{0}{2}\right)+k_2 \quad \text{i.e. } k_2 = 0$$

Hence solution of the equation (3) is given by,

$$\bar{z} = \frac{Q\bar{E}}{2m}t^2 = \frac{-0.3\times10^{-6}\times 30\bar{a}_z}{2\times3\times10^{-16}}t^2 \quad \ldots(8)$$

At t = 3 μsec,

$$\bar{z} = \frac{-0.3\times10^{-6}\times 30}{2\times3\times10^{-16}}\times(3\times10^{-6})^2 = -\ \mathbf{0.135\ m}$$

Let us consider initial constant velocity in x-direction, the charge attains x-coordinate of,

$$x = vt = (3\times10^5)(3\times10^{-6}) = \mathbf{0.9\ m}$$

Hence at t = 3 μsec, the position of charge is given by,

$$P(x, y, z) = \mathbf{(0.9, 0, -0.135)\ m}$$

(ii) To find velocity at t = 3 μ sec using equation (6), we get,

$$\bar{v} = \frac{Q\bar{E}}{m}t = \frac{-0.3\times10^{-6}\times 30\,\bar{a}_z)}{3\times10^{-16}}(3\times10^{-6})$$

$$= -\ \mathbf{9\times10^4\ \bar{a}_z\ m/sec}$$

The actual velocity of charge can be obtained by including initial constant velocity in x-direction as,

$$\bar{v} = \mathbf{(3\times10^5\ \bar{a}_x - 9\times10^4\ \bar{a}_z)\ m/sec}$$

(iii) The kinetic energy of the charge is given by,

$$K.E. = \frac{1}{2}m\,|\bar{v}|^2 = \frac{1}{2}\times3\times10^{-16}\times\left[\sqrt{(3\times10^5)^2+(-9\times10^4)^2}\right]^2$$

$$\therefore \quad K.E. = \mathbf{1.4715\times10^{-5}\ J}$$

Example 8.22 : *A current element 2 m in length lies along y-axis centered at origin. The current is 5 A in $\bar{a}_y$ direction. If it experiences a force 1.5 $\left(\frac{\bar{a}_x+\bar{a}_z}{\sqrt{2}}\right)$ N due to uniform field $\bar{B}$, determine $\bar{B}$.*

Solution : Force exerted on a straight current element in uniform magnetic field is given by,

$$\bar{F} = I\,\bar{L}\times\bar{B}$$

$$\therefore \quad \frac{1.5}{\sqrt{2}}(\bar{a}_x + \bar{a}_z) = [(5)(2\bar{a}_y) \times (B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z)]$$

$$\therefore \quad \frac{1.5}{\sqrt{2}}(\bar{a}_x + \bar{a}_z) = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 10 & 0 \\ B_x & B_y & B_z \end{vmatrix}$$

$$\therefore \quad 1.0606 (\bar{a}_x + \bar{a}_z) = (10 B_z) \bar{a}_x - (10 B_x) \bar{a}_z$$

Comparing coefficient of unit vectors on both the sides, we can write,

$$-10 B_x = 1.0606 \quad \text{i.e. } B_x = -0.10606 \text{ T}$$

and

$$10 B_z = 1.0606 \quad \text{i.e. } B_z = 0.10606 \text{ T}$$

Hence the uniform magnetic field is given by,

$$\bar{B} = +0.10606 (-\bar{a}_x + \bar{a}_z) \text{ T}$$

Example 8.23 : *A magnetic field $\bar{B} = 3.5 \times 10^{-2} \bar{a}_z$ exerts a force on a 0.3 m long conductor along x axis. If a current of 5 A flows in $-\bar{a}_x$ direction, determine what force must be applied to hold conductor in position.*

Solution : The force exerted on a straight conductor is given by,

$$\bar{F} = I \bar{L} \times \bar{B}$$

$$= 5(-0.3 \bar{a}_x) \times (3.5 \times 10^{-2} \bar{a}_z)$$

$$= -0.0525 (-\bar{a}_y) \qquad \ldots (\because \bar{a}_x \times \bar{a}_z = -\bar{a}_y)$$

$$= -0.0525 (-\bar{a}_y)$$

$$= 0.0525 \bar{a}_y \text{ N}$$

Hence the force applied to hold the conductor in position must be

$$\bar{F} = -0.0525 \bar{a}_y \text{ N}$$

Example 8.24 : *The magnetic flux density in a region of free space is given by $\bar{B} = -3x\bar{a}_x + 5y\bar{a}_y - 2z\bar{a}_z$ T. Find the total force on the rectangular loop as shown in the Fig. 8.29. If it lies in the plane z = 0 and is bounded by x = 1, x = 3, y = 2, y = 5, all dimensions in cm.*

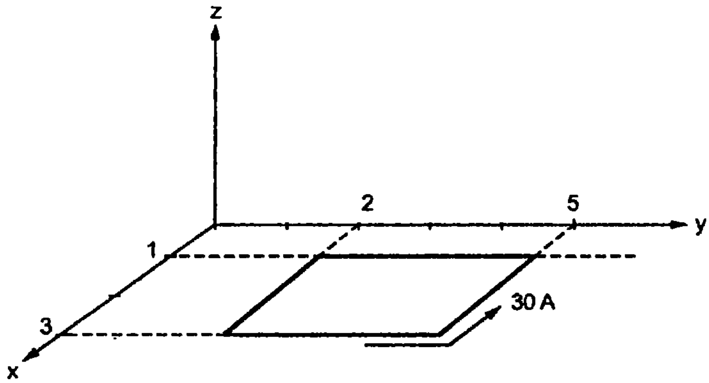


Fig. 8.29

Solution : In the plane z = 0, the z-component of the magnetic field $\bar{B}$ will be zero. In other words it will not contribute to the force. The force on the loop is given by,

$$\bar{F} = \int I\, d\bar{L} \times \bar{B}$$

Considering dimensions of all sides in meter, we can write,

$$\bar{F} = \int_{x=0.01}^{x=0.03} 30\, dx\, \bar{a}_x \times (-3x\, \bar{a}_x + 5y\, \bar{a}_y) + \int_{y=0.02}^{y=0.05} 30\, dy\, \bar{a}_y \times (-3x\, \bar{a}_x + 5y\, \bar{a}_y)$$

$$+ \int_{x=0.03}^{x=0.01} 30\, dx\, \bar{a}_x \times (-3x\, \bar{a}_x + 5y\, \bar{a}_y) + \int_{y=0.05}^{y=0.02} 30\, dy\, \bar{a}_y \times (-3x\, \bar{a}_x + 5y\, \bar{a}_y)$$

For side 1 and 3, the values of y are 0.02 and 0.05 respectively. While for sides 2 and 4, the values of x are 0.03 and 0.01 respectively.

$$\therefore \quad \bar{F} = \int_{0.01}^{0.03} 30\, dx\, (5)\, (0.02)\, \bar{a}_z + \int_{0.02}^{0.05} 30\, dy\, (-3)\, (0.03)\, (-\bar{a}_z)$$

$$+ \int_{0.03}^{0.01} 30\, dx\, (5)\, (0.05)\, \bar{a}_z + \int_{0.05}^{0.02} 30\, dy\, (-3)\, (0.01)\, (-\bar{a}_z)$$

$$\therefore \quad \bar{F} = 3\int_{0.01}^{0.03} dx\, \bar{a}_z + 2.7\int_{0.02}^{0.05} dy\, \bar{a}_z + 7.5\int_{0.03}^{0.01} dx\, \bar{a}_z + 0.9\int_{0.05}^{0.02} dy\, \bar{a}_z$$

$$\therefore \quad \bar{F} = 3[x]_{0.01}^{0.03}\, \bar{a}_z + 2.7\, [y]_{0.02}^{0.05}\, \bar{a}_y + 7.5\, [x]_{0.03}^{0.01}\, \bar{a}_z + 0.9\, [y]_{0.05}^{0.02}\, \bar{a}_z$$

$$\therefore \quad \bar{F} = [3(0.03-0.01) + 2.7(0.05-0.02) + 7.5(0.01-0.03) + 0.9(0.02-0.05)]\bar{a}_z$$

$$\therefore \quad \bar{F} = -0.036\, \bar{a}_z = -36\, \bar{a}_z \text{ mN}$$

Example 8.25 : *The region I and II interface each other. Region I has $\mu_{r1} = 1.5$, while region II $\mu_{r2} = 1$. The flux density $\bar{B}_I = 1.2\,\bar{a}_x + 0.8\,\bar{a}_y + 0.4\,\bar{a}_z$ T is incident at boundary from region 1. Calculate $\bar{B}_{II}$, angle of incidence, angle of reflection and angle of refraction by applying boundary conditions. Verify the angles computed with the help of Snell's law.*

Solution : Let us assume that z = 0 i.e. x-y plane forms the boundary such that region I is defined by z < 0 while region II by z > 0 as shown in the Fig. 8.30. Also the boundary is current free.

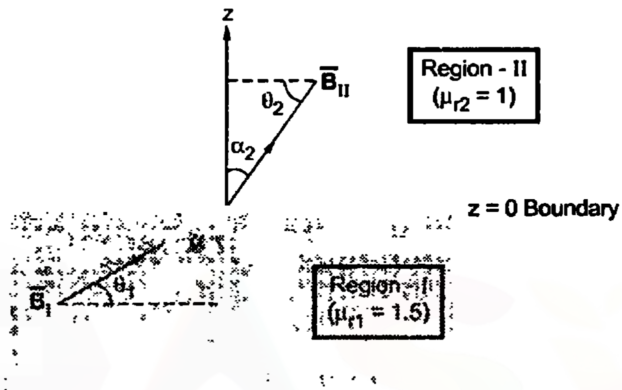


Fig. 8.30

In region I,

$\bar{B}_I = 1.2\,\bar{a}_x + 0.8\,\bar{a}_y + 0.4\,\bar{a}_z$ T

But $\bar{B} = \mu \bar{H}$. Using this relationship $\bar{H}_I$ in region I can be expressed as,

$$\bar{H}_I = \frac{1}{\mu_1}\bar{B}_I = \frac{1}{\mu_0 \mu_{r1}}\bar{B}_I$$

$$\therefore \quad \bar{H}_I = \frac{1}{\mu_0(1.5)}\left[1.2\bar{a}_x + 0.8\bar{a}_y + 0.4\,\bar{a}_z\right]$$

$$\therefore \quad \bar{H}_I = \frac{1}{\mu_0}\left[0.8\bar{a}_x + 0.5333\,\bar{a}_y + 0.2667\,\bar{a}_z\right] \quad \text{... (1)}$$

As z = 0 is the boundary, z axis is normal to the boundary. Thus the component of $\bar{H}_I$ along x-direction and y-direction both are tangential components while that in z-direction is the normal component.

Now in region II, the total magnetic flux density $\bar{B}_{II}$ is given by

$$\bar{B}_{II} = \bar{B}_{tanII} + \bar{B}_{NII} \quad \text{... (2)}$$

According to the boundary conditions for current free boundary, the normal component of $\bar{B}$ are continuous.

$$\therefore \quad \bar{B}_{NI} = \bar{B}_{NII} = 0.4\,\bar{a}_z \quad \text{... (3)}$$

Similarly for current free boundary, the tangential components of $\bar{H}$ are continuous at the boundary.

$$\therefore \quad \bar{H}_{tanI} = \bar{H}_{tanII} = \frac{1}{\mu_0}\left[0.8\bar{a}_x + 0.5333\bar{a}_y\right] \quad \text{... (4)}$$

Putting value of $\left(\bar{H}_1 - \bar{H}_2\right)$ in equation (1)

$$[(H_{tan1} - 14.5)\bar{a}_x + 2\bar{a}_z] \times \bar{a}_z = 9\,\bar{a}_y$$

$$\therefore \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ (H_{tan1} - 14.5) & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 9\,\bar{a}_y$$

$$\therefore -(H_{tan1} - 14.5)\bar{a}_y = 9\,\bar{a}_y$$

Comparing the components on both the sides,

$$\therefore \quad -H_{tan\,1} + 14.5 = 9$$

$$\therefore \quad H_{tan\,1} = 5.5 \;\; A/m \qquad \dots (3)$$

Hence $\bar{H}_1$ in medium is given by,

$$\therefore \quad \bar{H}_1 = 5.5\;\bar{a}_x + 6\;\bar{a}_z \quad A/m$$

Example 8.27 : *Consider an interface in y-z plane. The region $x < 0$ is medium 1 with $\mu_{r1} = 4.5$ and magnetic field $\bar{H}_1 = 4\bar{a}_x + 3\bar{a}_y - 6\bar{a}_z$ A/m. The region $x > 0$ is medium 2 with $\mu_{r2} = 6$. Find $\bar{H}_2$ in medium 2 and angle made by $\bar{H}_2$ with normal to the interface.*

Solution : An interface is in y-z plane at x = 0. For x < 0, there is medium 1 ($\mu_{r1} = 4.5$) and for x > 0, there is medium 2 ($\mu_{r2} = 6$) as shown in the Fig. 8.31. Let us assume that the interface is current free.

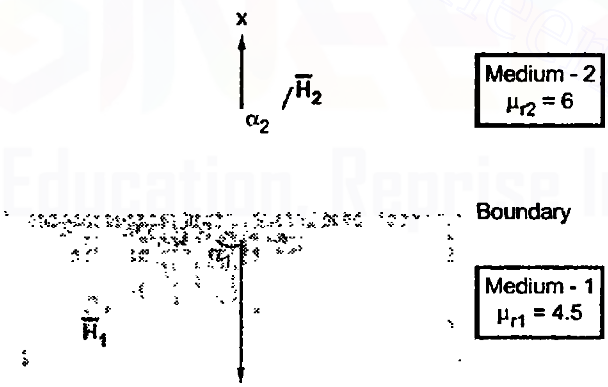


Fig. 8.31

From the Fig. 8.31 it is clear that, for y-z plane, x-axis is the normal to the interface. Thus the component of $\bar{H}_1$ along $\bar{a}_x$ is the normal component while the components along $\bar{a}_y$ and $\bar{a}_z$ are tangential components.

$$\left.\begin{aligned} \therefore \quad \bar{H}_{tan1} &= 3\bar{a}_y - 6\bar{a}_z \;\; A/m \\ \text{and} \quad \bar{H}_{N1} &= 4\,\bar{a}_x \;\; A/m \end{aligned}\right\} \qquad \dots (1)$$

From the boundary conditions, for current free boundary, the tangential component of $\bar{H}$ is continuous.

$$\therefore \quad \bar{H}_{tan1} = \bar{H}_{tan2} = 3\,\bar{a}_y - 6\bar{a}_z \ \text{ A/m} \quad \text{... (2)}$$

At the boundary normal component of $\bar{H}$ is discontinuous.

$$\therefore \quad \frac{H_{N1}}{H_{N2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_0\mu_{r2}}{\mu_0\mu_{r1}} = \frac{6}{4.5}$$

$$\therefore \quad H_{N2} = \frac{(4.5)}{6} H_{N1} = (0.75)(4) = 3 \quad \text{... (3)}$$

As the normal direction is along x-axis,

$$\bar{H}_{N2} = 3\,\bar{a}_x \ \text{ A/m} \quad \text{... (4)}$$

From equations (2) and (4), the field intensity $\bar{H}_2$ in region 2 is given by,

$$\bar{H}_2 = \bar{H}_{tan2} + \bar{H}_{N2} = 3\,\bar{a}_x + 3\,\bar{a}_y - 6\,\bar{a}_z \text{ A/m}$$

Let the angle made by $\bar{H}_2$ with the normal to the interface be α_2. Then we can write

$$\bar{H}_2 \cdot \bar{a}_x = |\bar{H}_2|\,|\bar{a}_x| \cos\alpha_2$$

$$\therefore \ (3\bar{a}_x + 3\bar{a}_y - 6\bar{a}_z)\cdot(\bar{a}_x) = \left(\sqrt{(3)^2+(3)^2+(-6)^2}\right)(1)\,(\cos\alpha_2)$$

$$\therefore \quad 3 = (7.3484)\cos\alpha_2 \quad \text{....}\ \bar{a}_x \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_x = 0$$

$$\therefore \quad \cos\alpha_2 = \frac{3}{7.3484} = 0.4082$$

$$\therefore \quad \alpha_2 = \cos^{-1}(0.4082) = 65.9^\circ$$

Example 8.28 : *Find the normal component of the magnetic field which traversed from medium 1 to medium 2 having $\mu_{r1} = 2.5$ and $\mu_{r2} = 4$.*
Given that $\bar{H}_1 = -30\bar{a}_x + 50\bar{a}_y + 70\bar{a}_z$ V/m.

Solution : Assume that x-y plane i.e. z = 0 plane is the interface. Below z = 0, there is medium 1 with $\mu_{r1} = 2.5$, while above z = 0, there is medium 2 with $\mu_{r2} = 4$.

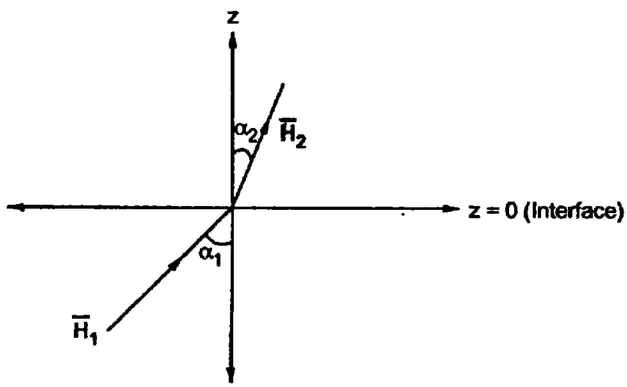


Fig. 8.32

From the Fig. 8.32 it is clear that, for x-y plane z-axis is the normal to the interface. Thus the component of $\bar{H}_1$ along $\bar{a}_z$ is

the normal component $\bar{H}_{N1}$. As $\bar{a}_x$ and $\bar{a}_y$ directions are tangential to the interface, the components of $\bar{H}_1$ along $\bar{a}_x$ and $\bar{a}_y$ are tangential.

$$\therefore \quad \bar{H}_{tan1} = -30\,\bar{a}_x + 50\,\bar{a}_y \quad A/m$$

$$\text{and} \quad \bar{H}_{N1} = 70\,\bar{a}_z \quad A/m$$

From the boundary conditions, for current free boundary, the tangential component of $\bar{H}$ is continuous.

$$\therefore \quad \bar{H}_{tan2} = \bar{H}_{tan1} = -30\,\bar{a}_x + 50\,\bar{a}_y \quad A/m$$

Also the normal component of $\bar{H}$ is discontinuous at the interface,

$$\therefore \quad \frac{H_{N1}}{H_{N2}} = \frac{\mu_2}{\mu_1} = \frac{\mu_{r2}}{\mu_{r1}}\frac{4}{2.5} = 1.6$$

$$\therefore \quad H_{N2} = \frac{H_{N1}}{1.6} = \frac{70}{1.6} = 43.75 \ A/m$$

As the normal direction is along z-axis,

$$\bar{H}_{N2} = 43.75\,\bar{a}_z \quad A/m$$

Hence the magnetic field intensity above z = 0, in medium 2, is given by

$$\bar{H}_2 = \bar{H}_{tan2} + \bar{H}_{N2} = -30\,\bar{a}_x + 50\,\bar{a}_y - 43.75\,\bar{a}_z \quad A/m$$

Example 8.29 : *If $\bar{B} = 0.05\,x\,\bar{a}_y$ T in a material for which $\chi_m = 2.5$, find ; a) μ_r ; b) μ ; c) $\bar{H}$; d) $\bar{M}$; e) $\bar{J}$; and f) $\bar{J}_b$.*

Solution : For the material,

magnetic susceptibility $\chi_m = 2.5$

a) The relative permeability interms of the magnetic susceptibility is given by

$$\mu_r = \chi_m + 1 = 2.5 + 1 = 3.5$$

b) The permeability of the material is given by,

$$\mu = \mu_0\mu_r = \left(4\times\pi\times10^{-7}\right)(3.5) = 4.3982\times10^{-6} \quad H/m$$

c) The magnetic flux density is given by,

$$\bar{B} = \mu\bar{H} = \mu_0\mu_r\bar{H}$$

$$\therefore \quad \bar{H} = \frac{\bar{B}}{\mu_0\mu_r} = \frac{(0.05x)\bar{a}_y}{4.3982\times10^{-6}} = \left(11.3682\times10^3\ x\right)\bar{a}_y \quad A/m$$

d) The magnetization is given by,

$$\bar{M} = \chi_m\bar{H} = 2.5\left[\left(11.3682\times10^3\right)x\right]\bar{a}_y$$

$$\therefore \quad \bar{M} = \left(28.42\times10^3\right)x\,\bar{a}_y \quad A/m$$

For a solenoid with large length as compared to small cross-section, the magnetic field intensity inside the coil can be assumed to be constant and zero at points just outside the solenoid.

Let the current flowing through solenoid 1 be I_1. Then the magnetic field intensity is given by,

$$H_1 = \frac{N_1 I_1}{l_1} = \frac{(1000) I_1}{50 \times 10^{-2}} = (2000)\ I_1 \text{ A/m}$$

The magnetic flux density is given by,

$$B_1 = \mu H_1 = \mu_0 \mu_r H_1 = (4 \times \pi \times 10^{-7})\ (2000)\ I_1$$

$$= (2.5132 \times 10^{-3})\ I_1 \text{ Wb/m}^2$$

The total flux produced is given by,

$$\phi_1 = (B_1)\ (A_1) \qquad \text{... where } A_1 = \text{area of cross-section of solenoid}$$

$$= (2.5132 \times 10^{-3})(I_1)[\pi \times (1 \times 10^{-2})^2]$$

$$= (0.7895 \times 10^{-6}) I_1 \text{ Wb}$$

The flux obtained above can only link with the second solenoid as H_1 and B_1 are zero outside solenoid 1.

Hence the mutual inductance between two solenoids can written as,

$$M_{12} = \frac{N_2\ \phi_1}{I_1} = \frac{(2000)(0.7895 \times 10^{-6}) I_1}{I_1} = 1.5791 \text{ mH}$$

Example 8.34 : *A solenoid has an inductance of 20 mH. If the length of the solenoid is increased by two times and the radius is decreased to half of its original value, find the new inductance.*

Solution : The inductance of the solenoid is given by

$$L = \frac{\mu N^2 A}{l} = 20 \text{ mH}$$

where l = Length of the solenoid

A = Area = πr^2

N = Number of turns

Now length is made $2l$ while the radius is made $\left(\frac{r}{2}\right)$. Then the inductance is given by

$$L_{new} = \frac{\mu N^2 \left[(\pi)\left(\frac{r}{2}\right)^2\right]}{(2l)} = \frac{\mu N^2 (\pi r^2)}{8l}$$

For region 1, the magnetic flux density is given by,

$$\bar{B}_1 = \frac{\mu I \rho}{2\pi a^2} \bar{a}_\phi \text{ ...for region 1 } (0 \le \rho \le a) \quad ...(1)$$

Similarly for region 2, the magnetic flux density is given by,

$$\bar{B}_2 = \frac{\mu I}{2\pi \rho} \bar{a}_\phi \text{ for region 2 } (0 \le \rho \le b) \quad ...(2)$$

Refer Fig. 8.34 (a). The flux leaving the differential shell of thickness dρ is given by,

$$d\psi_1 = B_1 \, d\rho \, dz = \frac{\mu I \rho}{2\pi a^2} \, d\rho \, dz \quad ...(3)$$

Thus the flux linkage $d\psi_1$ multiplied by the ratio of area within the path enclosing the flux to the total area we get,

$$d\phi_1 = d\psi_1 \cdot \frac{I_{enc}}{I} = d\psi_1 \cdot \frac{\pi \rho^2}{\pi a^2} = d\psi_1 \cdot \frac{\rho^2}{q^2} \quad ...(4)$$

Thus total flux linkages within differential element are.

$$d\phi_1 = \frac{\mu I \rho \, d\rho \, dz}{2\pi a^2} \cdot \frac{\rho^2}{a^2}$$

For length *l* of cable,

$$\phi_1 = \int_{z=0}^{l} \int_{\rho=0}^{a} \frac{\mu I \rho^2 \, d\rho \, dz}{2\pi a^4} = \frac{\mu I l}{8\pi} \quad ...(5)$$

$$\therefore \quad L_{in} = \frac{\phi_1}{I} = \frac{\mu l}{8\pi} \quad ...(A)$$

Similarly for external conductor, the flux linkages between the inner and outer conductor is given by,

$$d\phi_2 = B_2 \, d\rho \, dz = \frac{\mu I}{2\pi\rho} \, d\rho \, dz \quad ...(6)$$

Hence

$$\phi_2 = \int_{\rho=a}^{b} \int_{z=0}^{l} \frac{\mu I \, d\rho \, dz}{2\pi\rho} = \frac{\mu I l}{2\pi} ln\left(\frac{b}{a}\right) \quad ...(7)$$

Hence

$$L_{out} = \frac{\phi_2}{I} = \frac{\mu l}{2\pi} ln\left(\frac{b}{a}\right) \quad ...(B)$$

Hence

$$L = L_{in} + L_{out} = \frac{\mu l}{2\pi}\left[\frac{1}{4} + ln\frac{b}{a}\right]$$

Thus inductance per unit length of an air coaxial cable is given by,

$$L' = \frac{L}{l} = \frac{\mu}{2\pi}\left[\frac{1}{4} + ln\frac{b}{q}\right] \text{H/m}$$

Example 8.37 : *Calculate the internal and external inductances per unit length of a transmission line consisting of two long parallel conducting wires of radius 'a' that carry currents in opposite directions. The axes of the wires are separated by a distance d, which is much larger than a.*

Solution : Consider transmission line made up of two long parallel conductors of radius a carrying currents in opposite directions, with distance of separation as shown in the Fig. 8.35.

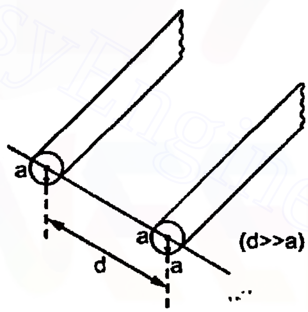


Fig. 8.35

Energy stored is given by,

$$W_m = \frac{1}{2} L I^2$$

i.e. $$L = \frac{2 W_m}{I^2} \quad ...(1)$$

But $$W_m = \int \bar{B}\cdot\bar{H}\, dv = \int \frac{B^2}{2\mu} dv \quad ...(2)$$

Substituting W_m in equation (1), we get,

$$L_{in} = \frac{2}{I^2}\int \frac{B^2}{2\mu} dv$$

$$= \frac{1}{I^2 \mu}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{\mu^2 I^2 \rho^2}{4\pi^2 a^4}\, \rho\, d\rho\, d\phi\, dz$$

$$= \frac{\mu}{4\pi^2 a^4} \int_0^l dz \int_0^{2\pi} d\phi \int_0^a \rho^3 \; d\rho$$

$$= \frac{\mu}{4\pi^2 a^4} [l] [2\pi] \left[\frac{a^4}{4}\right] = \frac{\mu l}{8\pi} \quad ...(3)$$

For outer conductor, we can write,

$$L_{out} = \frac{2}{I^2} \int \frac{B^2}{2\mu} dv = \frac{1}{I^2 \mu} \int\int\int \frac{\mu^2 I^2}{4\pi^2 \rho^2} \rho d\rho \, d\phi \, dz$$

$$= \frac{\mu}{4\pi^2} \int_0^l dz \int_0^{2\pi} d\phi \int_a^{d-a} \frac{1}{\rho} d\rho \;\; = \frac{\mu}{4\pi^2} [l][2\pi][ln \rho]_a^{da}$$

$$= \frac{\mu l}{2\pi} \, ln \left(\frac{d-a}{a}\right) \quad ...(4)$$

Hence total inductance for two symmetric wires is given by,

$$L = 2[L_{in} + L_{out}] = \frac{\mu l}{4\pi} + \frac{\mu l}{\pi} \, ln \left(\frac{d-a}{a}\right)$$

$$= \frac{\mu l}{\pi} \left[\frac{1}{4} + ln \left(\frac{d-a}{a}\right)\right]$$

As d >> a, d – a ≈ d. Hence we can write, the inductance is given by,

$$\therefore \quad L = \frac{\mu l}{\pi} \left[\frac{1}{4} + ln \left(\frac{d}{a}\right)\right] \text{ H}$$

Example 8.38 : *An air-core toroid with rectangular cross-section, has 700 turns, with inner radius of 1 cm and outer radius of 2 cm and height is 1.5 cm. Find inductance using (1) The formula for sq. cross section of toroids (2) The approximate formula for general toroid, which assumes a uniform H at mean radius.*

Solution : N = 700,

Inner radius = r_1 = 1 cm = 1×10^{-2} m

Outer radius = r_2 = 2 cm = 2×10^{-2} m

Height = h = 1.5 cm = 1.5 cm × 10^{-2} m

1) In general, inductance of a toroid of square cross section is given by,

$$L = \frac{\mu_0 N^2 h}{2\pi} \, ln \left(\frac{r_2}{r_1}\right)$$

$$= \frac{4 \times \pi \times 10^{-7} \times (700)^2 \times 1.5 \times 10^{-2}}{2 \times \pi} \, ln \left(\frac{2 \times 10^{-2}}{1 \times 10^{-2}}\right)$$

$$= \textbf{1.0189 mH}$$

$$\frac{H_{N_1}}{H_{N_2}} = \frac{\mu_{r2}}{\mu_{r1}}$$

$$\therefore \quad H_{N_2} = \frac{\mu_{r1}}{\mu_{r3}} H_{N_1} = \frac{4}{7} H_{N_1} = \frac{4}{7}(0.5) = \frac{2}{7} \text{ A/m} = 0.2857.$$

As the tangential components are same,

$$H_{\tan 2x} = H_{\tan 1x} = 0.5$$

$$H_{\tan 2y} = H_{\tan 1y} = -0.75$$

Hence $\overline{H}_2$ is given by,

$$\overline{H}_2 = \frac{1}{\mu_0}[H_{\tan 2x}\,\bar{a}_x + H_{\tan 2y}\,\bar{a}_y + H_{N_2}\,\bar{a}_z]$$

$$\therefore \quad \overline{H}_2 = \frac{1}{\mu_0}[0.5\,\bar{a}_x - 0.75\,\bar{a}_y + 0.2857\,\bar{a}_z] \text{ A/m}$$

Hence $$\overline{B}_2 = \mu_2 \overline{H}_2 = \mu_0 \cdot \mu_{r2}\,\overline{H}_2 = 7\mu_0 \times \frac{1}{\mu_0}[0.5\,\bar{a}_x - 0.75\,\bar{a}_y + 0.2857\,\bar{a}_z]$$

$$\therefore \quad \overline{B}_2 = 3.5\,\bar{a}_x - 5.25\,\bar{a}_y + 2\,\bar{a}_z \;\text{ T}$$

Example 8.40 : *A charged particle of mass 2 kg and charge 1 C starts at the origin with velocity 3 $\bar{a}_y$ m/sec and travels in a region of uniform magnetic field $\overline{B}$ = 10 $\bar{a}_z$ Wb/m^2. At t = 4 sec, calculate*

i) The velocity and acceleration of the particle.

ii) The magnetic force on it.

iii) Kinetic energy and location. **(UPTU : 2006-07)**

Solution : i) $\overline{F} = m\,\bar{a} = M\,\frac{d\bar{v}}{dt} = Q\,\bar{v}\times\overline{B}$

Hence acceleration is given by,

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{Q}{M}\,\bar{v}\times\overline{B}$$

$$\therefore \quad \frac{d}{dt}[v_x\bar{a}_x + v_y\bar{a}_y + v_z\bar{a}_z] = \frac{1}{2}\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ v_x & v_y & v_z \\ 0 & 0 & 10 \end{vmatrix}$$

$$\therefore \quad \frac{d}{dt}[v_x\bar{a}_x + v_y\bar{a}_y + v_z\bar{a}_z] = \frac{1}{2}[10v_y\bar{a}_x - 10v_x\bar{a}_y]$$

$$\therefore \quad \frac{d}{dt}[v_x\bar{a}_x + v_y\bar{a}_y + v_z\bar{a}_z] = 5v_y\bar{a}_x - 5v_x\bar{a}_y$$

At t = 4 sec,

$$\therefore \quad \bar{v} = 3 \sin 20\, \bar{a}_x + 3\cos 20\, \bar{a}_y = 2.7388\, \bar{a}_y + 1.224\, \bar{a}_y \text{ m/sec}$$

Now acceleration is given by,

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}[3 \sin 5t\, \bar{a}_x + 3\cos 5t\, \bar{a}_y]$$

$$\therefore \quad \bar{a} = 15 \cos 5t\, \bar{a}_x - 15 \sin 5t\, \bar{a}_y \text{ m/sec}^2$$

At t = 4 sec,

$$\bar{a} = 15 \cos 20\, \bar{a}_x - \sin 20\, \bar{a}_y = 6.1212\, \bar{a}_x - 13.6941\, \bar{a}_y \text{ m/sec}^2$$

ii) The force on a particle is given by,

$$\bar{F} = m\bar{a} = 2\,[6.1212\, \bar{a}_x - 13.6941\, \bar{a}_y]$$

$$= 12.2424\, \bar{a}_x - 27.3882\, \bar{a}_y \text{ N}$$

iii) The kinetic energy is given by,

$$\text{K.E.} = \frac{1}{2} m |\bar{v}|^2 = \frac{1}{2} \times 2 \times \left[\sqrt{(2.7388)^2 + (1.224)^2}\right]^2 = 8.999 \text{ J} \approx 9 \text{ J}$$

Now

$$v_x = \frac{dx}{dt} = 3 \sin 5t \quad \text{Hence } x = \frac{-3}{5} \cos 5t + K_1' \qquad \text{...(4)}$$

$$v_y = \frac{dy}{dt} = 3 \cos 5t \quad \text{Hence } y = \frac{3}{5} \sin 5t + K_2' \qquad \text{...(5)}$$

$$v_z = \frac{dz}{dt} = 0 \quad \text{Hence } z = K_3' \qquad \text{...(6)}$$

At t = 0, (x, y, z) = (0, 0, 0) and thus,

$x(t = 0) = 0$ i.e. $0 = \frac{-3}{5} \cdot 1 + K_1'$

i.e. $\quad K_1' = 0.6$

$y(t = 0) = 0$ i.e. $0 = \frac{3}{5}(0) + K_2'$

i.e. $\quad K_2' = 0$

$z(t = 0) = 0$ i.e. $0 = K_3'$

Hence putting values of K_1', K_2', K_3' in equations (4), (5) and (6),

we get,

$$(x, y, z) = (0.6 - 0.6 \cos 5t,\ 0.6 \sin 5t,\ 0)$$

Hence at t = 4 sec, the location of particle is obtained as,

$$(x, y, z) = (0.6 - 0.6 \cos 20t,\ 0.6 \sin 20t,\ 0)$$

$$\therefore \quad (x, y, z) = \mathbf{(0.3552,\ 0.5477,\ 0)}$$

Example 8.41 : *A very long solenoid with 2 × 2 cm cross-section has an iron core (μ_r = 1000) and 4000 turns/meter. If it carries a current of 500 mA, find its self inductance per meter.* **[UPTU : 2007-08, 10 Marks]**

Solution : By definition, the inductance of soleoid is given by,

$$L = \frac{\mu N^2 A}{l} = \mu N^2 A \text{ H/m}$$

where $n = \dfrac{N}{l}$ = Number of turns per unit length

Now A = area of cross-section = 2 × 2 cm^2 = 4 cm^2 = 4 × 10^{-4} m^2

Hence, we can write,

$$L = (\mu_0\ \mu_r)\ n^2\ A$$

$$= 1000 \times 4 \times \pi \times 10^{-7} \times (4000)^2 \times 4 \times 10^{-4}$$

$$= \mathbf{8.0424\ H/m}$$

Example 8.42 : *The toroidal core shown in the Fig. 8.37 has r_o = 10 cm and circular cross section with a = 1 cm. If the core is made up of steel (μ = 1000 μ_0) and a coil with 200 turns, calculate the amount of current that will produce a flux of 0.5 mWb in the core.*

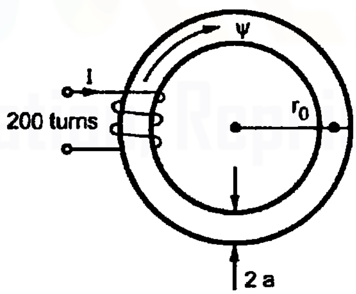


Fig. 8.37

[UPTU : 2007-08, 10 Marks]

Solution : For the given toroidal circuit, similar electric circuit can be drawn as shown in the Fig. 8.38.

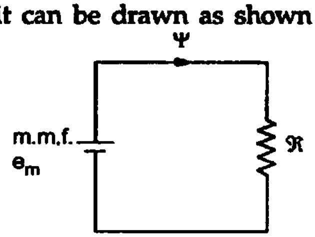


Fig. 8.38

Now the magnetomotive froce i.e. m.m.f. e_m is given by,

$$e_m = NI = \psi \cdot \Re = \psi \cdot \frac{\lambda}{\mu S}$$

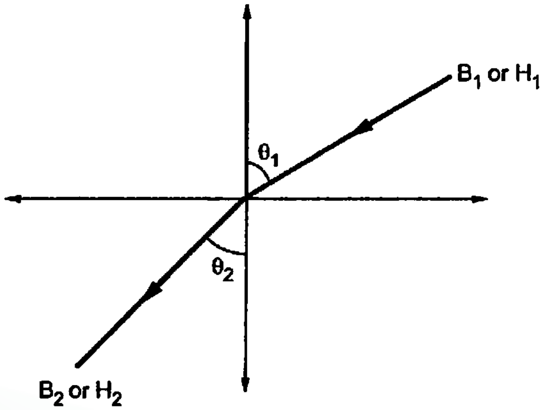


Fig. 8.39

$B_1 = 2U_x - 3U_y + 2U_z$, is incident on the xy plane (z = 0). The medium at z > 0 has $\mu_1 = 4\mu_0$ and z < 0 has $\mu_2 = 7\mu_0$. Find B_2 U_x, U_y and U_z indicate unit vectors in the respective directions.
[UPTU : 2002-2003, 10 Marks]

4. *Establish the boundary conditions for the tangential component of H at the boundary between two isotropic, homogeneous materials with parmeabilities μ_1 and μ_2.*

   Show, from given Fig. 8.40 that $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\mu_1}{\mu_2}$. **[UPTU : 2003-2004, 10 Marks]**

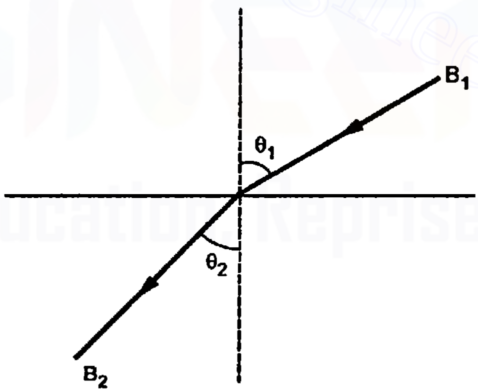


Fig. 8.40

5. *Discuss the energy stored in electric and magnetic fields.* **[UPTU : 2003-2004, 10 Marks]**
6. *Discuss the boundary condition for magnetic field.* **[UPTU : 2003-2004, 10 Marks]**
7. *A charged particle of mass 2 kg and charge 1 C starts at the origin with velocity 3 $\bar{a}_y$ m/sec and travels in a region of uniform magnetic field $\overline{B} = 10\ \bar{a}_z$ Wb/m^2. At t = 4 sec, calculate*

   i) The velocity and acceleration of the particle.

   ii) The magnetic force on it

   iii) Kinetic energy and location **[UPTU : 2006-07, 10 Marks]**

9 Time Varying Fields and Maxwell's Equations

9.1 Introduction

In the previous chapters we have studied the basic concepts in an electrostatic and magnetostatic fields. These fields can be considered as time invariant or static fields. In static electromagnetic fields, electric and magnetic fields are independent of each other. In this chapter, we shall concentrate on the time varying or dynamic fields. In dynamic electromagnetic fields, the electric and magnetic fields are interdependent. In general, static electric fields are produced by stationary electric charges. The static magnetic fields are produced due to the motion of the electric charges with uniform velocity or the magnetic charges. The time varying fields are produced due to the time varying currents.

In this chapter, we shall first study Faraday's law and Lenz's law. Then we shall discuss the concept of displacement current. We shall also study important equations of electromagnetic theory known as Maxwell's equations.

9.2 Faraday's Law

In year 1820, **Prof. Hans Christian Oersted** demonstrated that a compass needle deflected due to an electric current. After ten years, **Michael Faraday**, a British Scientist, proved that a magnetic field could produce a current.

According to Faraday's experiment, a static magnetic field cannot produce any current flow. But with a time varying field, an electromotive force (e.m.f.) induces which may drive a current in a closed path or circuit. This e.m.f. is nothing but a voltage that induces from changing magnetic fields or motion of the conductors in a magnetic field. Faraday discovered that the induced e.m.f. is equal to the time rate of change of magnetic flux linking with the closed circuit.

Faraday's law can be stated as,

$$\boxed{e = -N\frac{d\phi}{dt} \text{ volts.}} \qquad \text{... (1)}$$

where N = Number of turns in the circuit

e = Induced e.m.f.

Let us assume single turn circuit i.e. N = 1, then Faraday's law can be stated as,

$$e = -\frac{d\phi}{dt} \text{ volts} \qquad \text{... (2)}$$

The minus sign in equations (1) and (2) indicates that the direction of the induced e.m.f. is such that to produce a current which will produce a magnetic field which will oppose the original field.

In 1834, **Henri Frederic Emile Lenz** postulated the law. Thus according to **Lenz's law,** the induced e.m.f. acts to produce an opposing flux.

Let us consider Faraday's law. The induced e.m.f. is a scalar quantity measured in volts. Thus the induced e.m.f. is given by,

$$\boxed{e = \oint \bar{E} \cdot d\bar{L}} \qquad \text{... (3)}$$

The induced e.m.f. in equation (3) indicates a voltage about a closed path such that if any part of the path is changed, the e.m.f. will also change.

The magnetic flux ϕ passing through a specified area is given by,

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$

where B = Magnetic flux density

Using above result, equation (2) can be rewritten as,

$$e = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{S} \qquad \text{... (4)}$$

From equations (3) and (4), we get,

$$\boxed{e = \oint \bar{E} \cdot d\bar{L} = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{S}} \qquad \text{... (5)}$$

There are two conditions for the induced e.m.f. as explained below.

i) The closed circuit in which e.m.f. is induced is stationary and the magnetic flux is sinusoidally varying with time. From equation (5) it is clear that the magnetic flux density is the only quantity varying with time. We can use partial derivative to define relationship as $\bar{B}$ may be changing with the co-ordinates as well as time. Hence we can write,

$$\boxed{\oint \bar{E} \cdot d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S}} \qquad \text{... (6)}$$

This is similar to **transformer action** and e.m.f. is called **transformer e.m.f.** Using Stoke's theorem, a line integral can be converted to the surface integral as

$$\oint_S (\nabla \times \bar{E}) \cdot d\bar{S} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S} \qquad \text{... (7)}$$

Assuming that both the surface integrals taken over identical surfaces.

$$\therefore \qquad (\nabla \times \bar{E}) \cdot d\bar{S} = -\frac{\partial \bar{B}}{\partial t} \cdot d\bar{S}$$

Hence finally,

$$\boxed{\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}} \qquad \text{... (8)}$$

Equation (8) represents one of the Maxwell's equations. If $\bar{B}$ is not varying with time, then equations (6) and (8) give the results obtained previously in the electrostatics.

$$\oint \bar{E} \cdot d\bar{L} = 0, \text{ and}$$

$$\nabla \times \bar{E} = 0$$

ii) Secondly magnetic field is stationary, constant not varying with time while the closed circuit is revolved to get the relative motion between them. This action is similar to generator action, hence the induced e.m.f. is called **motional or generator e.m.f.**

Consider that a charge Q is moved in a magnetic field $\bar{B}$ at a velocity $\bar{v}$. Then the force on a charge is given by,

$$\bar{F} = Q\,\bar{v} \times \bar{B} \qquad \text{... (9)}$$

But the motional electric field intensity is defined as the force per unit charge. It is given by,

$$\therefore \qquad \bar{E}_m = \frac{\bar{E}}{Q} = \bar{v} \times \bar{B} \qquad \text{... (10)}$$

Thus the induced e.m.f. is given by,

$$\boxed{\oint \bar{E}_m \cdot d\bar{L} = \oint (\bar{v} \times \bar{B}) \cdot d\bar{L}} \qquad \text{... (11)}$$

Equation (11) represents total e.m.f. induced when a conductor is moved in a uniform constant magnetic field.

If the directions of velocity $\bar{v}$ with which conductor is moving and the magnetic field $\bar{B}$ are mutually perpendicular to each other, then the induced e.m.f. is given by,

$$\boxed{e = Blv \sin 90^\circ = Blv} \qquad \text{..(12)}$$

where $\qquad l$ = Length of straight conductor

iii) If in case, the magnetic flux density is also varying with time, then the induced e.m.f. is the combination of transformer e.m.f. and generator e.m.f. given by,

$$\boxed{\oint \bar{E} \cdot d\bar{L} = -\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S} + \oint (\bar{v} \times \bar{B}) \cdot d\bar{L}} \qquad \text{... (13)}$$

Example 9.1 : *A conductor 1 cm in length is parallel to z-axis and rotates at radius of 25 cm at 1200 r.p.m. Find induced voltage, if the radial field is given by,*

$$\bar{B} = 0.5\,\bar{a}_r \text{ T}$$

Solution : In above case, the magnetic flux is constant while the path is rotating at 1200 r.p.m. Under such condition, the field intensity is given by,

$$\bar{E} = \bar{v} \times \bar{B}$$

where $\bar{v}$ = Linear velocity

In 1 minute there are 1200 revolutions which corresponds to 20 revolutions in one second. In one revolution distance travelled is $(2\pi r)$ meter. Hence in 20 revolutions the distance travelled in one second is $(40\pi r)$ meter. The conductor rotates in ϕ-direction. Hence linear velocity is given by,

$$\bar{v} = (40\pi r)\bar{a}_\phi$$
$$= 40\pi(25\times10^{-2})\bar{a}_\phi$$
$$= 31.416\ \bar{a}_\phi\ \text{m/s}$$

Hence an electric field intensity is calculated as,

$$\bar{E} = [31.416\bar{a}_\phi] \times [0.5\bar{a}_r]$$
$$= 15.708\ (-\bar{a}_z) \qquad \dots\ \bar{a}_\phi \times \bar{a}_r = -\bar{a}_z$$

Induced voltage is given by,

$$e = \oint \bar{E} \cdot d\bar{L}$$

Now $d\bar{L} = (dz)\bar{a}_z$ as conductor is parallel to z-axis.

$$e = \int_{z=0}^{0.01} 15.708(-\bar{a}_z) \cdot (dz)\,\bar{a}_z$$
$$= -15.708\,[z]_0^{0.01} = -157.08\ \text{mV}$$

Negative sign indicates upper end of the conductor is positive while lower end is negative. Thus the magnitude of the induced voltage is 157.08 mV.

Example 9.2 : *A circular loop conductor lies in plane z = 0 and has a radius of 0.1 m and resistance of 5 Ω. Given $\bar{B} = 0.2 \sin 10^3 t\ \bar{a}_z$ J, determine the current in the loop.*

Solution : To find current in the loop, let us first calculate induced e.m.f.

A circular loop is in z = 0 plane. $\bar{B}$ is in z-direction which is perpendicular to the loop. So $\bar{B}$ is perpendicular to the circular loop.

Hence total flux is given by,

$$\Phi = \int_S \bar{B} \cdot d\bar{S}$$

With cylindrical co-ordinate system,

$$d\bar{S} = (r\,dr\,d\phi)\bar{a}_z$$

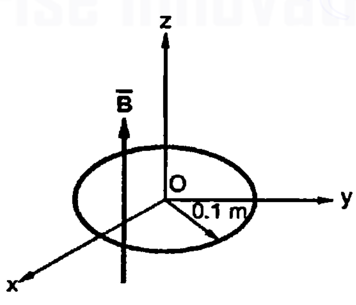


Fig. 9.1

From equation (3) it is clear that when $\frac{\partial \rho_v}{\partial t}=0$, then only equation (2) becomes true. Thus equations (2) and (3) are not compatible for time varying fields. We must modify equation (1) by adding one unknown term say $\bar{N}$.

Then equation (1) becomes,

$$\nabla \times \bar{H} = \bar{J}+\bar{N} \quad \text{... (4)}$$

Again taking divergence on both the sides

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}+\nabla \cdot \bar{N}=0$$

As $\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$, to get correct conditions we must write,

$$\nabla \cdot \bar{N} = \frac{\partial \rho_v}{\partial t}$$

But according to Gauss's law,

$$\rho_v = \nabla \cdot \bar{D}$$

Thus replacing ρ_v by $\nabla \cdot \bar{D}$

$$\nabla \cdot \bar{N} = \frac{\partial}{\partial t}(\nabla \cdot \bar{D})$$

$$= \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

Comparing two sides of the equation,

$$\bar{N} = \frac{\partial \bar{D}}{\partial t} \quad \text{... (5)}$$

Now we can write Ampere's circuital law in point form as,

$$\boxed{\nabla \times \bar{H} = \bar{J}_C+\frac{\partial \bar{D}}{\partial t}} \quad \text{... (6)}$$

The first term in equation (6) is **conduction current density** denoted by $\bar{J}_C$. Here attaching subscript C indicates that the current is due to the moving charges.

The second term in equation (6) represents current density expressed in ampere per square meter. As this quantity is obtained from time varying electric flux density. This is also called **displacement density.** Thus this is called **displacement current density** denoted by $\bar{J}_D$. With these definitions we can write equation (6) as,

$$\boxed{\nabla \times \bar{H} = \bar{J}_C+\bar{J}_D} \quad \text{... (7)}$$

Consider a parallel circuit of a resistor and capacitor driven by a time varying voltage V as shown in the Fig. 9.2.

Let the current flowing through resistor R be i_1 and the current flowing through capacitor C be i_2. The nature of the current flowing through the resistor R is different than

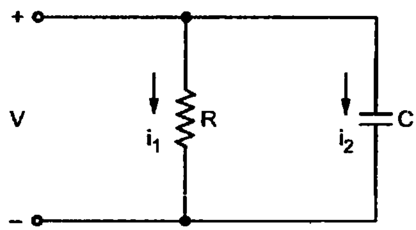


Fig. 9.2

that flowing through the capacitor. The current through resistor is due to the actual motion of charges. Thus the current through resistor can be written as,

$$i_1 = \frac{V}{R} \qquad \text{... (8)}$$

This current is called **conduction** current as the current is flowing because of actual motion of charges. Let it be denoted by i_C.

Let A be the cross-sectional area of resistor, then the conduction current density is given by,

$$\boxed{\bar{J}_C = \frac{i_C}{A} = \sigma \bar{E}} \qquad \text{... (9)}$$

Now assume that the initial charge on a capacitor is zero. Then for time varying voltage applied across parallel plate capacitor, the current through the capacitor is given by,

$$i_2 = C \frac{dv}{dt} \qquad \text{... (10)}$$

Let the two plates of area A are separated by distance d with dielectric having permittivity ε in between the plates. Then we can write

$$i_2 = \frac{\varepsilon A}{d} \frac{dv}{dt} \qquad \text{... (11)}$$

Now this current is called **displacement current** denoted by i_D. The electric field produced by the voltage applied between the two plates is given by,

$$E = \frac{V}{d}$$

or

$$V = (d)(E) \qquad \text{... (12)}$$

Substituting value of V in equation (11), we get,

$$i_D = i_2 = \frac{\varepsilon A}{d} \frac{d}{dt} (dE)$$

$$\therefore \quad i_D = \frac{\varepsilon A}{d} d \frac{dE}{dt} \qquad \text{... As distance d is not varying with time}$$

$$\therefore \quad i_D = \varepsilon A \frac{dE}{dt}$$

Now the ratio of current to the area of plate is the current density. In this case it is displacement current density denoted by J_D.

$$\therefore \quad \bar{J}_D = \frac{i_D}{A}$$

$$\therefore \quad \bar{J}_D = \frac{\varepsilon A}{d} \frac{d\bar{E}}{dt}$$

In other words, if the ratio of the magnitudes of the current densities is greater than 1, the medium is conductor and if the ratio of the magnitudes is less than 1 then the medium is dielectric

If	$\frac{\sigma}{\omega\varepsilon} >> 1,$	Medium is conductor
If	$\frac{\sigma}{\omega\varepsilon} << 1,$	Medium is dielectric

Also the ratio represented above depends on frequency, a medium which is conductor at low frequency may become insulator at very high frequency.

Example 9.3 : *In a given lossy dielectric medium, conduction current density* $J_C = 0.02 \sin 10^9 t\,(A/m^2)$. *Find the displacement current density if* $\sigma = 10^3$ *S/m and* $\varepsilon_r = 6.5$.

Solution : For lossy dielectric medium,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon}$$

$$\therefore \quad J_D = \frac{\omega\varepsilon J_C}{\sigma} = \frac{10^9 \times (\varepsilon_r \varepsilon_0) \times 0.02}{10^3}$$

$$\therefore \quad J_D = \frac{10^9 \times 6.5 \times 8.854 \times 10^{-12} \times 0.02}{10^3}$$

$$\therefore \quad J_D = 1.151 \times 10^{-6}\ A/m^2 = \mathbf{1.151\ \mu A/m^2}$$

As $\bar{J}_D$ and $\bar{J}_C$ are always at right angles to each other, we can write,

$$\bar{J}_D = \mathbf{1.151 \cos 10^9 t\ \mu A/m^2}$$

9.4 General Field Relations for Time Varying Electric and Magnetic Fields

The basic relation between an electric and magnetic field, starting from Faraday's law is given by,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \text{... (1)}$$

But we have already studied that,

$$\bar{B} = \nabla \times \bar{A} \qquad \text{where } \bar{A} \text{ is vector magnetic potential.}$$

$$\therefore \quad \nabla \times \bar{E} = -\frac{\partial}{\partial t}(\nabla \times \bar{A}) \qquad \text{... (2)}$$

Intercharging operators at R.H.S. of above equation, we get,

$$\nabla \times \bar{E} = -\nabla \times \frac{\partial \bar{A}}{\partial t}$$

$$\therefore \quad \nabla \times \bar{E} + \nabla \times \frac{\partial \bar{A}}{\partial t} = 0$$

$$\therefore \quad \nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t} \right) = 0 \quad \text{... (3)}$$

But according to vector identity 'curl' of a gradient of a scalar is always zero'. Hence we can write,

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = \nabla V \quad \text{... (4)}$$

As R.H.S. of the equation (3) including curl is zero, we can introduce negative sign at R.H.S. of the equation (4).

$$\therefore \quad \bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} \quad \text{... (5)}$$

Now when the field is static, $\frac{\partial \bar{A}}{\partial t} = 0$, hence we get basic gradient relationship as,

$$\bar{E} = -\nabla V \quad \text{... (6)}$$

Consider any closed surface. If the current is flowing out of the surface, we can write

$$I = \frac{dQ}{dt} \text{ A i.e. C/sec}$$

As current is flowing out of the surface, it indicates that positive charge is going out. So the positive charge is decreasing internally. Let Q_I be the internal charge,

$$\therefore \quad I = -\frac{dQ_I}{dt} \quad \text{... (7)}$$

If there is a volume charge ρ_v, then we can write,

$$Q_I = \int_v \rho_v \, dv \quad \text{... (8)}$$

$$\therefore \quad I = -\frac{d}{dt}\left[\int_v \rho_v \, dv \right]$$

Changing operations, we can write,

$$I = -\int_v \frac{d\rho_v}{dt} \, dv \quad \text{... (9)}$$

But current can be expressed as

$$I = \int_v \bar{J} \cdot d\bar{S} \quad \text{... (10)}$$

Equating equations (9) and (10),

$$\int_v \bar{J} \cdot d\bar{S} = -\int_v \frac{d\rho_v}{dt} \, dv$$

Using divergence theorem, converting surface integral to volume integral, assuming that the volume v is enclosed by the same surface S.

$$\therefore \quad \int_v \nabla \cdot J \, dv = -\int_v \frac{d\rho_v}{dt} dv$$

$$\therefore \quad \boxed{\nabla \cdot \bar{J} = -\frac{d\rho_v}{dt}} \qquad \text{... (11)}$$

Equation (11) is called **equation of continuity of current in point or differential form**

Consider Ampere's circuit law in point or differential form as,

$$\nabla \times \bar{H} = \bar{J}$$

Taking divergence on both sides of above equation, we get,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} \qquad \text{... (12)}$$

According to vector identity, **'divergence of curl of vector is zero'.**

But $\nabla \cdot \bar{J} = 0$ is valid only for static fields. For time varying field, we must modify above relation to have above property valid as,

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

$$\therefore \quad \nabla \cdot \bar{J} + \frac{\partial \rho_v}{\partial t} = 0 \qquad \text{... (13)}$$

Now for time varying fields we can write,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \frac{\partial \rho_v}{\partial t} \qquad \text{...(14)}$$

But we know that,

$$\nabla \cdot \bar{D} = \rho_v \qquad \text{... Gauss's law in point form,}$$

Putting in equation (14),

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \frac{\partial}{\partial t}[\nabla \cdot \bar{D}]$$

Interchanging operations of R.H.S. of above equation,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

$$\text{or} \quad \boxed{\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}} \qquad \text{... (15)}$$

Above equation is Ampere's circuit law for time varying fields. In this equation, $\bar{J}$ represents conduction current density while $\frac{\partial \bar{D}}{\partial t}$ represents displacement current density. So we can rewrite equation (15) as follows.

$$\boxed{\nabla \times \bar{H} = \bar{J}_C + \bar{J}_D} \qquad \text{... (16)}$$

9.5 Maxwell's Equations

We have previously studied that a static electric field $\bar{E}$ can exist without a magnetic field $\bar{H}$ demonstrated by a capacitor with a static charge Q. Similarly a conductor with a constant current I has a magnetic field $\bar{H}$ in the absence of an electric field $\bar{E}$. But in case of the time variable fields, $\bar{E}$ and $\bar{H}$ cannot exist without each other.

The valuable work done by James Clerk Maxwell helped in discovering electromagnetic waves. The time varying fields are involved in the experiments of Faraday, Hertz and the theoretical analysis done by Maxwell.

Maxwell's equations are nothing but a set of four expressions derived from Ampere's circuit law, Faraday's law, Gauss's law for electric field and Gauss's law for magnetic field. These four expressions can be written in following forms

i) Point or differential form, ii) Integral form

1) According to Ampere's circuit law, the line integral of magnetic field intensity $\bar{H}$ around a closed path is equal to the current enclosed by the path.

$$\therefore \quad \oint \bar{H} \cdot d\bar{L} = I_{enclosed}$$

Replacing current by the surface integral of conduction current density $\bar{J}$ over an area bounded by the path of integration of $\bar{H}$, we get more general relation as,

$$\oint \bar{H} \cdot d\bar{L} = \int_S \bar{J} \cdot dS \qquad \text{... (1)}$$

Above expression can be made further general by adding displacement current density to conduction current density as follows,

$$\boxed{\oint \bar{H} \cdot d\bar{L} = \int_S \left[\bar{J} + \frac{\partial \bar{D}}{\partial t}\right] \cdot d\bar{S}} \qquad \text{... (1-a)}$$

Equation (1-a) is Maxwell's equation derived from Ampere's circuit law. This equation is in integral form in which line integral of $\bar{H}$ is carried over the closed path bounding the surface S over which the integration is carried out on R.H.S. In the circuit theory, closed path is called **Mesh.** Hence the equation considered above is also called **Mesh equation or Mesh relation.**

Statement : "The total magnetomotive force around any closed path is equal to the surface integral of the conduction and displacement current densities over the entire surface bounded by the same closed path."

Applying Stoke's theorem to L.H.S. of the equation (1-a), we get,

$$\int_S (\nabla \times \bar{H}) \cdot d\bar{S} = \int_S \left[\bar{J} + \frac{\partial \bar{D}}{\partial t}\right] \cdot d\bar{S}$$

Assuming that the surface considered for both the integrations is same, we can write,

$$\boxed{\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}} \qquad \text{... (1-b)}$$

This is Maxwell's equation for electric fields derived from Gauss's law which is expressed in point form or differential form.

4) For magnetic fields, the surface integral of $\bar{B}$ over a closed surface S is always zero, due to non existance of monopole in the magnetic fields.

$$\boxed{\int_S \bar{B} \cdot d\bar{S} = 0} \quad \text{... (4-a)}$$

This is **Maxwell's magnetic field equation** expressed in **integral form.** This is derived for Gauss's law applied to the magnetic fields.

Statement : "The surface integral of magnetic flux density over a closed surface is always equal to zero."

Using divergence theorem, the surface integral can be converted to volume integral as,

$$\int_S (\nabla \cdot \bar{B}) dv = 0$$

But being a finite volume, $dv \neq 0$,

$$\therefore \quad \boxed{\nabla \cdot \bar{B} = 0} \quad \text{... (4-b)}$$

This is differential form or **point form of Maxwell's equation derived from Gauss's law applied to the magnetic fields.**

Table 9.1 summarizes Maxwell's equations

Differential form	Integral form	Significance
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \cdot d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S}$	Faraday's law
$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \cdot d\bar{L} = I + \int_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{S}$	Ampere's circuital law
$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot d\bar{S} = \int_S \rho_v dv$	Gauss's law
$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot d\bar{S} = 0$	No isolated magnetic charges.

Table 9.1 Maxwell's equations

9.5.1 Maxwell's Equations for Free Space

In the previous section, we have obtained Maxwell's equations in integral and point form. Let us consider now free space as a medium in which fields are present. Free space is a non-conducting medium in which volume charge density ρ_v is zero and conductivity σ is also zero.

The Maxwell's equation, in the free space are as mentioned below.

A) Point Form :

i) $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

$$\therefore \qquad \frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z} = 0$$

$$\therefore \qquad \frac{\partial}{\partial x}(10x)+\frac{\partial}{\partial y}(-4y)+\frac{\partial}{\partial z}(kz) = 0$$

$$\therefore \qquad 10-4+k = 0$$

$$\therefore \qquad k = -6\ \mu C/m^3$$

Example 9.5 : *If the magnetic field $\bar{H}=[3x\cos\beta+6y\sin\alpha]\bar{a}_z$, find current density $\bar{J}$ if fields are invariant with time.*

Solution : The point form of Maxwell's second equation is,

$$\nabla\times\bar{H} = \bar{J}+\frac{\partial\bar{D}}{\partial t}$$

But as fields are time invariant, we can write,

$$\frac{\partial\bar{D}}{\partial t} = 0$$

$$\therefore \qquad \nabla\times\bar{H} = \bar{J}$$

$$\therefore \qquad \bar{J} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (3x\cos\beta+6y\sin\alpha) \end{vmatrix}$$

$$\therefore \qquad \bar{J} = \frac{\partial}{\partial y}[3x\cos\beta+6y\sin\alpha]\,\bar{a}_x-\frac{\partial}{\partial x}[3x\cos\beta+6y\sin\alpha]\,\bar{a}_y$$

$$\therefore \qquad \bar{J} = 6\sin\alpha\,\bar{a}_x-3\cos\beta\,\bar{a}_y \quad A/m^2$$

9.6 Boundary Conditions for Time Varying Fields

The relationship between the electric flux density $\bar{D}$, electric field intensity $\bar{E}$, magnetic flux density $\bar{B}$ and magnetic field intensity $\bar{H}$ can be explained with the help of the point form or the integral form of Maxwell's equations. The field equations postulated by Maxwell are valid at a point in a continuous medium. The Maxwell's equations are useful in determining the conditions at the boundary surface of the two different media. We can apply the concepts of linear, isotropic and homogeneous medium. Consider the boundary between medium 1 with parameters ε_1, μ_1 and σ_1 and medium 2 with parameters ε_2, μ_2 and σ_2. In general, the boundary conditions for time varying fields are same as those for static fields. Thus at the boundary, refering boundary conditions for static electric magnetic fields, we can write,

i) The tangential component of electric field intensity $\bar{E}$ is continuous at the surface.

$$E_{tan1} = E_{tan2} \qquad \dots (1)$$

ii) The tangential component of the magnetic field intensity is continuous across the surface except for a perfect conductor.

$$H_{tan1} = H_{tan2} \quad \text{... (2)}$$

At the surface of the perfect conductor, the tangential component of the magnetic field intensity is discontinuous at the boundary.

$$\therefore \quad H_{tan1} - H_{tan2} = K \quad \text{... (2-a)}$$

iii) The normal component of the electric flux density is continuous at the boundary if the surface charge density is zero.

$$\therefore \quad D_{N1} = D_{N2} \quad \text{... (3)}$$

If the surface charge density is non zero, then the normal component of the electric flux density is discontinuous at the boundary.

$$D_{N1} - D_{N2} = \rho_S \quad \text{... (3-a)}$$

iv) The normal component of the magnetic flux density is continuous at the boundary.

$$B_{N1} = B_{N2} \quad \text{... (4)}$$

9.7 Retarded Potentials

For static electric fields, the electric scalar potential is given by,

$$V = \int_v \frac{\rho_v}{4\pi\varepsilon R} dv \quad \text{... (1)}$$

For static magnetic fields, the magnetic vector potential is given by,

$$\bar{A} = \int_v \frac{\mu \bar{J}}{4\pi R} dv \quad \text{... (2)}$$

Let us now study the behaviour of these potentials when the fields are time varying.

For time varying fields,

$$\bar{B} = \nabla \times \bar{A} \quad \text{... (3)}$$

As derived in earlier section, if we combine above relation with the expression of Faraday's law, we can write,

$$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} \quad \text{... (4)}$$

It is clear from equations (3) and (4) that we can determine fields $\bar{B}$ and $\bar{E}$ provided that potentials V and $\bar{A}$ are known. It is necessary to find expressions for V and $\bar{A}$, suitable for time varying fields.

From the general field relations for time varying electric and magnetic fields,

$$\bar{D} = \varepsilon \bar{E} \quad \text{and}$$

$$\nabla \cdot \bar{D} = \rho_v$$

By taking divergence of equation (4) and using above two relations which are valid for time varying fields, we get

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \bar{A})$$

$$\therefore \quad \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \bar{A}) = -\frac{\rho_v}{\varepsilon} \qquad \text{... (5)}$$

Consider Maxwell's equation given below

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

But $\quad \bar{B} = \mu \bar{H}, \quad \bar{D} = \varepsilon \bar{E} \quad$ and $\quad \bar{B} = \nabla \times \bar{A}$

$$\therefore \quad \frac{1}{\mu} \nabla \times \nabla \times \bar{A} = \bar{J} + \varepsilon \frac{\partial \bar{E}}{\partial t}$$

$$\therefore \quad \nabla \times \nabla \times \bar{A} = \mu \bar{J} + \varepsilon\mu \frac{\partial}{\partial t}\left[-\nabla V - \frac{\partial \bar{A}}{\partial t}\right]$$

$$\therefore \quad \nabla \times \nabla \times \bar{A} = \mu \bar{J} - \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \bar{A}}{\partial t^2} \qquad \text{... (6)}$$

But from vector identity,

$$\nabla \times \nabla \times \bar{A} = \nabla (\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$$

We can rewrite equation (6) as follows,

$$\nabla (\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu \bar{J} - \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \bar{A}}{\partial t^2}$$

$$\therefore \quad \nabla^2 \bar{A} - \nabla (\nabla \cdot \bar{A}) = -\mu \bar{J} + \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\varepsilon \frac{\partial^2 \bar{A}}{\partial t^2} \qquad \text{... (7)}$$

It is important that complicated equations (5) and (7) are not sufficient enough to define $\bar{A}$ and V completely. These two equations demonstrate necessary but not the sufficient conditions. In general any vector field can be uniquely defined if its curl and divergence are known and the value of the field is known at any one point.

The curl of $\bar{A}$ is already specified in the equation (3). Now we may choose the divergence of $\bar{A}$ from equation (7) as

$$\boxed{\nabla \cdot \bar{A} = -\mu\varepsilon \frac{\partial V}{\partial t}} \qquad \text{... (8)}$$

Equation (8) gives relationship between $\bar{A}$ and V. It is called **Lorentz condition for potentials.**

Using the Lorentz condition in equation (5), we get,

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu\varepsilon \frac{\partial V}{\partial t}\right) = -\frac{\rho_v}{\varepsilon}$$

$$\therefore \quad \nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \qquad \text{... (9)}$$

Similarly using Lorentz condition in equation (7), we get,

From above expression of $\bar{A}$,

$$A_x = y\left(\frac{x}{a}+t\right), \quad A_y = 0 \quad \text{and} \quad A_z = 0$$

$$\therefore \quad \nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Putting values of A_x, A_y and A_z, we can write,

$$\nabla \cdot \bar{A} = \frac{\partial}{\partial x}\left[y\left(\frac{x}{a}+t\right)\right]$$

$$\therefore \quad \nabla \cdot \bar{A} = \frac{y}{a} \qquad \text{... (1)}$$

Now

$$-\mu\varepsilon\frac{\partial v}{\partial t} = -\mu\varepsilon\frac{\partial}{\partial t}[-y(x+at)]$$

$$= -\mu\varepsilon(-ya)$$

$$= \mu\varepsilon y \cdot a = \frac{\mu\varepsilon}{\sqrt{\mu\varepsilon}} \cdot y \qquad \text{... } a = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$= \sqrt{\mu\varepsilon}\; y = \frac{y}{a} \qquad \text{... (2)}$$

From equations (1) and (2), we can write,

$$\nabla \cdot \bar{A} = -\mu\varepsilon\frac{\partial v}{\partial t}$$

b)

$$\bar{B} = \nabla \times \bar{A}$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

From expression of $\bar{A}$,

$$A_x = y\left(\frac{x}{a}+t\right), \quad A_y = 0, \quad A_z = 0$$

$$\therefore \quad \bar{B} = \frac{\partial}{\partial z}\left[y\left(\frac{x}{a}+t\right)\right]\bar{a}_y - \frac{\partial}{\partial y}\left[y\left(\frac{x}{a}+t\right)\right]\bar{a}_z$$

9.8 Phasor Representation of a Vector

In general, any complex number m can be written as,

$$m = a + jb = r \angle \theta^\circ \quad \text{... (1)}$$

or

$$m = re^{j\theta} = r(\cos\theta + j\sin\theta) \quad \text{... (2)}$$

In equations (1) and (2), a and b are the real and imaginary parts of complex number m. The symbol j represents complex operator. Its value is $\sqrt{-1}$. The magnitude of m is given by,

$$r = |m| = \sqrt{a^2 + b^2} \quad \text{... (3)}$$

The phase angle is given by,

$$\theta = \tan^{-1}\frac{b}{a} \quad \text{... (4)}$$

From above discussion, it is clear that any phasor can be represented in rectangular as well as polar form represented by equations (1) to (4). Note that the phasor representation is applicable only to the sinusoidal signals. Any sinusoidal signal can be defined with the help of three parameters namely amplitude, frequency and phase. Let the applied electric field is given by,

$$E = E_m \cos(\omega t + \phi)$$

where E_m = Amplitude, ωt = Angular frequency and

ϕ = Phase angle

According to Euler's identity, $e^{j\theta} = \cos\theta + j\sin\theta$

Thus the real and imaginary parts of $E_m\, e^{j\theta}$ (where $\theta = \omega t + \phi$) are given by,

$$\mathrm{Re}\left(E_m\, e^{j\theta}\right) = E_m \cos(\omega t + \phi) \quad \text{... (5)}$$

and

$$I_m\left(E_m\, e^{j\theta}\right) = E_m \sin(\omega t + \phi) \quad \text{... (6)}$$

Hence we can write,

$$E = \mathrm{Re}\left(E_m\, e^{j\theta}\right) = \mathrm{Re}\left(E_m e^{j\omega t} e^{j\phi}\right) \quad \text{... (7)}$$

The complex term $E_m\, e^{j\phi}$ is called **phasor**. Generally it is represented by attaching suffix s to the quantity of concern, such as E_s.

A phasor may be either scalar or vector.

Let the vector $\overline{M}$ is time varying field which varies with respect of x, y, z and t. Then the phasor form of $\overline{M}$ is obtained by dropping the time factor. Let it be $\overline{M}_s$ which depends only on x, y and z. Then the two quantities are related to each other by the relation.

$$\overline{M} = \mathrm{Re}\left(\overline{M}_s\, e^{j\omega t}\right) \quad \text{... (8)}$$

Differentiating $\overline{M}$ with respect to t partially,

$$\frac{\partial \overline{M}}{\partial t} = \frac{\partial}{\partial t}\mathrm{Re}(\overline{M}_s\, e^{j\omega t})$$

$$\therefore \quad \boxed{\frac{\partial \overline{M}}{\partial t} = \text{Re}(j\omega \overline{M}_s e^{j\omega t})} \quad \text{... (9)}$$

Similarly we can write,

$$\int \overline{M} \partial t = \int \text{Re}(\overline{M}_s e^{j\omega t}) \partial t$$

$$\therefore \quad \boxed{\int \overline{M} \partial t = \text{Re}\left(\frac{\overline{M}_s}{j\omega} e^{j\omega t}\right)} \quad \text{... (10)}$$

Key Point : *From equations (9) and (10) it is clear that, differentiating and integrating the quantity with respect to time is equivalent to multiplying and dividing the phasor of that quantity by factor jω respectively.*

Examples with Solutions

Example 9.7 : *A rectangular conducting loop with a resistance of 0.2 Ω rotates at 500 r.p.m. The vertical conductor at $r_1 = 0.03$ m is in the field $\overline{B}_1 = 0.25\ \overline{a}_r$ T and other conductor is at $r_2 = 0.05$ m and in the field $\overline{B}_2 = 0.8\ \overline{a}_r$ T. Find current flowing in the loop.*

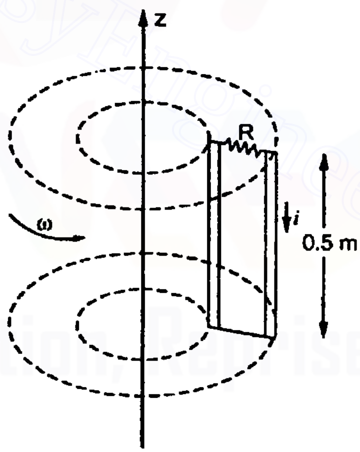


Fig. 9.4

Solution : The inner conductor which is at $r_1 = 0.03$ m rotates at 500 r.p.m. Thus inner conductor rotates with $\frac{500}{60}$ revolutions per second. As in one second, the distance covered is $(2\pi r)$ meter, for the inner conductor the distance covered is $\left(\frac{500}{60}\right)(2\pi r)$ meters.

Then the linear velocity for inner conductor is given by,

$$\overline{v}_1 = \left(\frac{500}{60}\right)(2\pi)(0.03)\ \overline{a}_\phi \quad \text{m/s}$$

$$= 1.5707\ \bar{a}_\phi \text{ m/s}$$

Similarly for outer conductor, linear velocity is given by,

$$\bar{v}_2 = \left(\frac{500}{60}\right)(2\pi)(0.05)\bar{a}_\phi \text{ m/s}$$

$$= 2.6179\ \bar{a}_\phi \text{ m/s}$$

Here $\bar{B}$ is not varying with time, it is constant in $\bar{a}_r$ direction. Thus under such condition, the induced e.m.f. is given by,

$$\text{e.m.f.} = \int \bar{E} \cdot d\bar{L}$$

where $\bar{E} = \bar{v} \times \bar{B}$

For inner conductor,

$$\bar{E}_1 = \bar{v}_1 \times \bar{B}_1$$

$$= [1.5707\ \bar{a}_\phi] \times [0.25\ \bar{a}_r]$$

$$= -0.3926\ \bar{a}_z \qquad \text{....... } \bar{a}_\phi \times \bar{a}_r = -\bar{a}_z$$

Both the conductors are vertical. Let us assume that length of each conductor be 0.5 m.

$$\therefore \quad d\bar{L}_1 = dz\bar{a}_z$$

$$\therefore \quad \text{e.m.f.}_1 = \int \bar{E}_1 \cdot d\bar{L}_1 = \int_{z=0}^{0.5\text{ m}} (-0.3926\bar{a}_z) \cdot (dz\bar{a}_z)$$

$$= -0.3925\ [z]_0^{0.5}$$

$$= -0.1963 \text{ V}$$

For outer conductor,

$$\bar{E}_2 = \bar{v}_2 \times \bar{B}_2$$

$$= [2.6179\ \bar{a}_\phi] \times [0.8\ \bar{a}_r]$$

$$= -2.09432\ \bar{a}_z \qquad \text{....... } \bar{a}_\phi \times \bar{a}_r = -\bar{a}_z$$

$$\therefore \quad \text{e.m.f.}_2 = \int \bar{E}_1 \cdot d\bar{L}_2$$

$$= \int_{z=0}^{0.5\text{ m}} (-2.09432)\ \bar{a}_z \cdot dz\bar{a}_z$$

$$= -2.09432\ [z]_0^{0.5}$$

$$= -1.04716 \text{ V}$$

Hence current in the loop is given by,

$$i = \frac{\text{e.m.f.}_1 - \text{e.m.f.}_2}{R} = \frac{-0.1963-(-1.04716)}{0.2}$$

$$\therefore \quad i = \frac{0.85086}{0.2} = 4.2543 \text{ A}$$

Example 9.8 : *The circular loop conductor having a radius of 0.15 m is placed in X-Y plane. This loop consists of a resistance of 20 Ω as shown in the Fig. 9.5. If the magnetic flux density is*

$$\bar{B} = 0.5 \sin 10^3 t \, \bar{a}_z \text{ T}$$

Find current flowing through this loop.

Solution : The circular loop conductor is in X-Y plane. $\bar{B}$ is in $\bar{a}_z$ direction which is perpendicular to X-Y plane.

Hence, we can write,

$$d\bar{S} = (r\,dr\,d\phi)\bar{a}_z$$

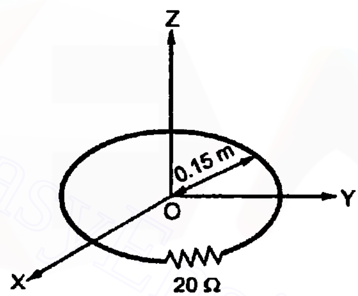


Fig. 9.5

Total flux is given by,

$$\Phi = \int_S \bar{B} \cdot d\bar{S}$$

$$= \int_{\phi=0}^{2\pi} \int_{r=0}^{0.15} [(0.5 \sin 10^3 t)\bar{a}_z] \cdot [(r\,dr\,d\phi)\bar{a}_z]$$

$$= (0.5 \sin 10^3 t)[\phi]_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{0.15}$$

$$= (0.5 \sin 10^3 t)\,[2\pi] \left[\frac{(0.15)^2}{2}\right]$$

$$= 35.3429 \sin 10^3 t \quad \text{mWb}$$

Now induced e.m.f. is given by,

$$e = -\frac{d\phi}{dt}$$

$$= -\frac{d}{dt}\left[35.3429\times10^{-3}\sin 10^3 t\right]$$

$$= -\left(35.3429\times10^{-3}\right)\left(10^3\right)\cos 10^3 t$$

$$= -35.3429 \cos 10^3 t \text{ V}$$

Hence current in the conductor is given by,

$$i = \frac{e}{R} = \frac{-35.3429\cos 10^3 t}{20}$$

$$\therefore \quad i = \mathbf{-1.7671 \cos 10^3 t \ \ A}$$

Example 9.9 : *An area of 0.65 m^2 in the plane z = 0 encloses a filamentary conductor. Find the induced voltage if,*

$$\bar{B} = 0.05 \cos 10^3 t\left(\frac{\bar{a}_y + \bar{a}_z}{\sqrt{2}}\right) \text{ Tesla.}$$

Solution : Here filamentary conductor is fixed and it is placed in z = 0 plane. It encloses area of 0.65 m^2.

$$\therefore \quad d\bar{S} = dS\, \bar{a}_z$$

Induced e.m.f. according to Faraday's law is given by,

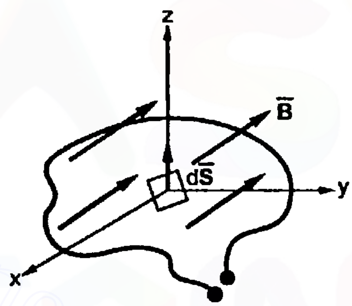


Fig. 9.6

$$e = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S}$$

$$= -\int_S \frac{\partial}{\partial t}\left[0.05\cos 10^3 t\left(\frac{\bar{a}_y + \bar{a}_z}{\sqrt{2}}\right)\right] \cdot (dS\, \bar{a}_z)$$

$$= -\int_S \frac{0.05\left(10^3\right)\left(-\sin 10^3 t\right)}{\sqrt{2}} dS \qquad \ldots \bar{a}_y \cdot \bar{a}_z = 0, \ \bar{a}_z \cdot \bar{a}_z = 1$$

$$= +35.355 \sin 10^3 t \left[\int_S dS\right]$$

But $\int_S dS$ is given as 0.65 m^2.

$$\therefore \quad e = 35.355 \sin 10^3 t\,(0.65)$$

$$\therefore \quad = \mathbf{22.98 \sin 10^3 t \ \ V}$$

Example 9.10 : *A conducting cylinder of radius 7 cm and height 50 cm rotates at 600 r.p.m. in a radial field $\bar{B} = 0.10\, \bar{a}_r$ T. Sliding contacts at the top and bottom are used to connect a voltmeter as shown in the Fig. 9.7. Calculate induced voltage.*

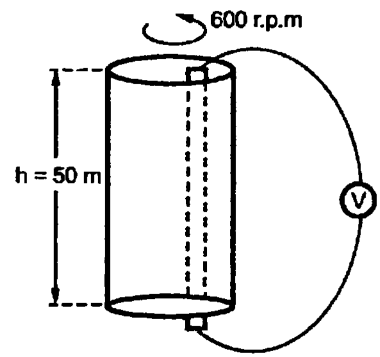


Fig. 9.7

Solution : A conducting cylinder rotates in the direction as shown in the Fig. 9.7. It rotates at 600 r.p.m. Means in 1 sec there are 10 revolutions. The radius of the cylinder is 0.07 m. In 1 revolution, the distance travelled by the cylinder is $(2\pi r)$ m i.e. $(2\times\pi\times 0.07)$ m. Hence in 10 revolutions, it travels $(2\times\pi\times 0.07\times 10)$ m distance. So the linear velocity is given by,

$$\bar{v} = (2\times\pi\times 0.07\times 10)\bar{a}_\phi \text{ m/s}$$

$$= 4.398\ \bar{a}_\phi \text{ m/s}$$

The electric field intensity is given by,

$$\bar{E} = \bar{v}\times\bar{B}$$

$$= (4.398\ \bar{a}_\phi)\times(0.20)\ \bar{a}_r$$

$$= 0.8796\ (-\bar{a}_z) \qquad \ldots\ \bar{a}_\phi\times\bar{a}_r = -\bar{a}_z$$

Here field is not varying with time. The cylindrical conductor is rotating. Each vertical element of it on the curved surface cuts same flux and thus the induced voltage is same. As these elements are as if in parallel, the e.m.f. induced in one element is same as that total e.m.f.

$$\therefore \qquad e = \int \bar{E}\cdot d\bar{L}$$

$$= \int_{z=0}^{0.5} 0.8796\ (-\bar{a}_z)\cdot(dz\,\bar{a}_z)$$

$$= -0.8796\ [z]_0^{0.5} \qquad \ldots\ \bar{a}_z\cdot\bar{a}_z = 1$$

$$= -0.4398 \text{ V}$$

Example 9.11 : *Find the frequency at which conduction current density and displacement current density are equal in a medium with* $\sigma = 2\times 10^{-4}$ ℧/m *and* $\varepsilon_r = 81$.

Solution : The ratio of amplitudes of the two current densities is given as 1, so we can write,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon} = 1$$

i.e. $$\omega = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_0 \varepsilon_r}$$

$\therefore$ $$\omega = \frac{2\times 10^{-4}}{(8.854\times 10^{-12})(81)} = 0.2788\times 10^{6} \text{ rad/sec}$$

But $$\omega = 2\pi f$$

$\therefore$ $$f = \frac{\omega}{2\pi} = \frac{0.2788\times 10^{6}}{2\pi} = 44.372 \text{ kHz}$$

Hence, the frequency at which the ratio of amplitudes of conduction and displacement current density is unity, is 44.372 kHz.

Example 9.12 : *In a material for which σ = 5.0 S/m and ε_r = 1, the electric field intensity is $E = 250 \sin 10^{10} t$ V/m. Find the conduction and displacement current densities, and the frequency at which both have equal magnitudes.*

Solution : The conduction current density is given by,

$$J_C = \sigma E$$
$$= 5(250 \sin 10^{10} t)$$
$$= 1250 \sin 10^{10} t \text{ A/m}^2$$

The displacement current density is given by,

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon E)$$
$$= \frac{\partial}{\partial t}[\varepsilon_0 \varepsilon_r E]$$
$$= \frac{\partial}{\partial t}[8.854\times 10^{-12} \times 1 \times 250 \sin 10^{10} t]$$
$$= (8.854\times 10^{-12} \times 250)(10^{10})(\cos 10^{10} t)$$
$$= 22.135 \cos 10^{10} t \text{ A/m}^2$$

For the two densities, the condition for magnitudes to be equal is,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\varepsilon\omega} = 1$$

Solution : a) The conduction current density is given by,

$$J_C = \sigma E = \sigma E_m \cos \omega t$$

The displacement current density is given by,

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial \varepsilon E}{\partial t} = \varepsilon \frac{\partial}{\partial t}[E_m \cos \omega t]$$

$$\therefore \quad J_D = -\omega \varepsilon E_m \sin \omega t$$

$\therefore$ The ratio of the amplitudes of the two densities is given by,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma E_m}{\omega \varepsilon E_m} = \frac{\sigma}{\omega \varepsilon}$$

b) Applied field $E = E_m e^{-t/\tau}$

$$\therefore \quad J_C = \sigma E = \sigma E_m e^{-t/\tau}$$

Also $$J_D = \varepsilon \frac{\partial E}{\partial t} = \varepsilon E_m \left(-\frac{1}{\tau}\right) e^{-t/\tau} = -\frac{\varepsilon E_m}{\tau} e^{-t/\tau}$$

Now the ratio of the amplitudes of the two densities is given by,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma E_m}{\frac{\varepsilon E_m}{\tau}} = \frac{\sigma \tau}{\varepsilon}$$

Example 9.15 : *Find the amplitude of the displacement current density,*

a) In the air near car antenna where the field strength of FM signal is,
$\bar{E} = 80 \cos(6.277 \times 10^8 t - 2.092 y)\bar{a}_z$ *V/m ;*

b) Inside a capacitor where $\varepsilon_r = 600$ *and*
$\bar{D} = 3 \times 10^{-6} \sin(6 \times 10^6 t - 0.3464 x)\bar{a}_z$ *C/m².*

Solution : $\bar{E} = 80 \cos(6.277 \times 10^8 t - 2.092 y)\bar{a}_z$

The displacement current density is given by,

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_0 \varepsilon_r \bar{E})$$

For air, $\varepsilon_r = 1$

$$\therefore \quad \bar{J}_D = \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$= \varepsilon_0 \frac{\partial}{\partial t}\left[80 \cos(6.277 \times 10^8 t - 2.092 y)\bar{a}_z\right]$$

$$= (8.854 \times 10^{-12})(80)(-6.277 \times 10^8) \sin(6.277 \times 10^8 t - 2.092 y)\bar{a}_z$$

$$\therefore \quad \bar{J}_D = -0.4446 \sin(6.277 \times 10^8 t - 2.092 y)\,\bar{a}_z \quad A/m^2$$

Thus the amplitude of the displacement current density is,

$$J_D = 0.4446 \ \ A/m^2$$

b) Inside capacitor $\varepsilon_r = 600$, the displacement current density is given by,

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$$

$$= \frac{\partial}{\partial t}\left[3\times10^{-6}\sin\left(6\times10^6\ t - 0.3464x\right)\bar{a}_z\right]$$

$$= \left(3\times10^{-6}\right)\left(6\times10^6\right)\cos\left(6\times10^6 t - 0.3464x\right)\bar{a}_z$$

$$= 18\cos\left(6\times10^6\ t - 0.3464x\right)\bar{a}_z \ \ A/m^2$$

Hence the amplitude of displacement current density is,

$$J_D = 18 \ \ A/m^2$$

Example 9.16 : *Two parallel conducting plates of area 0.05 m^2 are separated by 2 mm of a lossy dielectric for which $\varepsilon_r = 8.3$ and $\sigma = 8\times10^{-4}$ S/m. Given an applied voltage $v = 10\sin 10^7\ tV$. Find total r.m.s. current.*

Solution : Consider that voltage $v = 10\sin 10^7 t$ is applied across two parallel plates of a capacitor. These two plates are separated by a distance d = 2 mm as shown in the Fig. 9.8.

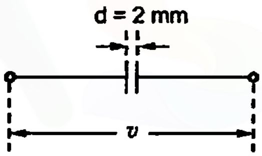


Fig. 9.8

The electric field produced due to the applied voltage v is given by,

$$E = \frac{v}{d} = \frac{10\sin 10^7\ t}{2\times10^{-3}} = 5000\sin 10^7 t \ \ V/m$$

$$J_C = \sigma E = \left(8\times10^{-4}\right)\left(5000\sin 10^7\ t\right) = 4\sin 10^7 t \ \ A/m^2$$

$$J_D = \varepsilon\frac{dE}{dt} = \varepsilon_0\varepsilon_r\frac{dE}{dt}$$

$$= 8.854\times10^{-12}\times8.3\ \frac{d}{dt}\left[5000\sin 10^7 t\right]$$

$$= 8.854\times10^{-12}\times8.3\times5000\times10^7\times\cos 10^7 t$$

$$= 3.6744\cos 10^7 t \ \ A/m^2$$

From the current densities we can get currents as given below.

The conduction current i_C is given by,

$$i_C = (J_C)(\text{Area}) = \left(4\sin 10^7\ t\right)(0.05) = 0.2\sin 10^7 t \ \ A$$

The displacement current i_D is given by,

$$i_D = (J_D)(\text{Area}) = \left(3.6744\cos 10^7 t\right)(0.05)$$

$$= 0.18372\cos 10^7 t \ \ A$$

Both the currents are at right angles to each other as shown in the Fig. 9.9.

$$I_T = \sqrt{i_C^2 + i_D^2}$$

$$= \sqrt{(0.2)^2 + (0.1837)^2}$$

$$= 0.2715 \text{ A}$$

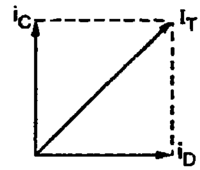


Fig. 9.9

Hence total r.m.s. current is given by,

$$I_{T(r.m.s.)} = \frac{I_T}{\sqrt{2}} = \frac{0.2715}{\sqrt{2}} = 0.1919 \text{ A}$$

Example 9.17 : *Find the displacement current density within a parallel plate capacitor having a dielectric with $\varepsilon_r = 10$, area of plates $A = 0.01\ m^2$, distance of separation $d = 0.05$ mm. Applied voltage is $V = 200 \sin 200t$.*

Solution : Current through a parallel plate capacitor is given by,

$$i_C = \left(\frac{\varepsilon \cdot A}{d}\right)\frac{dV}{dt} = \left(\frac{\varepsilon_0 \varepsilon_r \cdot A}{d}\right)\frac{dV}{dt}$$

Putting values of $\varepsilon_0, \varepsilon_r$, A, d and V,

$$i_C = \frac{(8.854\times10^{-12})(10)(0.01)}{0.05\times10^{-3}} \cdot \frac{d}{dt}[200 \sin 200t]$$

$$\therefore \quad i_C = 0.7083 \times 10^{-3} \cos 200t \text{ A}$$

As we know for parallel plate capacitor,

$$i_C = i_D,$$

The displacement current density is given by,

$$J_D = \frac{\text{Current}}{\text{Area}} = \frac{i_D}{A} = \frac{0.7083\times10^{-3} \cos 200t}{0.01}$$

$$\therefore \quad J_D = 70.832 \times 10^{-3} \cos 200t \text{ A/m}^2$$

Example 9.18 : *A parallel plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has voltage 50 sin 10^3t V applied to its plates. Calculate displacement current assuming $\varepsilon = 2\varepsilon_0$.*

Solution : $D = \varepsilon E = \varepsilon \frac{V}{d}$

Hence the displacement current density is given by,

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon \frac{V}{d}\right)$$

$$= \frac{\varepsilon}{d}\frac{dV}{dt}$$

Hence the displacement current is given by,

$$i_D = J_D \cdot \text{Area} = \left(\frac{\varepsilon}{d}\frac{dV}{dt}\right)(A) \quad \text{... Plate area = A}$$

$$\therefore \quad i_D = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

This current is same as conduction current.

$$\therefore \quad i_C = \frac{dQ}{dt} = A\frac{dD}{dt} = \varepsilon A\frac{dE}{dt} = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

Hence the conduction current and displacement current is same. The displacement current is given by

$$i_D = \frac{\varepsilon A}{d}\frac{dV}{dt}$$

$$= \frac{(2\varepsilon_0)(A)}{d}\frac{dV}{dt}$$

$$= \frac{2\times 8.854\times 10^{-12}\times 5\times 10^{-4}}{3\times 10^{-3}}\frac{d}{dt}\left(50\sin 10^3 t\right)$$

$$= \frac{2\times 8.854\times 10^{-12}\times 5\times 10^{-4}\times 50\times 10^3}{3\times 10^{-3}}\cos 10^3 t$$

$$= \mathbf{0.1475 \cos 10^3 t\ \mu A}$$

Example 9.19 : *A two dimensional electric field is given by $\bar{E} = x^2\,\bar{a}_x + x\bar{a}_y$ V/m. Show that this electric field cannot arise from a static distribution of charge.*

Solution : Consider Maxwell's equation for static fields,

$$\nabla \times \bar{E} = 0 \quad \text{... (1)}$$

Consider L.H.S. of equation (1),

$$\text{L.H.S.} = \nabla \times \bar{E}$$

$$= \nabla \times \left[x^2\,\bar{a}_x + x\,\bar{a}_y\right]$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & x & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right]\bar{a}_x - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2)\right]\bar{a}_z$$

$$+ \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^2)\right]\bar{a}_z$$

$$= [0] - [0] + (1)\,\bar{a}_z$$

$$= \bar{a}_z$$

But R.H.S. = 0. That means L.H.S. ≠ R.H.S.

Thus we have $\nabla \times \bar{E} \neq 0$ which indicates that the given electric field $\bar{E}$ is not static. But we can have a static field only if the charge distribution is static. From above calculation it is clear that $\bar{E}$ is not static implies this electric field can not arise from static distribution of charge.

Example 9.20 : *Do the fields* $\bar{E} = E_m \sin x \sin t\, \bar{a}_y$ *and* $\bar{H} = \dfrac{E_m}{\mu_0} \cos x \cos t\, \bar{a}_z$ *satisfy Maxwell's equations ?*

Solution : Consider Maxwell's equation derived from Faraday's law,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

We know that, $\bar{B} = \mu \bar{H} = \mu_0 \mu_r \bar{H}$

Let $\mu_r = 1$, so $\bar{B} = \mu_0 \bar{H}$

$$\therefore \quad \nabla \times \bar{E} = -\mu_0 \frac{\partial \bar{H}}{\partial t}$$

$$\text{L.H.S} = \nabla \times \bar{E}$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_m \sin x \sin t & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial z} E_m \sin x \sin t\right] \bar{a}_x + \left[\frac{\partial}{\partial x} E_m \sin x \sin t\right] \bar{a}_z$$

$$= E_m \sin t \cos x\, \bar{a}_z \qquad \text{... (i)}$$

$$\text{R.H.S.} = -\mu_0 \frac{\partial \bar{H}}{\partial t}$$

$$= -\mu_0 \frac{\partial}{\partial t}\left[\frac{E_m}{\mu_0} \cos x \cos t\right] \bar{a}_z$$

$$= -\mu_0 \left(\frac{E_m}{\mu_0}\right) \cos x \frac{\partial}{\partial t} (\cos t)\, \bar{a}_z$$

$$= -E_m \cos x\, (-\sin t)\, \bar{a}_z$$

$$= E_m \sin t \cos x\, \bar{a}_z \qquad \text{... (ii)}$$

From equations (i) and (ii), L.H.S. and R.H.S. are equal i.e.

$$\therefore \quad k\,\bar{a}_z = -\mu \frac{\partial}{\partial t}[x+20t]\,\bar{a}_z$$

$$\therefore \quad k\,\bar{a}_z = -20\,\mu\,\bar{a}_z$$

Comparing,

$$k = -20\mu = -20\,(0.5) = -5 \quad V/m^2$$

b) Consider Maxwell's equation derived from Gauss's law for electric fields,

$$\nabla \bullet \bar{D} = \rho_v$$

$$\therefore \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v = 0 \qquad \text{... Given}$$

From given expressions of $\bar{D}$,

$D_x = 5x, \quad D_y = -2y\,, \quad D_z = kx$

Putting values of D_x, D_y and D_z, we get,

$$\frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(kx) = 0$$

$$\therefore \quad 5-2+k = 0$$

$$\therefore \quad k = -3 \quad \mu C/m^3$$

Note that in part (a), k is unknown in the expression of $\bar{E}$ which is expressed in V/m. In the expression k is multiplied with x which is expressed in metres (m). Hence accordingly k is expressed in V/m^2. While in part (b), k is the part of expression of $\bar{D}$ which is expressed in $\mu C/m^2$. k is multiplied by z which is expressed in m, in expression of $\bar{D}$. Hence k is expressed in $\mu C/m^2$.

Example 9.22 : *Given $\bar{H} = H_m e^{j(\omega t+\beta z)}\,\bar{a}_x$ A/m in free space. Find $\bar{E}$.*

Solution : Using Maxwell's equation,

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

In a free space, conduction current density is zero. So $\bar{J} = 0$.

$$\therefore \quad \nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$$

But

$$\bar{D} = \varepsilon \bar{E}$$

$$\therefore \quad \nabla \times \bar{H} = \frac{\partial(\varepsilon \bar{E})}{\partial t}$$

$$\therefore \quad \nabla \times \bar{H} = \varepsilon \frac{\partial \bar{E}}{\partial t}$$

From given expression of $\bar{E}$,

$$E_x = 0$$

$$E_y = E_m \sin(\omega t - \beta z) = E_m [\sin \omega t \cos \beta z - \cos \omega t \sin \beta z]$$

$$E_z = 0$$

$$\therefore \left[\frac{\partial E_y}{\partial z}\right] \bar{a}_x + \left[\frac{\partial E_y}{\partial x}\right] \bar{a}_z = -\frac{\partial \bar{B}}{\partial t}$$

Also an electric field is varying with z only, and not with x and y.

$$\therefore \quad \frac{\partial E_y}{\partial x} = 0$$

Hence we can write,

$$\therefore -E_m \frac{\partial}{\partial z} [\sin \omega t \cos \beta z - \cos \omega t \sin \beta z] \bar{a}_x = -\frac{\partial \bar{B}}{\partial t}$$

$$\therefore -E_m [\sin \omega t(-\sin \beta z)(\beta) - \cos \omega t(\cos \beta z)(\beta)] \bar{a}_x = -\frac{\partial \bar{B}}{\partial t}$$

$$\therefore -\beta E_m [\cos \omega t \cos \beta z + \sin \omega t \sin \beta z] \bar{a}_x = -\frac{\partial \bar{B}}{\partial t}$$

$$\therefore \quad E_m \beta \cos(\omega t - \beta z) \bar{a}_x = -\frac{\partial \bar{B}}{\partial t}$$

or $$\frac{\partial \bar{B}}{\partial t} = -E_m \beta \cos(\omega t - \beta z) \bar{a}_x$$

Separating variables and integrating with respect to corresponding variables,

$$\therefore \quad \bar{B} = -\beta \cdot E_m \int \cos(\omega t - \beta z) \bar{a}_x \, dt$$

$$\therefore \quad \bar{B} = \frac{-\beta E_m}{\omega} \sin(\omega t - \beta z) \bar{a}_x \quad \text{Wb/m}^2 \qquad \text{... (3)}$$

$\bar{B}$ and $\bar{H}$ can be related as,

$$\bar{B} = \mu \bar{H} = \mu_0 \mu_r \bar{H}$$

For free space $\mu_r = 1$

$$\therefore \quad \bar{B} = \mu_0 \bar{H}$$

or $$\bar{H} = \frac{\bar{B}}{\mu_0} = \frac{-\beta E_m}{\omega \mu_0} \sin(\omega t - \beta z) \bar{a}_x \quad \text{A/m} \qquad \text{... (4)}$$

From equations (1) and (4) it is clear that $\bar{E}$ and $\bar{H}$ are mutually perpendicular to each other.

At t = 0,

$$\bar{E} = E_m \sin(-\beta z)\bar{a}_y$$

$$= -E_m \sin\beta z \, \bar{a}_y$$

or $$\bar{E} = E_m \sin\beta z \left(-\bar{a}_y\right)$$

Similarly,

$$\bar{H} = -\frac{\beta E_m}{\omega\mu_0} \sin(-\beta z)\, \bar{a}_x$$

$$\therefore \quad \bar{H} = \frac{\beta E_m}{\omega\mu_0} \sin \beta z \, \bar{a}_x$$

$$= H_m \sin\beta z \, \bar{a}_x$$

Thus $\bar{E}$ and $\bar{H}$ are perpendicular to each other along z-axis, with the assumption that β and E_m are positive, as shown in the Fig. 9.10.

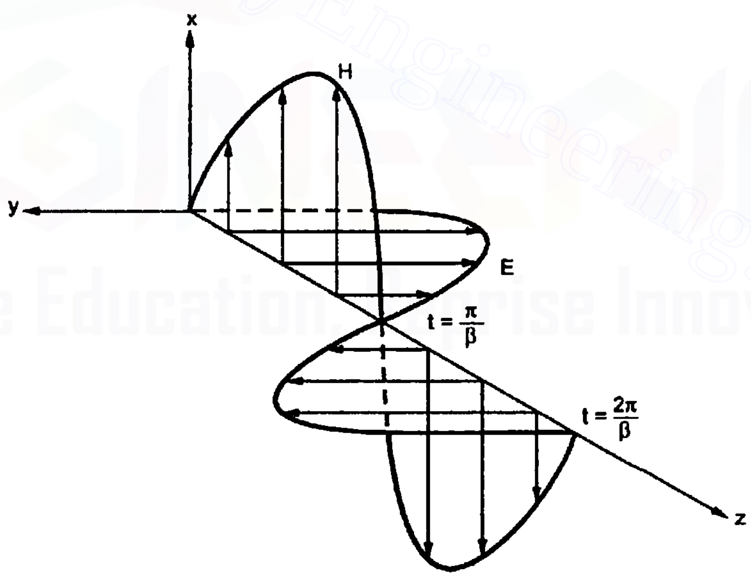


Fig. 9.10

Solution : a) The induced e.m.f is given by,

$$e = \oint \bar{E}.\, d\bar{L}$$

$$\text{But} \oint \bar{E}.\, d\bar{L} = \oint (\bar{v}\times\bar{B}).d\bar{L}$$

$$\therefore \quad e = \int_0^{0.2} [2.5 \sin 10^3\, t\, \bar{a}_z \times 0.04\, \bar{a}_y].[dx\, \bar{a}_x]$$

$$\therefore \quad e = \int_0^{0.2} [0.1 \sin 10^3\, t\,(-\bar{a}_x)].[dx\, \bar{a}_x]$$

$$\therefore \quad e = -0.1 \sin 10^3\, t \int_0^{0.2} dx$$

$$\therefore \quad e = -0.1 \sin 10^3\, t\, [x]_0^{0.2}$$

$$\therefore \quad e = -0.1 \sin 10^3\, t\, [0.2]$$

$$\therefore \quad e = -\mathbf{0.02 \sin 10^3\, t\ V}$$

b) If $\bar{B}$ is changed to $\bar{B} = 0.04\, \bar{a}_x$ T then the conductor can not cut field lines hence induced voltage will be zero. This can be verified mathematically as follows.

$$e = \int \bar{E}.\, d\bar{L} = \int (\bar{v}\times\bar{B}).\, d\bar{L}$$

But according to vector identity, $(\bar{A}\times\bar{B}).\,\bar{C} = \bar{A}.(\bar{B}\times\bar{C})$, above equation becomes,

$$e = \int \bar{v}.(\bar{B}\times d\bar{L}) = \int_0^{0.2} [2.5 \sin 10^3\, t\, \bar{a}_z].[0.04\, \bar{a}_x \times dx\, \bar{a}_x]$$

$$\therefore \quad e = 0 \qquad \ldots[\because \bar{a}_x \times \bar{a}_x = 0]$$

Example 9.27 : *A parallel plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage of 50 sin 10^{3}t Volts applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\varepsilon_0$.*

Solution : $D = \varepsilon E = \varepsilon \dfrac{V}{d}$

Hence the displacement current density is given by,

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon \frac{V}{d}\right)$$

$$= \frac{\varepsilon}{d}\frac{dV}{dt}$$

Hence the displacement current is given by

$$i_D = J_D \cdot \text{Area} = \left(\frac{\varepsilon}{d}\frac{dV}{dt}\right)(A) \qquad \ldots \text{Plate area} = A$$

$$\therefore \qquad i_D = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

This current is same as conduction current.

$$\therefore \qquad i_C = \frac{dQ}{dt} = A\frac{dD}{dt} = \varepsilon A\frac{dE}{dt} = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

Hence the conduction current and displacement current is same. The displacement current is given by

$$i_D = \frac{\varepsilon A}{d}\frac{dV}{dt}$$

$$= \frac{(2\varepsilon_0)(A)}{d}\frac{dV}{dt}$$

$$= \frac{2\times 8.854\times 10^{-12}\times 5\times 10^{-4}}{3\times 10^{-3}}\frac{d}{dt}(50\sin 10^3 t)$$

$$= \frac{2\times 8.854\times 10^{-12}\times 5\times 10^{-4}\times 50\times 10^3}{3\times 10^{-3}}\cos 10^3 t$$

$$= \mathbf{0.1475\ \cos 10^3 t\ \mu A}$$

Example 9.28 : *Given $\bar{E} = E_0 z^2 e^{-t}\,\bar{a}_x$ in free space, determine if there exists a magnetic field such that both Faraday's law and Ampere's circuital law are satisfied simultaneously.*

Solution :

1) $$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \qquad \text{....Faraday's Law}$$

But $$\bar{E} = E_0 z^2 e^{-t}\,\bar{a}_x$$

$$\therefore \qquad \nabla\times\bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_0 z^2 e^{-t} & 0 & 0 \end{vmatrix}$$

$$= (0-0)\,\bar{a}_x - \left[0-\frac{\partial}{\partial z}E_0 z^2 e^{-t}\right]\bar{a}_y + \left[0-\frac{\partial}{\partial y}E_0 z^2 e^{-t}\right]\bar{a}_z$$

$$= 2z E_0 e^{-t}\,\bar{a}_y \qquad \text{....(1)}$$

Hence according to Faraday's law ,

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

$$\therefore \qquad \bar{B} = -\int \nabla\times\bar{E}\,dt$$

$$\therefore \quad \bar{B} = -\int (2z\,E_0\,e^{-t}\,\bar{a}_y)\,dt$$

$$\therefore \quad \bar{B} = -(-2z\,E_0\,e^{-t}\,\bar{a}_y)$$

$$\therefore \quad \bar{B} = 2\,z\,E_0\,e^{-t}\,\bar{a}_y \qquad \text{....(2)}$$

But $\bar{B} = \mu\,\bar{H} = \mu_0\bar{H}$ ...... (for free space $\mu_r = 1$, hence $\mu = \mu_0$)

$$\therefore \quad \bar{H} = \frac{\bar{D}}{\mu_0} = \frac{2\,E_0}{\mu_0} z e^{-t}\,\bar{a}_y \text{ A/m} \qquad \text{....(3)}$$

2) According to Ampere's circuital law,

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

But for free space , σ = conductivity = 0 . Hence $\bar{J} = 0$ i.e. conduction current density is zero.

Now $\quad \bar{D} = \varepsilon_0 \bar{E}$

$$\therefore \quad \frac{\partial \bar{D}}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t}[E_0\,z^2\,e^{-t}\,\bar{a}_x]$$

$$\therefore \quad \frac{\partial \bar{D}}{\partial t} = -\varepsilon_0\,z^2\,E_0\,e^{-t}\,\bar{a}_x \qquad \text{....(4)}$$

Now,

$$\nabla \times \bar{H} = \nabla \times \frac{2E_0}{\mu_0} z\,e^{-t}\,\bar{a}_y$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \frac{2E_0}{\mu_0} z\,e^{-t} & 0 \end{vmatrix}$$

$$= \left[0 - \frac{\partial}{\partial z}\left(\frac{2\,E_0}{\mu_0} z\,e^{-t}\right)\right]\bar{a}_x + 0 + \left[\frac{\partial}{\partial x}\left(\frac{2\,E_0}{\mu_0}\,z\,e^{-t}\right) - 0\right]\bar{a}_z$$

$$= \frac{-2\,E_0}{\mu_0} e^{-t}\,\bar{a}_x + 0 + 0$$

$$\therefore \quad \nabla \times \bar{H} = \frac{-2\,E_0}{\mu_0} e^{-t}\,\bar{a}_x \qquad \text{....(5)}$$

Equating equations (5) and (4) , we get ,

$$\frac{-2\,E_0}{\mu_0} e^{-t}\,\bar{a}_x = -\varepsilon_0\,z^2\,E_0\,e^{-t}\,\bar{a}_x$$

$$\therefore \quad z = \frac{\sqrt{2}}{\sqrt{\mu_0 \varepsilon_0}} = \frac{\sqrt{2}}{\sqrt{4\times\pi\times10^{-7}\times8.854\times10^{-12}}}$$

$$\therefore \quad z = 4.2397\times10^{8}$$

Substituting value of z in equation (3), we get,

$$\bar{H} = \frac{2\,E_0}{\mu_0}(4.2397\times10^{8})e^{-t}\,\bar{a}_y$$

$$\therefore \quad \bar{H} = \frac{2\times4.2397\times10^{8}}{4\times\pi\times10^{-7}}\,E_0\,e^{-t}\,\bar{a}_y$$

$$\therefore \quad \bar{H} = \mathbf{0.6747\times10^{15}\,E_0 e^{-t}\bar{a}_y\ A/m}$$

Example 9.29 : *A square coil with loop area 0.01 m^2 and 50 turns rotated about its axis right angle to a uniform magnetic field B = 1 T. Calculate the instantaneous value of e.m.f. induced in the coil when its plane is*

i) At right angle to the field

ii) In the plane of the field

iii) When the plane of coil is 45° to the field.

Solution : Given : N = 50 , A = 0.01 m^2 , B = 1T

Induced e.m.f. is given by

$$e = NAB\ \sin\theta$$

i) Coil at right angles to field :

$$\theta = 90^\circ$$

$$\therefore \quad e = 50\times0.01\times1\times\sin 90^\circ = \mathbf{0.5\ V}$$

ii) Coil in the plane of the field :

$$\theta = 0^\circ$$

$$\therefore \quad e = 50\times0.01\times1\times\sin 0 = \mathbf{0\ V}$$

iii) Coil is at 45° with the plane of field :

$$\therefore \quad \theta = 45^\circ$$

$$\therefore \quad e = 50\times0.01\times1\times\sin 45^\circ = \mathbf{0.353\ V}$$

Example 9.30 : *Electric vector $\bar{E}$ of a wave in a free space is given by $E_x = 0$, $E_z = 0$ and $E_y = A\cos\omega\left(t-\frac{z}{c}\right)$ where c = velocity of light . Using Maxwell's equation for free space, determine expression for $\bar{H}$.*

Solution : According to Maxwell's equation,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

But $\bar{B} = \mu \bar{H} = \mu_0 \mu_r \bar{H} = \mu_0 \bar{H}$ .... (For free space $\mu_r = 1$)

$$\therefore \quad \nabla \times \bar{E} = -\mu_0 \frac{\partial \bar{H}}{\partial t} \qquad \text{....(1)}$$

Consider L.H.S. of equation (1) ,

$$\therefore \quad \nabla \times \bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & A\cos\omega\left(t-\frac{z}{c}\right) & 0 \end{vmatrix}$$

$$= \left[0-\frac{\partial}{\partial z}\left\{A\cos\omega\left(t-\frac{z}{c}\right)\right\}\right]\bar{a}_x - 0 + \left[\frac{\partial}{\partial x}\left\{A\cos\omega\left(t-\frac{z}{c}\right)-0\right\}\right]\bar{a}_z$$

$$= +\frac{\omega}{c} A \sin\omega\left(t-\frac{z}{c}\right)\bar{a}_x$$

$$= +\frac{\omega A}{c} \sin\omega\left(t-\frac{z}{c}\right)\bar{a}_x \qquad \text{....(A)}$$

Consider R.H.S of equation (1),

$$\therefore \quad -\mu_0 \frac{\partial \bar{H}}{\partial t} = -\mu_0 \frac{\partial}{\partial t}\left[H_x\, \bar{a}_x + H_y\, \bar{a}_y + H_z\, \bar{a}_z\right] \qquad \text{... (B)}$$

Equating equations (A) and (B), we get,

$$-\mu_0 \frac{\partial H_x}{\partial t}\bar{a}_x - \mu_0 \frac{\partial H_y}{\partial t}\bar{a}_y - \mu_0 \frac{\partial H_z}{\partial t}\frac{\bar{a}_z}{\bar{a}_x} = +\frac{\omega A}{c}\sin\omega\left(t-\frac{z}{c}\right)\bar{a}_x$$

Equating coefficients of unit vectors from both the sides

(as H_y and H_z both are zero) , we get,

$$-\mu_0 \frac{\partial H_x}{\partial t} = \frac{\omega A}{c}\sin\omega\left(t-\frac{z}{c}\right)$$

$$\therefore \quad \frac{\partial H_x}{\partial t} = -\frac{\omega A}{\mu_0 c}\sin\omega\left(t-\frac{z}{c}\right)$$

Separating variables and integrating both the sides with respect to corresponding variables , we get,

$$H_x = \frac{-\omega A}{\mu_0 c}\int \sin\omega\left(t-\frac{z}{c}\right)dt$$

$$\therefore \quad H_x = \frac{-\omega A}{\mu_0 c}\left[\frac{-\cos\omega\left(t-\frac{z}{c}\right)}{\omega}\right]$$

$$\therefore \quad H_x = \frac{A}{\mu_0 c}\cos\omega\left(t-\frac{z}{c}\right)$$

But in free space, c = velocity of light $= \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$$\therefore \quad H_x = \frac{A}{\mu_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}\cos\omega\left(t-\frac{z}{c}\right)$$

$$\therefore \quad H_x = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, A\cos\omega\left(t-\frac{z}{c}\right) \text{ A/m}$$

$$H_y = 0$$

$$H_z = 0$$

Hence $$\bar{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, A\cos\omega\left(t-\frac{z}{c}\right)\bar{a}_x \text{ A/m}$$

Example 9.31 : *A No 10 copper wire carries a conduction current of 1 amp at 60 Hz. Calculate the displacement current in the wire. For copper assume,*

$$\varepsilon = \varepsilon_0 = \frac{1}{36\times\pi\times 10^9} \text{ F/m} = 8.854\times 10^{-12} \text{ F/m}$$

$$\mu = \mu_0 = 4\pi\times 10^{-7} \text{ H/m}$$

$$\sigma = 5.8\times 10^7 \text{ ℧/m}$$

(UPTU : 2005-06)

Solution : By definition,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon} = \frac{5.8\times 10^7}{2\times\pi\times 60\times 8.854\times 10^{-12}} = 1.7376\times 10^{16}$$

But $$|\bar{J}_C| = \frac{i_C}{A} \quad \text{and} \quad |\bar{J}_D| = \frac{i_D}{A}$$

$$\therefore \quad \frac{i_C / A}{i_D / A} = 1.7376\times 10^{16}$$

$$\therefore \quad i_D = \frac{i_C}{1.7376\times 10^{16}} = \frac{1}{1.7376\times 10^{16}} = 0.05755\times 10^{-15} \text{ A}$$

Example 9.32 : *Consider a loop as shown in the Fig. 9.12. If $\bar{B} = 0.5\ \bar{a}_z$ Wb/m², $R = 20\Omega$, l = 10 cm and rod is moving with constant velocity of $8\ \bar{a}_x$ m/sec, find*

i) the induced e.m.f. in the rod,

ii) the current through the resistance

iii) the motional force on the rod,

iv) the power dissipated by the resistance

(UPTU : 2006-07, 10 Marks)

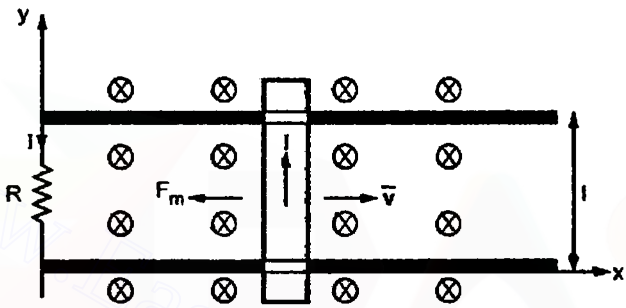


Fig. 9.12

Solution : i) The induced motional e.m.f. is given by,

$$e = \int (\bar{v}\times\bar{B})\bullet d\bar{L} = \int_{l=10\times10^{-2}}^{0} (8\ \bar{a}_x \times 0.5\ \bar{a}_z)\bullet(dy\ \bar{a}_y)$$

$$= 4\int_{10\times10^{-2}}^{0} (-\bar{a}_y)\bullet(dy\ \bar{a}_y) = -4\int_{10\times10^{-2}}^{0} dy$$

$$= -4\,[0 - 10\times10^{-12}]$$

$$= 0.4\ \text{V}$$

ii) The current through resistor $= \dfrac{V_{induced}}{R} = \dfrac{e}{R} = \dfrac{0.4}{20} = 20$ mA

iii) The motional force on the bar is given by,

$$F = BIl = (0.5)\,(20\times10^{-3})\,(10\times10^{-2}) = 1\times10^{-3}\,\text{N} = 1\ \text{mN}$$

iv) The power dissipated by resistance is given by,

$$P_D = I^2R = (20\times10^{-3})^2 \times 20 = 8\times10^{-3}\ \text{W} = 8\ \text{mW}$$

Example 9.33 : *A loop shown in the Fig. 9.13 is inside a uniform magnetic field $\bar{B} = 50\ \bar{a}_x$ mWb/m^2. If side d.c. of the loop cuts flux lines at frequency of 50 Hz and the loop lies in the y-z plane at t = 0, find*

i) the induced e.m.f. at t = 1 ms

ii) the induced current at t = 3 ms **(UPTU : 2007-08, 10 Marks)**

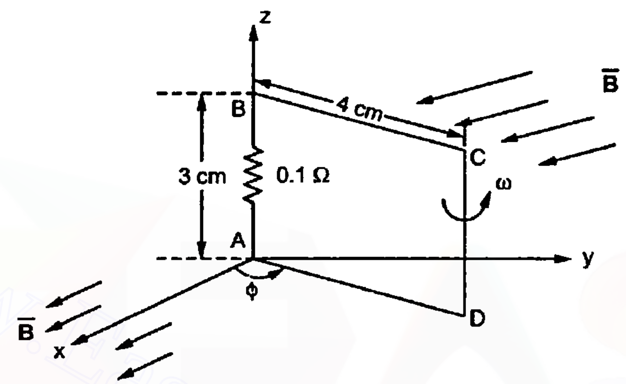


Fig. 9.13

Solution : a) As the magnetic field $\bar{B}$ is constant with respect to time, the induced e.m.f. is motional and is given by,

$$e = \int (\bar{v} \times \bar{B}) \bullet d\bar{L}$$

Now $$d\bar{L} = d\bar{L}_{DC} = dz\ \bar{a}_z \quad ...(i)$$

$$\bar{v} = \frac{dL'}{dt} = \frac{\rho\, d\phi}{dt}\bar{a}_\phi = \rho\,\omega\,\bar{a}_\phi \quad ...(ii)$$

But from Fig. 9.13, $$\rho = 4\text{ cm} = 4 \times 10^{-2}\text{ m}$$

$$\omega = 2\pi f = 2 \times \pi \times 50 = 100\,\pi\text{ rad/sec}$$

As velocity $\bar{v}$ and $d\bar{L}$ are expressed in cylindrical co-ordinates, transforming $\bar{B}$ into cylindrical co-ordinates.

$$\bar{B} = B_0\,\bar{a}_x = B_0(\cos\phi\, a_\rho - \sin\bar{a}_\phi) \quad ...(iii)$$

Now $$\bar{v} \times \bar{B} = \begin{vmatrix} \bar{a}_\rho & \bar{a}_\phi & \bar{a}_z \\ 0 & \rho\omega & 0 \\ B_0\cos\phi & -B_0\sin\phi & 0 \end{vmatrix}$$

$\therefore$ $$\bar{v} \times \bar{B} = -\rho\omega B_0\cos\phi\,\bar{a}_z \quad ...(iv)$$

Hence

$$e = \int_{z=0}^{z=3\times10^{-2}} (-\rho\omega B_0 \cos\phi \bar{a}_z) \bullet (dz\, \bar{a}_z)$$

$$\therefore \quad e = \int_{z=0}^{z=3\times10^{-2}} -\rho\omega B_0 \cos\phi\, dz \qquad \ldots(v)$$

We now calculate cos ϕ. we know that,

$$\omega = \frac{d\phi}{dt} \rightarrow \phi = \omega t + k_0$$

where k_0 is constant of integration.

At t = 0, $\phi = \frac{\pi}{2}$ as loop is in y-z plane. Henc $k_0 = \pi/2$.

Thus $\qquad \phi = \omega t + \frac{\pi}{2}$

Putting value of ϕ in equation (v), we get

$$e = \int_{z=0}^{z=3\times10^{-2}} -\rho\,\omega B_0 \cos\left(\omega t + \frac{\pi}{2}\right) dz$$

$$= \int_{z=0}^{z=3\times10^{-2}} +\rho\,\omega B_0 \sin\omega t\, dz \qquad \ldots(vi)$$

Putting values of ρ_1 ω, B_0, we get,

$$e = +4\times10^{-2}\times100\times\pi\times50\times10^{-3}\times\sin\omega t \int_{z=0}^{z=3\times10^{-2}} dz$$

$$\therefore \quad e = +0.2\times\pi\times[z]_0^{3\times10^{-2}} \sin\omega t$$

$$\therefore \quad e = +0.2\pi\,[3\times10^{-2}]\sin\omega t$$

$$\therefore \quad e = +6\,\pi\sin\omega t \text{ mV} = +6\,\pi\sin(100\,\pi t) \text{ mV} \qquad \ldots(vii)$$

Hence at t = 1 ms

$$e = 6\,\pi\sin(100\,\pi\times1\times10^{-3}) \text{ mV} = 5.8248 \text{ mV}$$

[Note that angle should be expressed in radian.]

ii) The current induced is obtained as,

$$I = \frac{e}{R} = \frac{6\pi \sin(100\pi t)\times 10^{-3}}{0.1} = 60\,\pi \sin(100\,\pi t)\text{ mA}$$

Hence at t = 3 ms induced current is given by,

$$I = 60\,\pi \sin(100\times\pi\times 3\times 10^{-3})\times 10^{-3}$$

$$= 152.496 \text{ mA} = 0.15249 \text{ A}$$

Review Questions

1. *State and explain Faraday's law for induced e.m.f.*
2. *Write a short note on :*

   i) Equation of continuity

   ii) Displacement current.
3. *Show that the ratio of the amplitudes of the conduction current density and displacement current density is $\sigma/\omega\varepsilon$ for the applied field $E = E_m \cos\omega t$. Assume $\mu = \mu_0$.*
4. *Write a note on Maxwell's equations.*
5. *State Maxwell's equations for static fields. Explain how they are modified for time varying electric and magnetic fields.*
6. *Write Maxwell's equations in point form and explain physical significance of the equations.*
7. *Write Maxwell's equations in integral form and give their physical significance.*
8. *Explain following*

   i) Motional e.m.f. ii) Transformer e.m.f.
9. *Show that for a capacitor the conduction current in the wire equals the displacement current in the dielectric if subjected to a time changing field.*
10. *Show that*

$$\nabla . \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

where $\bar{J}$ = *Conduction current density A/m²*

ρ = *Volume charge density in C/m³*

11. *Write a short note on retarded potential.*
12. *Find the amplitude of the displacement current density in air space within a large power transformer where $\bar{H} = 10^6 \cos(377t + 1.2566\times 10^{-6} z)\,\bar{a}_y$ A/m.*

    [Ans. : $\bar{J}_D = 1.258 \sin(377t + 1.2566\times 10^{-6} z)\,\bar{a}_x A/m^2$]
13. *Find the amplitude of the displacement current density inside a typical metallic conductor where $\sigma = 5\times 10^7$ ℧/m, f = 1 kHz, ε_r = 1.* **[Ans. : $\bar{J}_D = 11.126 \cos(6283t - 444z)\,\bar{a}_x A/m^2$]**

14. *A straight conductor of 0.2 m lies on the x-axis. With one end at origin. The conductor is subjected to a magnetic flux density $B = 0.04\ \bar{a}_y$ T and velocity $v = 2.5 \sin^3 t\ \bar{a}_z$ m/s. Calculate the motional electric field intensity and e.m.f. induced in the conductor.*

[Ans. : $\bar{E} = -0.10 \sin 10^3 t\ \bar{a}_x$ V/m, $V = -0.20 \sin 10^3 t$ V]

15. *A rectangular loop shown in Fig. 9.14, moves toward the origin at a velocity $V = -200\bar{a}_y$ m/s in a magnetic field $B = 0.75 e^{-0.5y} \bar{a}_z$ T.*

Find the current at the instant when the coil sides are at y = 0.50 m and 0.60 m if R = 3 Ω.

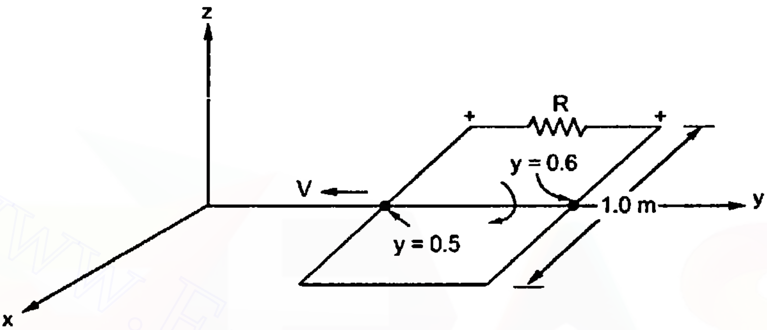


Fig. 9.14

[Ans. : i = 1.9 A]

16. *A square coil with a loop area 0.01 m^2 and 50 turns is rotated about its axis at right angle to a uniform magnetic field β = 1 T. Calculate the instantaneous value of e.m.f. induced in the coil when its plane is -*

a) At right angle to the field.

b) In the plane of the field.

c) When the plane of coil is 45° to the field. **[Ans. : a) zero b) 52.38 V c) 37 V]**

17. *A voltage of V(t) = 0.1 sin 120 π t volts is applied to a capacitor of 1 pF. Find the displacement current at t = 0.* **[Ans : $I_D = 0.03768$ ηA]**

18. *Show that in a capacitor the conduction current and displacement current are equal.*

19. *A capacitor has a capacitance of 1.5 pF. Find the displacement current at t = 0, if a voltage $5 \sin 100\pi t$ is applied to it.* **[Ans. : $I_D = 2.3562$ ηA]**

20. *Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma = 2\times10^{-4}$ mho/m and $\varepsilon_R = 81$.* **[Ans. : f = 44.384 kHz]**

21. *If σ = 0, ε = 2.5, ε_0 and $\mu = 10\mu_0$ determine whether or not the following pairs of fields satisfy Maxwell's equation,*

a) $\bar{E} = 2y\bar{a}_y$, $\bar{H} = 5\ x\bar{a}_x$

b) $\bar{E} = 100 \sin 6\times10^7\ t \sin z\ \bar{a}_y$, $\bar{H} = -0.1328 \cos 6\times10^7\ t \cos z\ \bar{a}_x$

c) $\bar{D} = \left(z + 6\times10^7 t\right)\bar{a}_x$, $\bar{B} = \left(-754z - 452\times10^{10} t\right)\bar{a}_y$ **[Ans. : a) No b) Yes c) Yes]**

22. *What values of A and B are required of two fields* $\bar{E} = 120\pi\cos\left(10^6\pi t - \beta x\right)\bar{a}_y (V/m)$ *and* $\bar{H} = A\cos\left(10^6\pi t - \beta x\right)\bar{a}_z (A/m)$?

    To satisfy Maxwell's equation in medium where

    $\varepsilon_r = \mu_r = 4$ *and* $\sigma = 0$. **[Ans. : A = 1 and B = 0.0425]**

23. *The sides of a square loop in the z = 0 plane are located at x = ± 0.6 m and y = ± 6 m. There exists a uniform time varying magnetic field given by*

    $\bar{B} = \left(0.2\bar{a}_x - 0.4\bar{a}_y + 0.8\bar{a}_z\right) \cos 2000t \left(Wb/m^2\right)$.

    If the total resistance of square loop is 1 KΩ , find the current through the loop.

    [Ans. : i = 2.304 sin 2000 t A]

University Questions

1. *Derive the equation of continuity for time varying fields and point out the inconsistency of Ampere's law for time varying fields.* **[UPTU : 2002-2003, 10 Marks]**
2. *Uniform E and B fields are oriented right angles to each other. An electron moves with a speed of* 8×10^6 *m/s at right angles to both fields and passes undeflected through the field. If the magnitude of B is 0.5 m Wb/m*^{2}*, find the value of E.* **[UPTU : 2003-2004, 5 Marks]**
3. *Write the Maxwell's equation in the integral form and explain the physical significance.* **[UPTU : 2003-2004, 10 Marks]**
4. *State and prove Maxwell's equations and give their physical interpretation.* **[UPTU : 2003-2004, 10 Marks]**
5. *A no. 10 copper wire carries a conduction current of 1 amp at 60 Hz. What is the displacement current in the wire. For copper assume,*

   $\varepsilon = \varepsilon_0 = \dfrac{1}{36 \times \pi \times 10^9}$ *F/m*

   $\mu = \mu_0 = 4\pi \times 10^{-7}$ *H/m*

   $\sigma = 5.8 \times 10^7$ *Ʊ/m* **[UPTU : 2005-06, 10 Marks]**
6. *Consider a loop as shown in the Fig. 9.15. If* $\bar{B} = 0.5\,\bar{a}_z$ *Wb/m*^{2}*, R = 20 Ω, l = 10 cm and the rod is moving with constant velocity of 8* $\bar{a}_x$ *m/sec, find*

   i) the induced e.m.f. in the rod, *ii) the current through the resistance*

   iii) the motional force on the rod, *iv) the power dissipated by the resistance*

   [UPTU : 2006-07, 10 Marks]

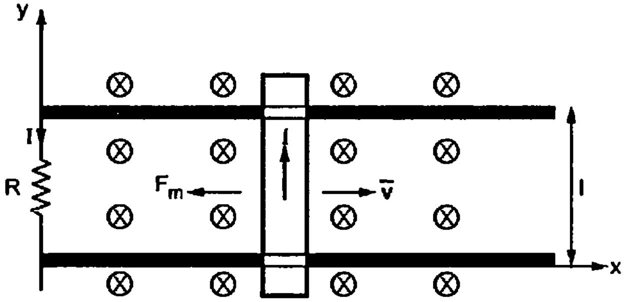


Fig. 9.15

7. *The loop shown in the Fig. 9.16 is inside a uniform magnetic field* $\bar{B} = 50\ \bar{a}_x$ *mWb/m*^{2}. *If side DC of the loop cuts the flux lines at frequency of 50 Hz and the loop lies in the y-z plane at t = 0, find*

   i) the induced e.m.f. at t = 1 ms

   ii) the induced current at t = 3 ms

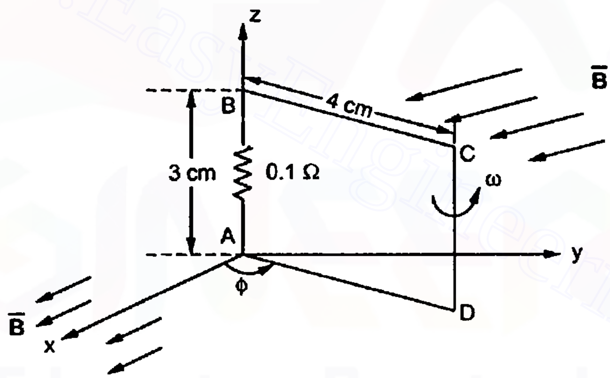


Fig. 9.16

[UPTU : 2007-08, 10 Marks]

8. *Derive Maxwell's first and second equations in integral and differential forms.*

[UPTU : 2008-09, 10 Marks]

1 Transmission Line Theory

1.1 Introduction

The electrical lines which are used to transmit the electrical waves along them are called **transmission lines.** The transmission line theory is the theory of propagation of electric waves along the transmission lines. The practical examples of the electric waves, which are transmitted along the transmission lines are the telephone messages and electrical power signals. The transmission lines are assumed to consist of a pair of wires which are uniform throughout their whole length.

The transmission line parameters like resistance, inductance and capacitance are not physically separable unlike circuit elements of a lumped circuit. The transmission parameters are **distributed** all along the length of the transmission line. Hence the method of analysing the transmission lines is different than the method of analysing the lumped circuits. In the analysis of the transmission line, only steady state currents and voltages are considered. The analysis includes the finding of current and voltage at any point along the length of the line, when a known voltage is continuously applied at one end. The end to which the voltage is applied is called **sending end** while the end at which the signals are received is called **receiving end** of the transmission line.

1.2 Types of Transmission Lines

The various types of the transmission lines are,

1. Open-wire line : These lines are the parallel conductors open to air hence called open wire lines. The conductors are separated by air as the dielectric and mounted on the posts or the towers. The telephone lines and the electrical power transmission lines are the best examples of the open wire lines.

There are certain disadvantages of the open wire lines which are, requirement of telephone posts and towers hence high initial cost, affected by atmospheric conditions like wind, air, ice etc., maintenance is difficult and possibility of shorting due to flying

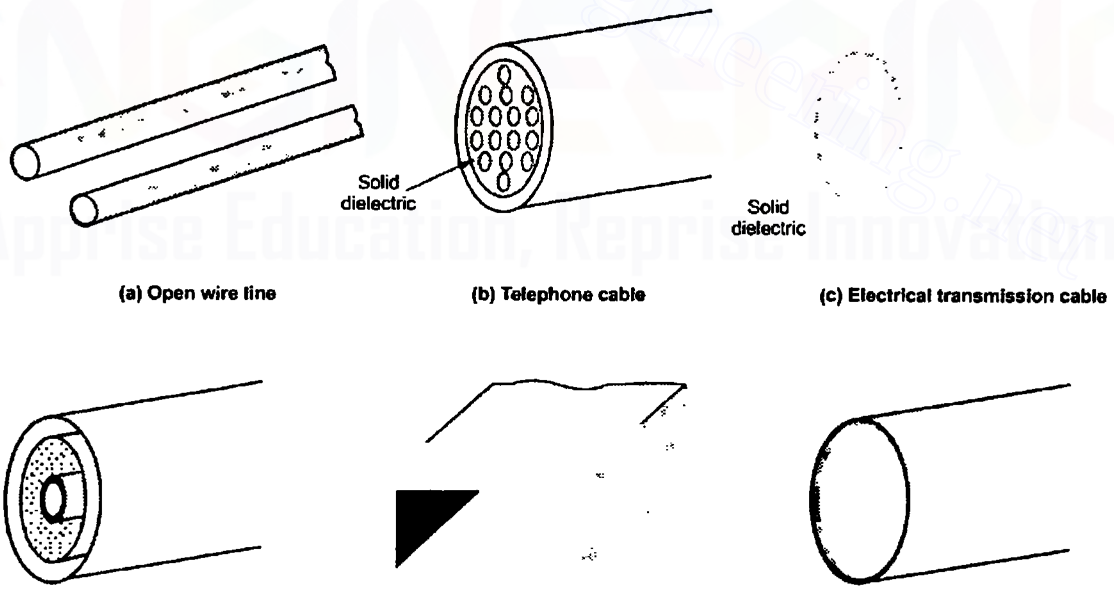
objects and birds. But less capacitance compared to underground cable is the advantage of open wire line.

2. Cables : These are underground lines. The telephone cables consist of hundred of conductors which are individually insulated with paper. These are twisted in pairs and combined together and placed inside a protective lead or plastic sheath. While underground electrical transmission cables consist of two or three large conductors which are insulated with oil impregnated paper or other solid dielectric and placed inside protective lead sheath. Both these types are still considered as parallel conductors separated bv a solid dielectric.

3. Co-axial line : As the name suggests, there are two conductors which are co-axially placed. One conductor is hollow and other is placed co-axially inside the first conductor. The dielectric may be solid or gaseous. These lines are used for high voltage levels.

4. Wave guides : These types of transmission lines are used to transmit the electrical waves at microwave frequencies. Constructionally these are the hollow conducting tubes having uniform cross section. The energy is transmitted from inner walls of the tube by the phenomenon of total internal reflection.

Different types of the transmission line are as shown in the Fig. 1.1.



(a) Open wire line (b) Telephone cable (c) Electrical transmission cable

(d) Coaxial cable (e) Rectangular waveguide (f) Circular waveguide

Fig. 1.1 Types of the transmission line

1.3 Transmission Line Parameters

For the analysis and the design of the transmission lines, it is necessary to have the knowledge of the electric circuit parameters, associated with the transmission lines. The various electric parameters associated with the transmission lines are,

1. Resistance : Depending upon the cross sectional area of the conductors, the transmission lines have the resistance associated with them. The resistance is uniformly distributed all along the length of the transmission line. Its total value depends on the overall length of the transmission line. Hence its value is given per unit length of the transmission line. It is denoted as R and given in ohms per unit length.

2. Inductance : When the conductors carry the current, the magnetic flux is produced around the conductors. It depends on the magnitude of the current flowing through the conductors. The flux linkages per ampere of current, gives rise to the effect called inductance of the transmission line. It is also distributed all along the length of the transmission line. It is denoted as L and measured in henry per unit length of the transmission line.

3. Capacitance : The transmission line consists of two parallel conductors, separated by a dielectric like air. Such parallel conductors separated by an insulating dielectric produces a capacitive effect. Due to this, there exists a capacitance associated with the transmission line which is also distributed along the length of the conductor. It is denoted as C and measured in farads per unit length of the transmission line.

4. Conductance : The dielectric in between the conductors is not perfect. Hence a very small amount of current flows through the dielectric called displacement current. This is nothing but a leakage current and this gives rise to a leakage conductance associated with the transmission line. It exists between the conductors and distributed along the length of the transmission line. It is denoted as G and measured in mho per unit length of the transmission line.

Thus the four important transmission line parameters are R, L, C and G. As the current flows from one conductor and completes the path through other conductor, the resistance of both the wires is included while specifying the resistance per unit length of the line. These line parameters are constants and are called **primary constants** of the transmission line. These constants are assumed to be independent of frequency for the transmission line. These primary constants can be obtained by the measurements on a sample of the transmission line.

1.4 Properties of Symmetrical Networks - Characteristic Impedance and Propagation Constant

The analysis of the transmission of the electric waves along a line can be done by considering a uniform and symmetrical transmision line. Before starting analysis of the symmetrical transmission line, let us take a brief review of the electrical properties of the symmetrical network.

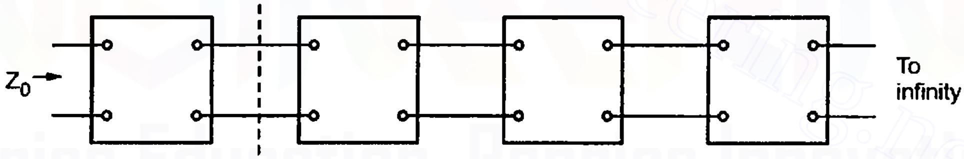
Any symmetrical network has two important electrical properties namely,

1. Characteristics impedance (Z_0)
2. Propagation constant (γ)

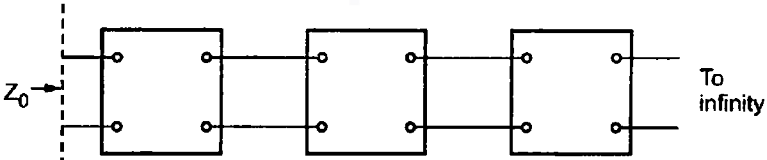
1.4.1 Characteristic Impedance (Z_0)

Consider that infinite number of identical symmetrical networks are connected in **cascade or tandem** as shown in the Fig. 1.2 (a). The input impedance measured at the input terminals of the first network in the chain of infinite networks will have some finite value which depends on the network composition. This impedance is the important property of a symmetrical network. Thus the characteristic impedance of a symmetrical network is the impedance measured at the input terminals of the first network in the chain of infinite networks in cascade and it is represented by Z_0.

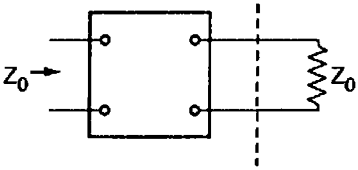
If first network is disconnected from the chain as shown in the Fig. 1.2 (b), then also the input impedance measured at the input terminals of second network will be



(a) Input impedance at input terminals of first network of chain of infinite network



(b) Input impedance at input terminals of second network of chain of infinite networks



(c) Input impedance of first network terminated in characteristic impedance

Fig. 1.2

Z_0 again as number of networks in the chain are still infinite. That means we can replace this chain by impedance Z_0 at the output port of the first network as shown in the Fig. 1.2 (c). Then the impedance at input terminals of the first network will be still Z_0.

Thus in general when any symmetrical network is terminated in its characteristic impedance Z_0, the input impedance will also be Z_0.

This property is true for output impedance if the symmetrical network terminated in Z_0 is driven by a generator with internal impedance equal to Z_0. In such network, the output impedance will be Z_0 only. The network terminated in characteristic impedance at input as well as output terminals is said to be **correctly terminated** or **properly terminated** symmetrical network as shown in the Fig. 1.3.

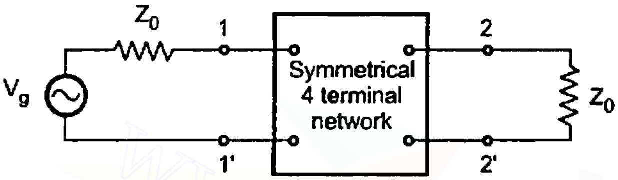


Fig. 1.3 Correctly terminated symmetrical 4 terminal network

1.4.2 Propagation Constant (γ)

Consider a chain of identical symmetrical networks connected in cascade as shown in the Fig. 1.4.

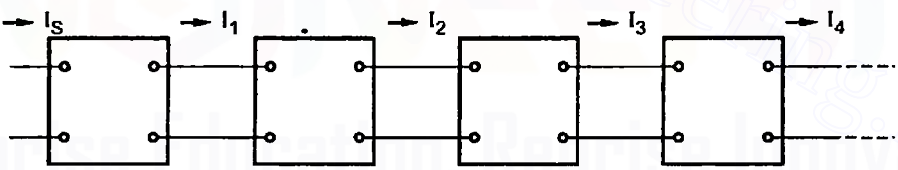


Fig. 1.4

The current leaving any section will be definite proportion of that entering section and in general will be out of phase with it. Thus the relationship between the currents entering and leaving the section is a vector quantity with both modulus and angle. This quantity is represented in the form e^{γ} for convinence where γ is a complex number given by $\gamma = \alpha + j\beta$

Let the ratio of input to output current be given by

$$\frac{I_S}{I_1} = e^{\gamma} \qquad \text{... (1)}$$

Since all the sections are identical, we can write

$$\frac{I_S}{I_1} = \frac{I_1}{I_2} = \frac{I_2}{I_3} = \frac{I_3}{I_4} \ldots = e^{\gamma}$$

Thus,

$$\frac{I_S}{I_2} = \frac{I_S}{I_1} \cdot \frac{I_1}{I_2} = e^{\gamma} \cdot e^{\gamma} = e^{2\gamma}$$

$$\frac{I_S}{I_3} = \frac{I_S}{I_1} \cdot \frac{I_1}{I_2} \cdot \frac{I_2}{I_3} = e^{\gamma} \cdot e^{\gamma} \cdot e^{\gamma} = e^{3\gamma}$$

Hence for n identical sections connected is cascade the ratio of input to output current is given by

$$\frac{I_S}{I_R} = e^{n\gamma} \qquad \ldots (2)$$

Note that input current is represented by sending end current, I_S; while the output current is represented by receiving end current, I_R . Above equation can be written as

$$\frac{I_S}{I_R} = e^{(\alpha + j\beta)} = e^{n\alpha} \cdot e^{jn\beta}$$

$$= e^{n\alpha} (\cos n\beta + j \sin n\beta)$$

$$= e^{n\alpha} \sqrt{\cos^2 n\beta + \sin^2 n\beta} \angle \tan^{-1} \frac{\sin n\beta}{\cos n\beta}$$

$$\therefore \quad \frac{I_S}{I_R} = e^{n\alpha} \angle n\beta \qquad \ldots (3)$$

where $e^{n\alpha}$ gives ratio of absolute magnitudes of sending end current to receiving end current and $n\beta$ gives the phase angle between these two currents.

If the network is correctly terminated, then we can write,

$$\frac{I_S}{I_R} = \frac{I_S \cdot Z_0}{I_R \cdot Z_0} = \frac{E_S}{E_R}$$

Also

$$\frac{E_S}{E_R} = e^{n\gamma} = e^{n\alpha} \angle n\beta \qquad \ldots (4)$$

The real part α of the propagation constant γ is called **attenuation constant** and it is measured in nepers.

$$\therefore \quad e^{\alpha} = \left| \frac{I_S}{I_R} \right| \qquad \ldots \text{ for one section i.e. } n = 1$$

$$\therefore \quad \alpha = ln \left| \frac{I_S}{I_R} \right| \text{ neper} \qquad \ldots (5)$$

Similarly for n-sections,

$$e^{n\alpha} = \left|\frac{I_S}{I_R}\right| \quad \text{... for n-sections}$$

$$n\alpha = ln\left|\frac{I_S}{I_R}\right| \text{ neper}$$

The imaginary part β of propagation constant γ is called **phase constant** and is equal to the angle in radians by which output current leaving section lags that input current entering section. For n-sections, the phase constant will be $n\beta$ radians.

1.5 The Infinite Line

The analysis of the transmission of the electric waves along any uniform and symmetrical transmission line can be done interms of the results existing for an imaginary line of infinite length having electrical constants per unit length identical to that of the line under consideration. Hence let us study the transmission of electric waves along a line of infinite length first.

The Fig. 1.5 shows the transmission line of infinite length.

Fig. 1.5 The infinite line

The alternating voltage applied to the sending end is E_S. A finite current will flow which depends on the capacity of the line and the leakage conductance between the two wires constituting the line. This finite current is denoted as I_S.

The ratio of the voltage applied E_S and the current flowing I_S is the input impedance of the line. This input impedance of the infinite line is called **characteristic impedance** of the transmission line and is denoted by Z_0. This parameter plays an important role in the analysis of lines. Infact the characteristic impedance of any practical line is defined as the impedance looking into an infinite line having same electrical properties. The characteristic impedance is a phasor quantity having a magnitude of $|Z_0|$ and an angle ϕ. Both magnitude and angle of the characteristic impedance vary with the frequency. Hence the frequency at which the characteristic impedance is measured, must be specified while specifying the value of the characteristic impedance.

1.5.1 Important Properties of the Infinite Line

In addition to the characteristic impedance, the infinite line has the following two important properties.

1. As the line has an infinite length, no waves will ever reach the receiving end and hence there is no possibility of the reflection at the receiving end. Thus there can not be any reflected waves, returning to the sending end. The complete power applied at the sending end is absorbed by the line.
2. As the reflected waves are absent, the characteristic impedance Z_0 at the sending end will decide the current flowing, when a voltage is applied to the sending end. The current will not be affected by the terminating impedance Z_R at the receiving end. This condition is fulfilled by the long lines in practice.

1.6 Short Line

The short line means a practical line of finite length. Short word does not indicate the information related to the actual length of the line. As it is a practical line with finite length, it is also called **finite line**. Let us see how finite line is related with an infinite line.

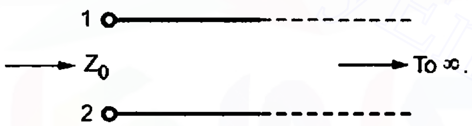


Fig. 1.6 Infinite line

Consider an infinite line as shown in the Fig. 1.6.

Its input impedance looking in at the terminals 1 and 2 is Z_0 which is its characteristic impedance.

Now let the section AB at the near end of the line is removed as shown in the Fig. 1.7.

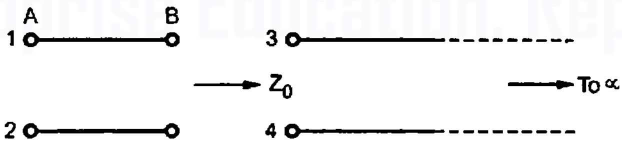


Fig. 1.7

Now section AB is a short section and compared to infinite length of the line, it is having negligible length. Hence the remaining line from the terminals 3 and 4 represents an infinite nature of the line as it is before. Hence as per the definition, impedance looking in at the terminals 3 and 4 is Z_0.

From electrical point of view, the impedance at the terminating end of section AB must be Z_0, at the terminal B.

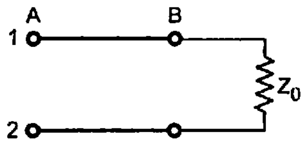


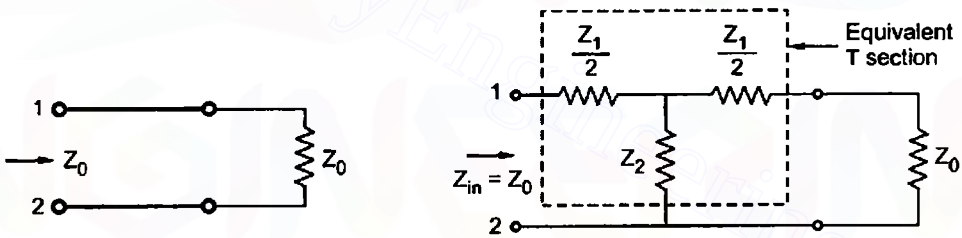
Fig. 1.8 Short section terminated in Z_0

Thus if the short section AB is now terminated in an actual impedance Z_0 as shown in the Fig. 1.8, then all the properties of such a line will be exactly same as that of an infinite line. The current and voltage at all the points along the length of the short section will be exactly the same as if that section has an infinite length.

Thus it can be concluded that, **a finite line which is terminated in its characteristic impedance behaves as an infinite line.** This means that its input impedance will be Z_0 and there will be no reflection.

1.6.1 Determination of Z_0 for Finite Line Terminated in Z_0

Consider a short line terminted in its characteristic impedance Z_0 as shown in the Fig. 1.9 (a). The short line is a symmetrical network and hence can be represented by the equivalent T-section as shown in the Fig. 1.9 (b). Such a representation is the property of a symmetrical network.



(a) Short line

(b) Equivalent T section

Fig. 1.9

It is known that finite line terminated in Z_0 behaves as an infinite line hence the input impedance Z_{in} of the equivalent T-section network also must be Z_0.

The input impedance Z_{in} of the equivalent T section network can be obtained as,

$$Z_{in} = \frac{Z_1}{2} + \left\{ Z_2 \,||\left(\frac{Z_1}{2} + Z_0 \right) \right\}$$

$$\therefore \qquad Z_{in} = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_0\right)}{Z_2 + \frac{Z_1}{2} + Z_0}$$

But $\quad Z_{in} = Z_0$

$$\therefore \quad Z_0 = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2}+Z_0\right)}{Z_2+\frac{Z_1}{2}+Z_0}$$

$$2Z_0\left[Z_2+\frac{Z_1}{2}+Z_0\right] = Z_1\left(Z_2+\frac{Z_1}{2}+Z_0\right)+2Z_2\left(\frac{Z_1}{2}+Z_0\right)$$

$$\therefore\ 2Z_0 Z_2 + Z_0 Z_1 + 2Z_0^2 = Z_1 Z_2 + \frac{Z_1^2}{2} + Z_0 Z_1 + Z_1 Z_2 + 2Z_0 Z_2$$

$$\therefore \quad 2Z_0^2 = 2Z_1 Z_2 + \frac{Z_1^2}{2}$$

$$\therefore \quad Z_0^2 = \frac{Z_1^2}{4} + Z_1 Z_2 \qquad \text{... (1)}$$

This is the result applicable for equivalent symmetrical T circuit.

Hence Z_0 for the equivalent T section of the finite line is,

$$Z_0 = \sqrt{\frac{Z_1^2}{4}+Z_1Z_2} \qquad \text{... (2)}$$

But to obtain Z_1 and Z_2, practically two measurements are done. The input impedance is measured under two conditions. These two conditions are open circuit and short circuits.

In open circuit, the line is kept open and input impedance is measured which is denoted as Z_{OC}. This is shown in the Fig. 1.10.

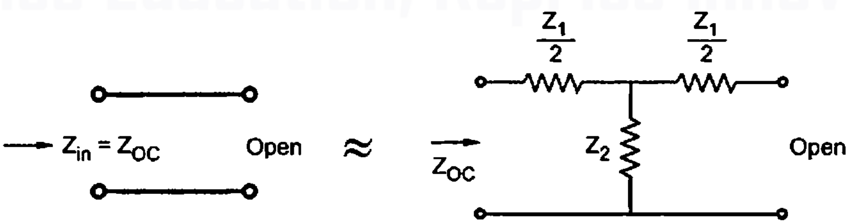


Fig. 1.10

From the Fig. 1.10 we can write,

$$Z_{OC} = \frac{Z_1}{2} + Z_2 \qquad \text{... (3)}$$

In short circuit case, the second end of the line is shorted and the input impedance is measured. It is denoted as Z_{SC}. This is shown in the Fig. 1.11.

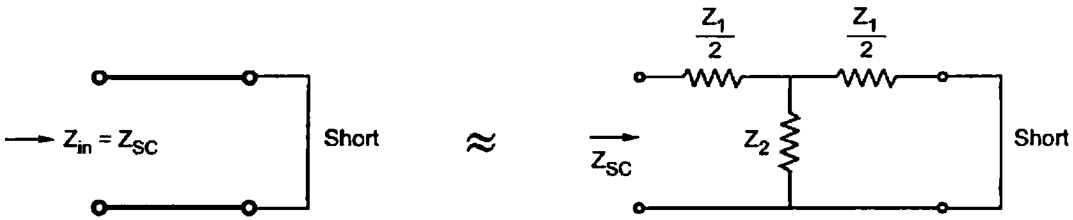


Fig. 1.11

From the Fig. 1.11 we can write,

$$Z_{SC} = \frac{Z_1}{2} + \left[Z_2 || \frac{Z_1}{2}\right] = \frac{Z_1}{2} + \frac{\frac{Z_1}{2} \times Z_2}{\frac{Z_1}{2} + Z_2}$$

$$= \frac{\frac{Z_1}{2}\left(\frac{Z_1}{2} + Z_2\right) + \frac{Z_1}{2} \times Z_2}{\frac{Z_1}{2} + Z_2}$$

$$= \frac{\frac{Z_1^2}{4} + \frac{Z_1 Z_2}{2} + \frac{Z_1 Z_2}{2}}{\frac{Z_1}{2} + Z_2} = \frac{\frac{Z_1^2}{4} + Z_1 Z_2}{\frac{Z_1}{2} + Z_2}$$

$$\therefore \quad Z_{SC} = \frac{Z_0^2}{Z_{OC}} \qquad \text{... (4)}$$

$$\therefore \quad Z_0^2 = Z_{OC} Z_{SC}$$

$$\therefore \quad Z_0 = \sqrt{Z_{OC} Z_{SC}} \qquad \text{... (5)}$$

Thus the characteristic impedance of a finite line is the geometric mean of the open and short circuit impedances.

Example 1.1: *Find the characteristic impedance of a line at 1600 Hz if the following measurements have been made on the line at 1600 Hz,*

$Z_{OC} = 750 \angle -30° \Omega$ *and* $Z_{SC} = 600 \angle -20° \Omega$

Solution : The characteristic impedance is given by,

$$Z_0 = \sqrt{Z_{OC}\,Z_{SC}} = \sqrt{750\angle -30° \times 600\angle -20°}$$

$$= \sqrt{4.5\times 10^5 \angle -50°} = \sqrt{4.5\times 10^5} \angle -\frac{50°}{2}$$

$$= 670.82 \angle -25°\ \Omega$$

1.7 Currents and Voltages Along an Infinite Line

A line terminated in Z_0 behaves as an infinite line. Consider a line, terminated in Z_0 and divided into number of identical sections of unit length as shown in the Fig.1.12.

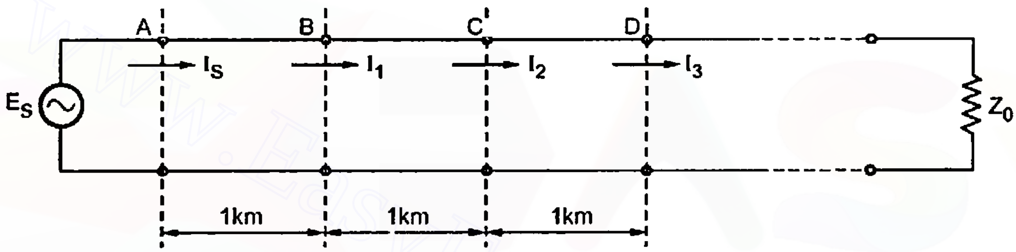


Fig. 1.12

The voltage E_S is applied at the sending end of the line at A. The AB, BC, CD etc. are the identical number of sections, of unit length. At point B which is unit length down the line, let the current be I_1. Due to the losses in the line, this current is less than the sending end current I_S. Similarly there will be some phase shift between I_S and I_1, due to the line parameters. Hence the ratio I_S / I_1 will be a phasor quantity.

The phasor quantity can be represented in the form of an exponential term. Thus the ratio I_S / I_1 can be represented as e^{γ} where γ is a complex quantity. This γ is called the **propagation constant** per unit length of the line.

The line is terminated in its characteristic impedance Z_0, while the input impedance at the sending end is also Z_0. Each section is terminated by the input impedance of the following section. And last section is terminated in Z_0 hence each section is terminated in Z_0. As seen earlier, as all sections are identical, each section can be represented by its symmetrical T section equivalent. Thus line can be represented as shown in the Fig. 1.13.

Hence the line is cascade connection of symmetrical T section equivalent circuits.

For the further analysis, consider the equivalent T section for the first unit length of the line, to find γ interms of Z_1 and Z_2. It is shown in the Fig. 1.14.

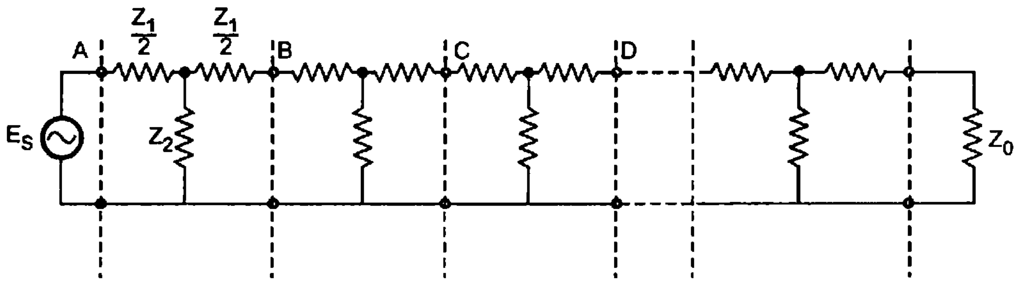


Fig. 1.13 T section equivalent of the line

As per the current division in parallel circuit we can write,

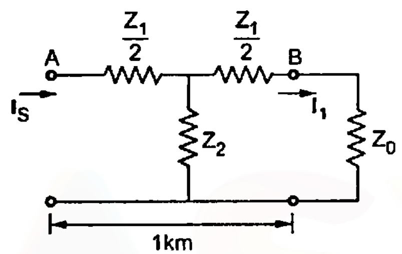


Fig. 1.14 Equivalent T section for unit length of line

$$I_1 = I_S \times \frac{Z_2}{\frac{Z_1}{2} + Z_0 + Z_2}$$

$$\therefore \quad \frac{I_S}{I_1} = \frac{\frac{Z_1}{2} + Z_2 + Z_0}{Z_2}$$

But $$\frac{I_S}{I_1} = e^{\gamma}$$

$$\therefore \quad e^{\gamma} = \frac{Z_2 + \frac{Z_1}{2} + Z_0}{Z_2} = 1 + \frac{Z_1}{2Z_2} + \frac{Z_0}{Z_2}$$

$$\therefore \quad \gamma = ln\left[1 + \frac{Z_1}{2Z_2} + \frac{Z_0}{Z_2}\right] \quad \text{... (1)}$$

This is the expression for γ interms of Z_1 and Z_2.

Now the current at point C is I_2 which is less than I_1. But the section BC is exactly identical to section AB hence the expressions derived for section AB of the line are equally applicable to all the sections. All the sections are represented by same equivalent T section. Hence the ratio I_1 / I_2 is same as I_S / I_1 i.e. e^{γ}.

$$\therefore \quad \frac{I_1}{I_2} = e^{\gamma} = 1 + \frac{Z_1}{2Z_2} + \frac{Z_0}{Z_2} \quad \text{... (2)}$$

where I_2 is current at point C which is 2 unit lengths down the line.

Same logic can be extended to section CD and so on.

$$\therefore \quad \frac{I_2}{I_3} = e^{\gamma}$$

where I_3 is the current at D which is 3 unit lengths down the line.

In general for the n^{th} section,

$$\frac{I_{n-1}}{I_n} = e^{\gamma} \quad \text{... (2)}$$

where I_{n-1} = Current at distance (n – 1) unit lengths down the line

I_n = Current at distance n unit lengths down the line

Now $$\frac{I_S}{I_1} = e^{\gamma}$$

$\therefore$ $$\frac{I_S}{I_2} = \frac{I_S}{I_1} \times \frac{I_1}{I_2} = e^{\gamma} \times e^{\gamma} = e^{2\gamma}$$

$\therefore$ $$\frac{I_S}{I_3} = \frac{I_S}{I_1} \times \frac{I_1}{I_2} \times \frac{I_2}{I_3} = e^{\gamma} \times e^{\gamma} \times e^{\gamma} = e^{3\gamma}$$

Thus in general

$$\frac{I_S}{I_n} = \frac{I_S}{I_1} \times \frac{I_1}{I_2} \times \frac{I_2}{I_3} \times \; .... \times \frac{I_{n-1}}{I_n} = e^{n\gamma}$$

$\therefore$ $$I_n = I_S \, e^{-n\gamma} \quad \text{... (3)}$$

This is an important relation which gives current at any point on the line interms of the sending end current and the propagation constant of the line. The I_n represents current at a point which is n unit lengths down the infinite line and γ is propagation constant per unit length. The equation (3) is applicable for any value of n.

At all the points along the line, the ratio of the voltage and the corresponding current is equal to the characteristic impedance Z_0. Hence similar to equation (3) an equation for voltage can be derived.

Now $$\frac{E_S}{I_S} = \frac{E_1}{I_1} = \frac{E_2}{I_2} = \frac{E_3}{I_3} = ... = \frac{E_n}{I_n} = Z_0$$

$\therefore$ $$\frac{E_S}{I_S} = \frac{E_n}{I_n}$$

$\therefore$ $$\frac{E_S}{E_n} = \frac{I_S}{I_n} = e^{n\gamma}$$

$\therefore$ $$E_n = E_S \, e^{-n\gamma} \quad \text{... (4)}$$

where E_n is the voltage at a distance of n unit lengths down the line and E_S is the sending end voltage. Note that the equations (3) and (4) are applicable for an infinite line or a finite line terminated in Z_0 only.

1.7.1 Attenuation and Phase Constant

It is seen that γ is the propagation constant which is a complex quantity. It can be represented as,

$$\gamma = \alpha + j\beta$$

$$\therefore \quad e^{\gamma} = e^{\alpha + j\beta} = e^{\alpha}\, e^{j\beta}$$

Now $e^{j\beta}$ can be expressed using Euler's relation as,

$$e^{\gamma} = e^{\alpha}\left[\cos\beta + j\sin\beta\right]$$

$$= e^{\alpha}\sqrt{\cos^2\beta + \sin^2\beta}\ \angle \tan^{-1}\left[\frac{\sin\beta}{\cos\beta}\right] \qquad \text{... polar form}$$

This is obtained by expressing rectangular form into polar form.

$$\therefore \quad e^{\gamma} = e^{\alpha}\angle \tan^{-1}[\tan\beta]$$

$$\therefore \quad e^{\gamma} = e^{\alpha}\angle\beta \qquad \text{... (5)}$$

$$\therefore \quad \frac{I_S}{I_1} = e^{\alpha}\angle\beta \qquad \text{... (6)}$$

Hence, $\left|\dfrac{I_S}{I_1}\right| = e^{\alpha}$ and $\angle\dfrac{I_S}{I_1} = \beta$

$$\therefore \quad \alpha = ln\left[\left|\frac{I_S}{I_1}\right|\right] \qquad \text{... (7)}$$

This α is known as the **attenuation constant** per unit length of the line and it is measured in **nepers** per km while β is known as the **phase constant** or **wavelength constant** per unit length of the line and it is measured in **radians** per unit length. The α indicates the rate at which the signal gets attenuated along the line while the β indicates the rate at which phase of the signal gets changed along the line.

Thus if the length of the line is n units then

$$\frac{I_S}{I_n} = \frac{I_S}{I_1}\times\frac{I_1}{I_2}\times \ldots\times\frac{I_{n-1}}{I_n} = e^{n\gamma}$$

$$= e^{n(\alpha + j\beta)} = e^{n\alpha}\cdot e^{jn\beta}$$

$$\therefore \quad \frac{I_S}{I_n} = e^{n\alpha}\angle n\beta \qquad \text{... (8)}$$

The attenuation of such line is thus n α nepers while the phase shift is n β radians.

Practically instead of nepers, the attenuation is measured in decibels (dB). The conversion is,

$$1 \text{ Neper} = 8.686 \text{ dB}$$

While the phase shift is measured in degrees where

$$1 \text{ radian} = 57.3 \text{ degrees}$$

The results derived in this section can be summarized as,

In case of an infinite line or a **short line terminated in its characteristic impedance** and having propagation constant γ, the current at any point which is 'x' units from the sending end is given by,

$$I_x = I_S e^{-\gamma x}$$

$$= I_S e^{-\alpha x} \angle -\beta x$$

The negative sign to the angle shows that the phase goes on lagging, down the line away from the sending end.

Similarly the voltage E at any point at a distance 'x' units from the sending end is given by,

$$E_x = E_S e^{-\gamma x}$$

$$= E_S e^{-\alpha x} \angle -\beta x$$

where I_S and E_S are the sending end current and voltage respectively.

1.8 Propagation Constant Interms of Z_{OC} and Z_{SC}

We know that,

$$e^{\gamma} = 1 + \frac{Z_1}{2 Z_2} + \frac{Z_0}{Z_2}$$

Substituting $\quad Z_0 = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$

$$e^{\gamma} = 1 + \frac{Z_1}{2 Z_2} + \sqrt{\frac{Z_1^2}{4 Z_2^2} + \frac{Z_1}{Z_2}} \quad \text{... dividing by } Z_2^2 \text{ inside root}$$

$$\therefore \quad e^{\gamma} = 1 + \frac{Z_1}{2 Z_2} + \sqrt{\left(\frac{Z_1}{2 Z_2}\right)^2 + \frac{Z_1}{Z_2}} \quad \text{... (1)}$$

Mathematically it can be shown that approximately,

$$e^{-\gamma} \approx 1+\frac{Z_1}{2\,Z_2}-\sqrt{\left(\frac{Z_1}{2\,Z_2}\right)^2+\frac{Z_1}{Z_2}} \qquad \text{... (2)}$$

Adding (1) and (2),

$$e^{\gamma}+e^{-\gamma} = 2+\frac{Z_1}{Z_2} \qquad \text{... (3)}$$

$$\therefore \quad \frac{e^{\gamma}+e^{-\gamma}}{2} = 1+\frac{Z_1}{2\,Z_2} \qquad \text{... (4)}$$

But $$\frac{e^{\gamma}+e^{-\gamma}}{2} = \cosh\gamma$$

$$\therefore \quad \cosh\gamma = 1+\frac{Z_1}{2\,Z_2} \qquad \text{... (5)}$$

Now $$e^{\gamma} = \cosh\gamma+\sinh\gamma$$

$$\therefore \quad (\cosh\gamma+\sinh\gamma)-\cosh\gamma = e^{\gamma}-\left(1+\frac{Z_1}{2\,Z_2}\right)$$

$$\therefore \quad \sinh\gamma = 1+\frac{Z_1}{2\,Z_2}+\frac{Z_0}{Z_2}-\left[1+\frac{Z_1}{2\,Z_2}\right]$$

$$\therefore \quad \sinh\gamma = \frac{Z_0}{Z_2} \qquad \text{... (6)}$$

$$\therefore \quad \tanh\gamma = \frac{\sinh\gamma}{\cosh\gamma} = \frac{\frac{Z_0}{Z_2}}{1+\frac{Z_1}{2\,Z_2}} = \frac{Z_0}{Z_2+\frac{Z_1}{2}}$$

But $$Z_0 = \sqrt{Z_{SC}\,Z_{OC}} \quad \text{and} \quad Z_2+\frac{Z_1}{2} = Z_{OC}$$

$$\therefore \quad \tanh\gamma = \frac{\sqrt{Z_{SC}\,Z_{OC}}}{Z_{OC}}$$

$$\therefore \quad \tanh\gamma = \sqrt{\frac{Z_{SC}}{Z_{OC}}} \qquad \text{... (7)}$$

This is very important result which is already derived at the time of discussion of symmetrical networks.

1.8.1 T Section Values Interms of Z_0 and γ

Now $$\cosh \gamma = 1 + \frac{Z_1}{2 Z_2}$$

$$\therefore \quad \frac{Z_1}{Z_2} = 2(\cosh \gamma - 1) = 2 \times 2 \sinh^2 \frac{\gamma}{2}$$

$$\therefore \quad \sinh \frac{\gamma}{2} = \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} \qquad \text{... (8)}$$

Now $$\sinh \gamma = \frac{Z_0}{Z_2} \qquad \text{from (6)}$$

$$\therefore \quad \frac{\sinh \gamma}{2} = \frac{Z_0}{2 Z_2}$$

$$\therefore \quad \frac{2 \sinh \frac{\gamma}{2} \cosh \frac{\gamma}{2}}{2} = \frac{Z_0}{2 Z_2}$$

$$\therefore \quad \cosh \frac{\gamma}{2} = \frac{Z_0}{2 Z_2} \times \frac{1}{\sinh \frac{\gamma}{2}} = \frac{Z_0}{2 Z_2} \times \frac{1}{\left(\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}\right)}$$

$$\therefore \quad \cosh \frac{\gamma}{2} = \frac{Z_0}{\sqrt{Z_1 Z_2}} \qquad \text{... (9)}$$

$$\therefore \quad \tanh \frac{\gamma}{2} = \frac{\sinh\left(\frac{\gamma}{2}\right)}{\cosh\left(\frac{\gamma}{2}\right)} = \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} \times \frac{\sqrt{Z_1 Z_2}}{Z_0} = \frac{Z_1}{2 Z_0}$$

$$\therefore \quad \frac{Z_1}{2} = Z_0 \tanh\left(\frac{\gamma}{2}\right) \qquad \text{... (10)}$$

and $$Z_2 = \frac{Z_0}{\sinh(\gamma)} \qquad \text{... (11)}$$

Thus if γ and Z_0 of the transmission line are known then the elements of T section equivalent circuit can be obtained for the line.

Note : If the length of the transmission line is 'l' and γ is defined per unit length then the expressions (5), (6), (7), (10) and (11) can be used by replacing γ with γl.

Thus the T section equivalent for a line of length 'l' can be shown as in the Fig.1.15.

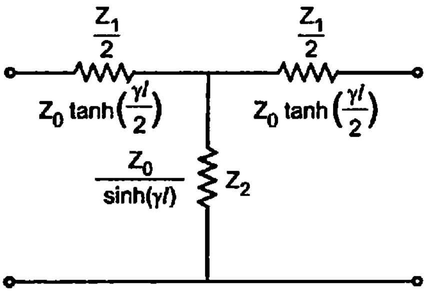


Fig. 1.15 T section equivalent

In a similar manner using T to π conversion, the π section equivalent for a line of length 'l' can be shown as in the Fig. 1.16.

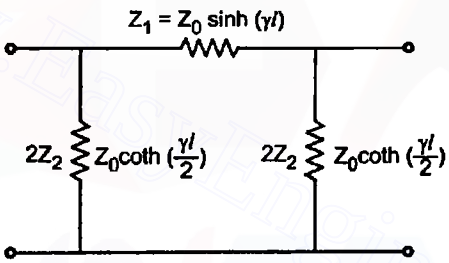


Fig. 1.16 π section equivalent

Example 1.2 : *A cable has an attenuation of 3.5 dB/km and a phase constant of 0.28 radians/km. If 3 volts are applied to the sending end, what will be the voltage at point 10 km down the line when the line is terminated in its characteristic impedance ?*

Solution : $E_S = 3V$, $\alpha = 3.5$ dB/km, $\beta = 0.28$ radians/km

Now $\quad 1 \text{ Neper} = 8.686 \text{ dB}$

$$\therefore \quad 3.5 \text{ dB/km} = \frac{3.5}{8.686} \text{ Neper/km} = 0.4029 \text{ Neper/km}$$

$$\text{Now} \quad E_x = E_S e^{-\gamma x} = E_S e^{-\alpha x} \angle -\beta x \qquad \text{... Use } \alpha \text{ in Neper/km}$$

$$x = 10 \text{ km}$$

$$\therefore \quad E_x = 3 e^{-0.4029 \times 10} \angle -0.28 \times 10 \text{ rad}$$

$$= 0.05337 \angle -2.8 \text{ rad} \qquad \text{... 1 rad} = 57.3°$$

$$= \mathbf{0.05337 \angle -160.44° \text{ V}}$$

This is the voltage at a point 10 km down the line.

Example 1.3 : *Calculate the characteristic impedance, the attenuation constant and phase constant of a transmission line if the following measurements have been made on the line.*

$Z_{OC} = 550 \angle -60° \, \Omega$ *and* $Z_{SC} = 500 \angle -14° \, \Omega$

Solution : For a transmission line,

$$Z_0 = \sqrt{Z_{OC} Z_{SC}} = \sqrt{550 \times 500 \angle -60° - 14°}$$

$$= 524.404 \angle -\frac{74°}{2} = 524.404 \angle -37° \, \Omega$$

$$= \mathbf{418.807 - j\, 315.594 \;\; \Omega}$$

Now

$$\tanh \gamma = \sqrt{\frac{Z_{SC}}{Z_{OC}}} = \sqrt{\frac{500 \angle -14°}{550 \angle -60°}} = 0.9534 \angle \frac{-14 + 60°}{2}$$

$$= 0.9534 \angle 23° = 0.877 + j\, 0.3725$$

$$\tanh \gamma = \frac{e^{2\gamma} - 1}{e^{2\gamma} + 1}$$

$$\therefore \quad \frac{e^{2\gamma} - 1}{e^{2\gamma} + 1} = 0.877 + j\, 0.3725$$

$$\therefore \quad e^{2\gamma} - 1 = (0.877 + j\, 0.3725)\,(e^{2\gamma} + 1)$$

$$\therefore \quad e^{2\gamma} - 1 = 0.877\, e^{2\gamma} + j\, 0.3725\, e^{2\gamma} + 0.877 + j\, 0.3725$$

$$e^{2\gamma}\, [1 - 0.877 - j\, 0.3725] = 1 + 0.877 + j\, 0.3725$$

$$e^{2\gamma}\, (0.123 - j\, 0.3725) = 1.877 + j\, 0.3725$$

$$\therefore \quad e^{2\gamma} = \frac{1.877 + j\, 0.3725}{0.123 - j\, 0.3725} = \frac{1.9136 \angle 11.22°}{0.3922 \angle -71.72°}$$

$$\therefore \quad e^{2\gamma} = 4.8791 \angle 82.94°$$

$$\therefore \quad 2\gamma = \ln [4.8791 \angle 82.94°]$$

$$\therefore \quad \gamma = \frac{1}{2} \ln [4.8791 \angle 82.94°]$$

Mathematically, $\ln [a \angle b] = \ln a + j\, b$

$$\therefore \quad \gamma = \frac{1}{2} \{\ln 4.8791 + j\, 82.94°\}$$

But $\gamma = \alpha + j\beta$

$$\therefore \qquad \alpha + j\beta = \frac{1}{2}\, ln\ 4.8791 + j\left(\frac{82.94}{2}\right)^{\circ}$$

$$\therefore \qquad \alpha = \frac{1}{2}\, ln\ 4.8791 = 0.7924 \quad \text{Nepers/km}$$

$$\text{and} \qquad \beta = \left(\frac{82.94}{2}\right)^{\circ} = 41.47^{\circ}/\text{km} = 0.7237 \quad \text{rad/km}$$

Note : Another mathematical result can be used to solve this problem.

$$\tanh \gamma = \tanh(\alpha + j\beta) = A + jB$$

$$\text{then} \qquad \alpha = \frac{1}{2}\tanh^{-1}\left\{\frac{2A}{1 + A^2 + B^2}\right\} \text{ Nepers}$$

$$\text{and} \qquad \beta = \frac{1}{2}\tan^{-1}\left\{\frac{2B}{1 - (A^2 + B^2)}\right\}$$

Units of β depends on $\tan^{-1}$ function mode. If $\tan^{-1}$ is calculated in degree mode, will be in degrees and if $\tan^{-1}$ is calculated in radian mode, β will be in radians.

1.9 Wavelength and Velocity

It is seen earlier that the current and voltage decreases along the transmission line as electric wave propagates from the sending end towards the receiving end. Graphically the variation of current with respect to distance can be shown as in the Fig. 1.17, assuming current is maximum at the sending end.

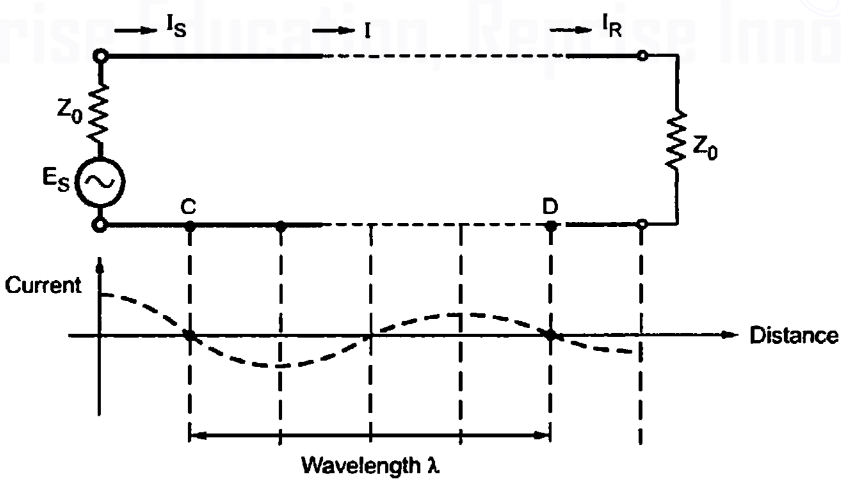


Fig. 1.17 Variation of current against distance

The current amplitude and phase decreases down the transmission line from the sending end. The voltage also varies similarly.

The distance between two points along the line at which currents or voltages differ in phase by 2π radians is called wavelength. It is denoted by λ.

It can also be defined as the distance between any point and the next point along the line at which the current or voltage is in the same phase.

The distance between points C and D along the line shown in the Fig. 1.17 is wavelength λ.

The phase constant of the line β is defined as radians per unit length of line. So if β is defined as rad/km it indicates that there is a phase change of β radians for a distance of 1 km of the line. Hence for a phase shift of 2π radians, the distance will be $2\pi/\beta$ km. This distance corresponding to phase shift of 2π radians is wavelength λ.

$$\therefore \qquad \lambda = \frac{2\pi}{\beta} \qquad \text{... (1)}$$

In one wavelength, one electrical cycle is completed. If the frequency is f Hz i.e. cycles/sec then for one cycle the time required is called time period given by,

$$T = \frac{1}{f} \text{ sec/cycle}$$

The wave travels distance of λ in one cycle, for which the time required is $1/f$ sec. Hence the velocity of propagation v can be written as,

$$v = \frac{\text{distance travelled}}{\text{time taken}} = \frac{\lambda}{\left(\frac{1}{f}\right)} = f\lambda$$

$$\therefore \qquad v = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} \qquad \text{... (2)}$$

It is measured in km/sec if β is in rad/km and in m/sec if β is in rad/m and so on. As it is related to phase constant of the line, the velocity is called **phase velocity.**

When β is a function of ω then velocity is produced by introduction of a group of frequencies travelling through the system. Such a velocity is called **group velocity** and can be obtained as,

$$v_g = \frac{d\omega}{d\beta}$$

This velocity plays an important role in wave guides and not in transmission line.

Example 1.4 *For an open wire overhead line β = 0.04 rad/km. Find the wavelength and velocity at a frequency of 1600 Hz. Hence calculate the time taken by the wave to travel 90 km.*

Solution : $\beta = 0.04$ rad/km, $f = 1600$ Hz

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.04} = 157.079 \text{ km}$$

and

$$v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\pi \times 1600}{0.04}$$

$$= 2.5132 \times 10^5 \text{ km/sec}$$

So time required to travel 90 km distance is,

$$t = \frac{90 \text{ km}}{2.5132 \times 10^5 \text{ km / sec}} = 3.581 \times 10^{-4} \text{ sec}$$

1.10 Relationship between Primary and Secondary Constants

It is seen that the practical line has following constants,

R = Resistance per unit length, measured in ohm (Ω)

G = Conductance per unit length, measured in mho ($\mho$)

L = Inductance per unit length, measured in henry (H)

C = Capacitance per unit length, measured in farad (F)

All these constants are assumed to be independent of frequency and are called **primary constants** of the transmission line. All these constants are measured considering both the wires of transmission line.

Apart from R, G, L and C few other constants related to the transmission line are characteristic impedance Z_0, the propagation constant γ, attenuation constant α and phase constant β. All these constants are fixed at one particular frequency but change their values as the frequency changes. Hence these constants are called **secondary constants**. Let us obtain the relationship between primary and secondary constants of the transmission line.

Consider a short length of line 'l' km. This section will have resistance of Rl Ω, conductance of Gl mho, inductance of Ll H and capacitance of Cl F. Its characteristic impedance is Z_0. The line is shown in the Fig. 1.18 (a).

This short line can be represented by symmetrical T network as shown in the Fig.1.18 (b).

If the length of the line is small, then the total series impedance of the section represents Z_1 and the total parallel impedance of the section represents Z_2.

$$\therefore \qquad Z_1 = (R + j\omega L)l$$

$$\text{and} \qquad Z_2 = \frac{1}{(G + j\omega C)l}$$

The total series impedance per unit length is denoted as Z while the total parallel impedance per unit length is denoted as Y.

$$\therefore \qquad Z = R + j\omega L \qquad \text{... per unit length}$$

$$Y = Z_{shunt} = G + j\omega C \qquad \text{... per unit length}$$

Hence the equivalent T network can be shown as in the Fig. 1.18 (c).

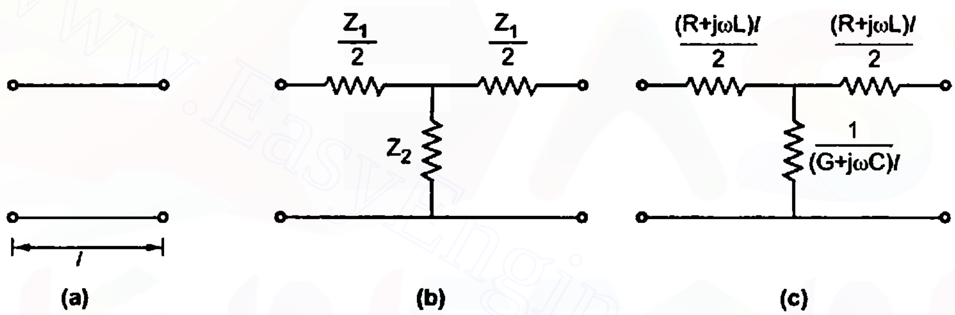


Fig. 1.18 Short transmission line representation

Note that this assumption is valid only when length of the line is small i.e. $l \to 0$.

1.10.1 Determination of Z_0 Interms of Primary Constants

For the T section we can write,

$$Z_0 = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} = \sqrt{\frac{(R+j\omega L)^2 l^2}{4} + \frac{(R+j\omega L)l}{(G+j\omega C)l}}$$

$$= \sqrt{\frac{R+j\omega L}{G+j\omega C} + \frac{(R+j\omega L)^2 l^2}{4}}$$

But as $l \to 0$, $l^2 \to 0$ and can be neglected.

$$\therefore \qquad Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{Z}{Y}} \qquad \text{... (1)}$$

Representing in polar form we can write,

$$Z_0 = \sqrt{\frac{\sqrt{R^2+\omega^2 L^2}}{\sqrt{G^2+\omega^2 C^2}} \angle \tan^{-1}\frac{\omega L}{R} - \tan^{-1}\frac{\omega C}{G}}$$

$$\therefore \quad Z_0 = \sqrt[4]{\frac{R^2+\omega^2 L^2}{G^2+\omega^2 C^2}} \angle \frac{1}{2}\left[\tan^{-1}\frac{\omega L}{R} - \tan^{-1}\frac{\omega C}{G}\right] \quad \text{... (2)}$$

When frequency is very small, $\omega \to 0$ hence

$$\therefore \quad \boxed{Z_0 = \sqrt{\frac{R}{G}}} \quad \text{... (3)}$$

When frequency is very large $R^2 << \omega^2 L^2$ and $G^2 << \omega^2 C^2$

$$\therefore \quad \boxed{Z_0 = \sqrt{\frac{L}{C}}} \quad \text{... (4)}$$

For all the practical lines R/G is always greater than L/C hence variation of Z_0 with frequency for a short practical line can be shown as in the Fig. 1.19.

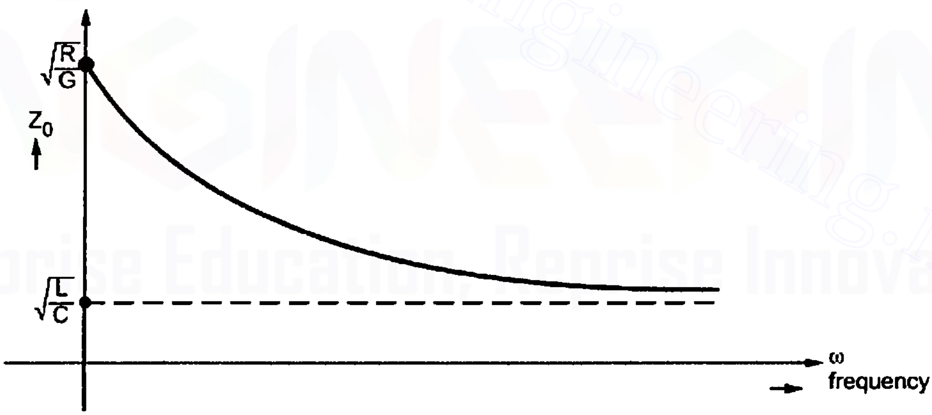


Fig. 1.19 Variation of Z_0 with frequency

1.10.2 Determination of γ Interms of Primary Constants

If γ is the propagation constant per unit length defined for a line then its value for a line of length l is γl. Hence for the T section considered we can write,

$$e^{\gamma l} = 1 + \frac{Z_1}{2 Z_2} + \frac{Z_0}{Z_2}$$

Substituting values of Z_1 and Z_2 for a short line of length l,

$$e^{\gamma l} = 1 + \frac{(R+j\omega L)l}{2\left[\frac{1}{(G+j\omega C)l}\right]} + \frac{Z_0}{\left[\frac{1}{(G+j\omega C)l}\right]}$$

$$\therefore \quad e^{\gamma l} = 1 + \frac{l^2 (R+j\omega L)(G+j\omega C)}{2} + Z_0\, l (G+j\omega C)$$

$$= 1 + \frac{l^2 (R+j\omega L)(G+j\omega C)}{2} + \sqrt{\frac{R+j\omega L}{G+j\omega C}} \times l(G+j\omega C)$$

$$\therefore \quad e^{\gamma l} = 1 + \frac{l^2 (R+j\omega L)(G+j\omega C)}{2} + l\sqrt{(R+j\omega L)(G+j\omega C)} \qquad \text{... (3)}$$

But mathematically exponential term $e^{\gamma l}$ can be expanded in a series as,

$$e^{\gamma l} = 1 + \gamma l + \frac{\gamma^2 l^2}{2!} + \ldots$$

As $l \to 0$, neglecting higher order terms,

$$e^{\gamma l} = 1 + \gamma l + \frac{\gamma^2 l^2}{2} \qquad \text{... (4)}$$

Comparing equations (3) and (4),

$$\boxed{\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{ZY}} \qquad \text{... (5)}$$

If γ is represented in the polar form as $|\gamma| \angle \theta$ then the attenuation and phase constants can be obtained as,

$$\alpha = |\gamma| \cos\theta \quad \text{and} \quad \beta = |\gamma| \sin\theta$$

Thus by representing γ in the rectangular form as $\alpha + j\beta$ the values of α and β can be directly obtained. While expressing γ from **rectangular form to polar form, β must be in radians.**

Example 1.5 : *A generator of 1V, 1 kHz supplies power to a 100 km long line terminated in Z_0 and having the following constants,*

$R = 10.4\ \Omega/km$, $L = 0.00367\ H/km$

$G = 0.8 \times 10^{-6}$ mho/km, $C = 0.00835 \times 10^{-6}$ F/km

Calculate Z_0, attenuation constant α, phase constant β, wavelength λ and velocity v, received current, voltage and power.

Solution : The Z_0 is given by,

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad \text{where} \quad \omega = 2\pi f = 2\pi \times 1 \times 10^3 \ \text{rad/sec}$$

$$Z_0 = \sqrt{\frac{10.4 + j\left(2\pi \times 10^3 \times 0.00367\right)}{0.8 \times 10^{-6} + j\left(2\pi \times 10^3 \times 0.00835 \times 10^{-6}\right)}}$$

$$= \sqrt{\frac{10.4 + j\,23.059}{0.8 \times 10^{-6} + j\,5.246 \times 10^{-5}}} = \sqrt{\frac{25.29 \angle 65.72^\circ}{5.246 \times 10^{-5} \angle 89.126^\circ}}$$

$$= \sqrt{4.8208 \times 10^5 \angle 65.72^\circ - 89.126^\circ}$$

$$\therefore \quad Z_0 = \mathbf{694.32 \ \angle -11.703^\circ \Omega}$$

Now $$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{25.29 \angle 65.72^\circ \times 5.246 \times 10^{-5} \angle 89.126^\circ}$$

$$= \sqrt{0.001326 \angle 154.846^\circ}$$

$$= 0.03641 \ \angle 77.423^\circ = 0.007928 + j\,0.03553$$

$$= \alpha + j\,\beta$$

$$\therefore \quad \alpha = \mathbf{0.007928 \ \ Nepers/km}$$

and $$\beta = \mathbf{0.03553 \ \ radians/km}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.03553} = \mathbf{176.841 \ \ km}$$

$$v = \frac{\omega}{\beta} = \frac{2\pi \times 1 \times 10^3}{0.03553} = \mathbf{1.95 \times 10^4 \ \ km/sec}$$

Now $$Z_0 = \frac{E_S}{I_S} \qquad \ldots E_S = 1V$$

$$\therefore \quad I_S = \frac{E_S}{Z_0} = \frac{1 \angle 0^\circ}{694.32 \angle -11.703^\circ}$$

$$= 1.4402 \times 10^{-3} \ \angle +11.703^\circ \ A$$

Now $$I_R = I_S \, e^{-\gamma x} \quad \text{where} \quad x = l = 100 \ \text{km}$$

$$\therefore \quad I_R = I_S \ e^{-(\alpha + j\beta) l}$$

$$= I_S \, e^{-\alpha l} \ \angle -\beta l$$

$$= 1.4402 \times 10^{-3} \times e^{-0.007928 \times 100} \ \angle -0.03553 \times 100 \ \ \text{rad}$$

$$\therefore \quad I_R = 6.518 \times 10^{-4} \ \angle -3.553 \ \text{rad} = \mathbf{6.518 \times 10^{-4} \ \angle -203.58^\circ \ A}$$

Now $$\frac{E_R}{I_R} = Z_0$$

$\therefore$ $$E_R = I_R \; Z_0 = 6.518 \times 10^{-4} \angle -203.58° \times 694.32 \angle -11.703°$$

$\therefore$ $$E_R = 0.4525 \angle -215.283° \text{ V}$$

Thus I_R and E_R are receiving end current and voltage respectively.

The power received is given by,

$$P_R = E_R \; I_R \cos\left(E_R \,\hat{}\, I_R \right)$$

where $E_R \,\hat{}\, I_R$ = angle between E_R and I_R

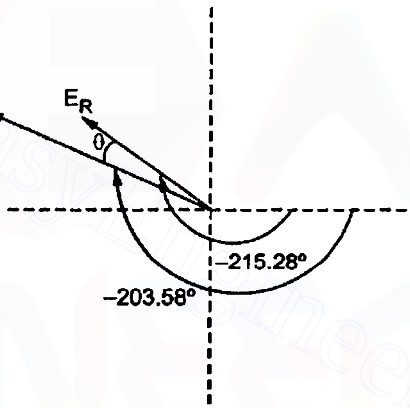


Fig. 1.20

$$\theta = E_R \,\hat{}\, I_R = 215.28° - 203.58°$$

$$= +11.703°$$

This is nothing but angle of Z_0

$\therefore$ $$P_R = 6518 \times 10^{-4} \times 0.4525 \times \cos(+11.703°)$$

$$= 288 \times 10^{-6} \text{ watts}$$

1.10.3 Determination of α and β Interms of Primary Constants

It is derived that,

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{\sqrt{R^2+\omega^2 L^2}\angle\tan^{-1}\frac{\omega L}{R}\sqrt{G^2+\omega^2 C^2}\angle\tan^{-1}\frac{\omega C}{G}}$$

$$= \sqrt[4]{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}\angle\frac{1}{2}\left[\tan^{-1}\frac{\omega L}{R}+\tan^{-1}\frac{\omega C}{G}\right] \quad \text{.. (6)}$$

But

$$\gamma = \alpha+j\beta = \sqrt{\alpha^2+\beta^2}\angle\tan^{-1}\frac{\beta}{\alpha} \quad \text{... (7)}$$

Equating magnitudes,

$$\sqrt{\alpha^2+\beta^2} = \sqrt[4]{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}$$

$$\therefore \quad \alpha^2+\beta^2 = \sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)} \quad \text{... (8)}$$

Now

$$\gamma = \alpha+j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\therefore \quad (\alpha+j\beta)^2 = (R+j\omega L)(G+j\omega C)$$

$$\therefore \quad \alpha^2+2j\alpha\beta+j^2\beta^2 = RG+j\omega LG+j\omega RC+j^2\omega^2 LC$$

$$\therefore \quad \alpha^2-\beta^2+j2\alpha\beta = (RG-\omega^2 LC)+j\omega(LG+RC) \quad \text{... (9)}$$

Equating real parts,

$$\alpha^2-\beta^2 = (RG-\omega^2 LC) \quad \text{... (10)}$$

Adding (8) and (10),

$$2\alpha^2 = \sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}+(RG-\omega^2 LC)$$

$$\therefore \quad \alpha = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}+(RG-\omega^2 LC)\right\}} \quad \text{... (11)}$$

Subtracting (10) from (8),

$$2\beta^2 = \sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}-(RG-\omega^2 LC)$$

$$\therefore \quad \beta = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}-(RG-\omega^2 LC)\right\}} \quad \text{... (12)}$$

1.10.4 Practical Formulae for Underground Cables

The underground cables consist of number of conductors which are closely spaced to each other. The diameter of the conductors is also small to accomodate large number of conductors. Now the diameter is small means cross sectional area is small hence the resistance R per km of such underground cables is very large. Due to close spacing of conductors, the capacitance C is also large. The inductance L is very small. The insulation between the conductors is good and hence leakage is very small. Thus the conductance G is very small.

Hence for the **unloaded** underground cables, at audio frequencies, it can be assumed that

1. $\omega C >> G$ hence G can be neglected
2. $\omega L << R$ hence ωL can be neglected

With these approximations, let us obtain the values of Z_0, α and β.

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$= \sqrt{\frac{R}{j\omega C}} \quad \text{... for underground cables}$$

$$\therefore \quad Z_0 = \sqrt{\frac{R}{\omega C}\angle -90°} = \sqrt{\frac{R}{\omega C}}\ \angle -45°$$

while

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} \approx \sqrt{(R)(j\omega C)}$$

$$= \sqrt{R\omega C \angle 90°} = \sqrt{\omega RC}\ \angle +45°$$

$$\therefore \quad \alpha = |\gamma|\cos\theta = \sqrt{\omega RC}\cos 45$$

$$= \sqrt{\frac{\omega RC}{2}} \text{ nepers/km}$$

and

$$\beta = |\gamma|\sin\theta = \sqrt{\omega RC}\sin 45$$

$$\approx \sqrt{\frac{\omega RC}{2}} \text{ radians/km}$$

Example 1.6 : *An unloaded underground cable has the following constants :*

R = 40 Ω/km, G = 0.5 μmho/km, L = 1 μH/km, C = 0.08 μF/km

Find the approximate values of Z_0, α and β at 400 Hz and 1600 Hz.

Solution : For an unloaded underground cable neglect G and L.

Case 1) $f = 400$ Hz i.e. $\omega = 2\pi f = 2.5132 \times 10^3$ rad/sec

$$\therefore \quad Z_0 = \sqrt{\frac{R}{\omega C}} \angle -45° \,\Omega = \sqrt{\frac{40}{2.5132 \times 10^3 \times 0.08 \times 10^{-6}}} \angle -45°$$

$$= 446.03 \angle -45° \,\Omega$$

$$\alpha = \sqrt{\frac{\omega RC}{2}} = 0.06341 \text{ neper/km}$$

$$\beta = \sqrt{\frac{\omega RC}{2}} = 0.06341 \text{ rad/km}$$

Case 2) $f = 1600$ Hz i.e. $\omega = 2\pi f = 1 \times 10^4$ rad/sec

$$\therefore \quad Z_0 = \sqrt{\frac{R}{\omega C}} \angle -45° = 223.606 \angle -45° \,\Omega$$

$$\alpha = \sqrt{\frac{\omega RC}{2}} = 0.1264 \text{ neper/km}$$

$$\beta = \sqrt{\frac{\omega RC}{2}} = 0.1264 \text{ rad/km}$$

1.11 General Solution of a Transmission Line

A transmission line is a circuit with distributed parameters hence the method of analysing such circuit is different than the method of analysis of a circuit with lumped parameters. It is seen that the current and voltage varies from point to point along the transmission line. The general solution of a transmission line includes the expressions for current and voltage at any point along a line of any length having uniformly distributed constants.

The various notations used in this derivation are,

R = Series resistance, ohms per unit length, including both the wires

L = Series inductance, henry per unit length

C = Capacitance between the conductors, farads per unit length

G = Shunt leakage conductance between the conductors, mhos per unit length

ωL = Series reactance per unit length

ωC = Shunt susceptance in mhos per unit length

$Z = R + j\omega L$ = Series impedance in ohms per unit length

$Y = G + j\omega C$ = Shunt admittance in mhos per unit length

s = Distance upto point of consideration, measured from receiving end

I = Current in the line at any point

E = Voltage between the conductors at any point

l = Length of the line

The transmission line of length l can be considered to be made up of infinitesimal T sections. One such section of length ds is shown in the Fig. 1.21. It carries a current I.

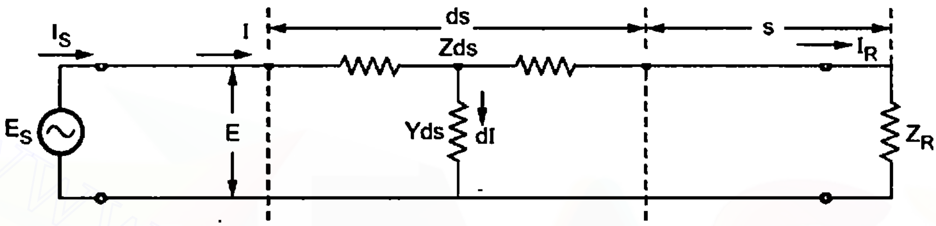


Fig. 1.21 Infinitesimal T section of a long line

The point under consideration is at a distance s from the receiving end. The length of section is ds hence its series impedance is Zds and shunt admittance is Yds. The current is I and voltage is E at this section.

The elemental voltage drop in the length ds is,

$$dE = I\,Z\,ds$$

$$\therefore \quad \frac{dE}{ds} = I\,Z \qquad \text{... (1)}$$

The leakage current flowing through shunt admittance from one conductor to other is given by,

$$dI = EY\,ds$$

$$\therefore \quad \frac{dI}{ds} = EY \qquad \text{... (2)}$$

Differentiating equations (1) and (2) with respect to s we get

$$\frac{d^2E}{ds^2} = Z\frac{dI}{ds}$$

$$\text{and} \quad \frac{d^2I}{ds^2} = Y\frac{dE}{ds}$$

This is because both E and I are functions of s.

$$\frac{d^2E}{ds^2} = ZEY \qquad \text{... (3)}$$

and $$\frac{d^2I}{ds^2} = YIZ \qquad \text{... (4)}$$

The equations (3) and (4) are the second order differential equations describing the transmission line having distributed constants, all along its length. It is necessary to solve these equations to obtain the expressions of E and I.

Replace the operator d/ds by m hence we get,

$$\left(m^2 - ZY\right) E = 0 \qquad \text{but } E \neq 0$$

$$\therefore \quad m = \pm\sqrt{ZY} \qquad \text{... (5)}$$

Same result is true for the current equation.

So there exists two solutions for positive sign of m and negative sign of m. The general solution of the equations for E and I are,

$$E = A\,e^{\sqrt{ZY}\,s} + B\,e^{-\sqrt{ZY}\,s} \qquad \text{... (6)}$$

$$I = C\,e^{\sqrt{ZY}\,s} + D\,e^{-\sqrt{ZY}\,s} \qquad \text{... (7)}$$

where A, B, C and D are arbitrary constants of integration.

It is now necessary to obtain the values of A, B, C and D.

As distance is measured from the receiving end, s = 0 indicates the receiving end.

$$E = E_R \quad \text{and} \quad I = I_R \qquad \text{... at } S = 0$$

Substituting in the solution,

$$E_R = A + B \qquad \text{... (8 (a))}$$

$$I_R = C + D \qquad \text{... (8(b))}$$

Same condition can be used in the equations obtained by differentiating the equations (6) and (7) with respect to s.

$$\frac{dE}{ds} = A\sqrt{ZY}\,e^{\sqrt{ZY}\,s} + B\left(-\sqrt{ZY}\right)e^{-\sqrt{ZY}\,s}$$

and $$\frac{dI}{ds} = C\sqrt{ZY}\,e^{\sqrt{ZY}\,s} + D\left(-\sqrt{ZY}\right)e^{-\sqrt{ZY}\,s}$$

But $$\frac{dE}{ds} = IZ \quad \text{and} \quad \frac{dI}{ds} = EY$$

$$\therefore \quad IZ = A\sqrt{ZY}\,e^{\sqrt{ZY}\,s} - B\sqrt{ZY}\,e^{-\sqrt{ZY}\,s} \qquad \text{... (9)}$$

and $$EY = C\sqrt{ZY}\, e^{\sqrt{ZY}\, s} - D\sqrt{ZY}\, e^{-\sqrt{ZY}\, s} \qquad \text{... (10)}$$

$$I = \frac{A}{Z}\sqrt{ZY}\, e^{\sqrt{ZY}\, s} - \frac{B}{Z}\sqrt{ZY}\, e^{-\sqrt{ZY}\, s}$$

i.e. $$I = A\sqrt{\frac{Y}{Z}}\, e^{\sqrt{ZY}\, s} - B\sqrt{\frac{Y}{Z}}\, e^{-\sqrt{ZY}\, s} \qquad \text{... (11)}$$

And $$E = C\sqrt{\frac{Z}{Y}}\, e^{\sqrt{ZY}\, s} - D\sqrt{\frac{Z}{Y}}\, e^{-\sqrt{ZY}\, s} \qquad \text{... (12)}$$

Now use $s = 0$, $E = E_R$ and $I = I_R$

$\therefore$ $$I_R = A\sqrt{\frac{Y}{Z}} - B\sqrt{\frac{Y}{Z}} \qquad \text{... (13(a))}$$

and $$E_R = C\sqrt{\frac{Z}{Y}} - D\sqrt{\frac{Z}{Y}} \qquad \text{... (13(b))}$$

The equations 8a, 8b, 13a and 13b are to be solved simultaneously to obtain the values of the constants A, B, C and D.

Now while solving these equations use the results,

$$Z_R = \frac{E_R}{I_R} \quad \text{and} \quad Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{Z}{Y}}$$

Hence the various constants obtained, after solving the equations simultaneously are,

$$A = \frac{E_R}{2} + \frac{I_R}{2}\sqrt{\frac{Z}{Y}} = \frac{E_R}{2}\left(1 + \frac{Z_0}{Z_R}\right) \qquad \text{... (14)}$$

$$B = \frac{E_R}{2} - \frac{I_R}{2}\sqrt{\frac{Z}{Y}} = \frac{E_R}{2}\left(1 - \frac{Z_0}{Z_R}\right) \qquad \text{... (15)}$$

$$C = \frac{I_R}{2} + \frac{E_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2}\left(1 + \frac{Z_R}{Z_0}\right) \qquad \text{... (16)}$$

$$D = \frac{I_R}{2} - \frac{E_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2}\left(1 - \frac{Z_R}{Z_0}\right) \qquad \text{... (17)}$$

Hence the general solution of the differential equations is,

$$E = \frac{E_R}{2}\left[\left(1 + \frac{Z_0}{Z_R}\right) e^{\sqrt{ZY}\, s} + \left(1 - \frac{Z_0}{Z_R}\right) e^{-\sqrt{ZY}\, s}\right] \qquad \text{... (18)}$$

$$I = \frac{I_R}{2}\left[\left(1 + \frac{Z_R}{Z_0}\right) e^{\sqrt{ZY}\, s} + \left(1 - \frac{Z_R}{Z_0}\right) e^{-\sqrt{ZY}\, s}\right] \qquad \text{... (19)}$$

Taking LCM as Z_R and taking $\frac{(Z_R + Z_0)}{Z_R}$ out from equation (18),

$$\therefore \quad E = \frac{E_R (Z_R + Z_0)}{2 Z_R}\left[e^{\sqrt{ZY}\,s} + \frac{(Z_R - Z_0)}{(Z_R + Z_0)}\, e^{-\sqrt{ZY}\,s}\right] \quad \text{... (20)}$$

Taking LCM as Z_0 and taking $\frac{(Z_R + Z_0)}{Z_0}$ out from equation (19)

$$\therefore \quad I = \frac{I_R (Z_R + Z_0)}{2 Z_0}\left[e^{\sqrt{ZY}\,s} - \frac{(Z_R - Z_0)}{(Z_R + Z_0)}\, e^{-\sqrt{ZY}\,s}\right] \quad \text{... (21)}$$

The negative sign is used to convert $Z_0 - Z_R$ to $Z_R - Z_0$.

The equations (20) and (21) is the **general solution** of a transmission line.

Another way of representing the equations is,

$$E = \frac{E_R}{2 Z_R}\left[(Z_R + Z_0)\, e^{\sqrt{ZY}\,s} + (Z_R - Z_0)\, e^{-\sqrt{ZY}\,s}\right]$$

$$\therefore \quad E = \frac{E_R}{2 Z_R}\left[Z_R\, e^{\sqrt{ZY}\,s} + Z_0\, e^{\sqrt{ZY}\,s} + Z_R\, e^{-\sqrt{ZY}\,s} - Z_0\, e^{-\sqrt{ZY}\,s}\right]$$

$$\therefore \quad E = E_R\left(\frac{e^{\sqrt{ZY}\,s} + e^{-\sqrt{ZY}\,s}}{2}\right) + \frac{Z_0\, E_R}{Z_R}\left(\frac{e^{\sqrt{ZY}\,s} - e^{-\sqrt{ZY}\,s}}{2}\right)$$

But $\frac{E_R}{I_R} = Z_R$ hence $\frac{E_R}{Z_R} = I_R$

$$\therefore \quad E = E_R\left[\frac{e^{\sqrt{ZY}\,s} + e^{-\sqrt{ZY}\,s}}{2}\right] + I_R\, Z_0\left[\frac{e^{\sqrt{ZY}\,s} - e^{-\sqrt{ZY}\,s}}{2}\right] \quad \text{... (22)}$$

and

$$I = I_R\left[\frac{e^{\sqrt{ZY}\,s} + e^{-\sqrt{ZY}\,s}}{2}\right] + \frac{E_R}{Z_0}\left[\frac{e^{\sqrt{ZY}\,s} - e^{-\sqrt{ZY}\,s}}{2}\right] \quad \text{... (23)}$$

But $\frac{e^{\sqrt{ZY}\,s} + e^{-\sqrt{ZY}\,s}}{2} = \cosh\sqrt{ZY}\,s$ and $\frac{e^{\sqrt{ZY}\,s} - e^{-\sqrt{ZY}\,s}}{2} = \sinh\sqrt{ZY}\,s$

$$\therefore \quad E = E_R \cosh\left(\sqrt{ZY}\,s\right) + I_R\, Z_0 \sinh\left(\sqrt{ZY}\,s\right) \quad \text{... (24)}$$

and

$$I = I_R \cosh\left(\sqrt{ZY}\,s\right) + \frac{E_R}{Z_0}\sinh\left(\sqrt{ZY}\,s\right) \quad \text{... (25)}$$

The equations (24) and (25) give the values of E and I at any point along the length of the line.

Important Note : The similar equations can be obtained interms of sending end voltage E_S and I_S. If x is the distance measured down the line from the sending end then,

$$x = l - s$$

And the equations (24) and (25) get transferred interms of E_S and I_S as,

$$E = E_S \cosh\left(\sqrt{ZY}\, x\right) - I_S\, Z_0 \sinh\left(\sqrt{ZY}\, x\right) \quad \text{... (26)}$$

$$I = I_S \cosh\left(\sqrt{ZY}\, x\right) - \frac{E_S}{Z_0} \sinh\left(\sqrt{ZY}\, x\right) \quad \text{... (27)}$$

And $\sqrt{ZY} = \gamma$ as derived earlier and hence equations can be written interms of propagation constant γ.

Summarizing,

If receiving end parameters are known and s is distance measured from the receiving end then,

$$E = E_R \cosh(\gamma s) + I_R\, Z_0 \sinh(\gamma s)$$

$$I = I_R \cosh(\gamma s) + \frac{E_R}{Z_0} \sinh(\gamma s)$$

And if sending end parameters are known and x is distance measured from the sending end then,

$$E = E_S \cosh(\gamma x) - I_S\, Z_0 \sinh(\gamma x)$$

$$I = I_S \cosh(\gamma x) - \frac{E_S}{Z_0} \sinh(\gamma x)$$

Any set of equations can be used to solve the problems depending on the values given.

1.12 Physical Significance of General Solution

From the equations, the sending end current can be obtained by substituting s = l measured from the receiving end.

$$E_S = E_R \cosh(\gamma l) + I_R\, Z_0 \sinh(\gamma l) \quad \text{... (1)}$$

and

$$I_S = I_R \cosh(\gamma l) + \frac{E_R}{Z_0} \sinh(\gamma l) \quad \text{... (2)}$$

Now

$$Z_R = \frac{E_R}{I_R}$$

$$\therefore \quad I_S = I_R \cosh(\gamma l) + \frac{Z_R}{Z_0} I_R \sinh(\gamma l)$$

$$\therefore \quad I_S = I_R \left[\cosh(\gamma l) + \frac{Z_R}{Z_0} \sinh(\gamma l)\right] \quad \text{... (3)}$$

Now if the line is terminated in its characteristic impedance Z_0 then,

$$I_S = I_R \left[\cosh(\gamma l) + \sinh(\gamma l)\right] \quad \text{... as } Z_R = Z_0$$

$$\therefore \quad \frac{I_S}{I_R} = \left[\cosh(\gamma l) + \sinh(\gamma l)\right] = e^{\gamma l} \quad \text{... (4)}$$

This is the equation which is already derived for the line terminated in Z_0.

Using $E_R = I_R Z_R$ in equation (1),

$$E_S = Z_R I_R \cosh(\gamma l) + I_R Z_0 \sinh(\gamma l)$$

$$\therefore \quad E_S = I_R \left[Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)\right] \quad \text{... (5)}$$

Dividing (5) by (3),

$$\frac{E_S}{I_S} = \frac{I_R \left[Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)\right]}{I_R \left[\cosh(\gamma l) + \frac{Z_R}{Z_0} \sinh(\gamma l)\right]}$$

But $$\frac{E_S}{I_S} = Z_S$$

$$\therefore \quad Z_S = \frac{Z_0 \left[Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)\right]}{\left[Z_0 \cosh(\gamma l) + Z_R \sinh(\gamma l)\right]} \quad \text{... (6)}$$

When the line is terminated in Z_0 then $Z_R = Z_0$. So substituting in equation (6) we get,

$$Z_S = Z_0 \quad \text{... (7)}$$

This shows that for a line terminated in its characteristic impedance its input impedance is also its characteristic impedance.

Now consider an infinite line with $l \to \infty$. Using this in equation (6) we get,

$$Z_S = \frac{Z_0 \left[Z_R + Z_0 \tanh(\gamma l)\right]}{\left[Z_0 + Z_R \tanh(\gamma l)\right]}$$

and $\tanh(\gamma l) \to 1$ as $l \to \infty$,

$$\therefore \quad Z_S = Z_0 \quad \text{... (8)}$$

This shows that finite line terminated in its characteristic impedance behaves as an infinite line, to the sending end generator.

Thus the equations for E_x and I_x are applicable for the finite line terminated in Z_0. The equations are reproduced here for the convenience of the reader.

$$E_x = E_S e^{-\gamma x} \quad \text{and} \quad I_x = I_S e^{-\gamma x}$$

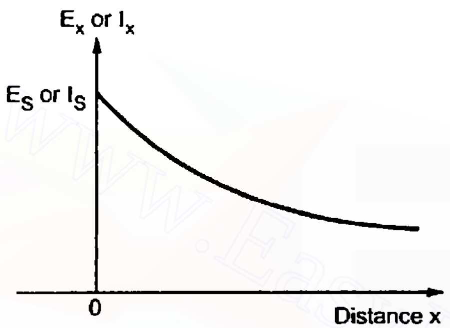


Fig. 1.22

If in practice instruments are connected along the line then the instruments will show the magnitudes $E_S e^{-\alpha x}$ and $I_S e^{-\alpha x}$ while the phase angles can not be obtained. If the graph of E_x or I_x is plotted against x then it can be shown as in the Fig. 1.22.

This is the physical significance of the general solution of a transmission line. Its use will be more clear by studying the various cases of the line.

1.13 Application of General Solution to The Particular Cases

The current and voltage at any point, x distance from the sending end are,

$$E = E_S \cosh(\gamma x) - I_S Z_0 \sinh(\gamma x) \quad \text{... (1)}$$

$$I = I_S \cosh(\gamma x) - \frac{E_S}{Z_0} \sinh(\gamma x) \quad \text{... (2)}$$

And if the distance s is measured from the receiving end then $s = l - x$ and the equations of E and I are,

$$E = E_R \cosh[\gamma(l-x)] + I_R Z_0 \sinh[\gamma(l-x)] \quad \text{... (3)}$$

$$I = I_R \cosh\left[\gamma(l-x) + \frac{E_R}{Z_0} \sinh[\gamma(l-x)]\right] \quad \text{... (4)}$$

Let us discuss application of these equations to the particular cases.

1.13.1 Finite Line Terminated in Z_0

The case is equally applicable for an infinite line.

At $x = l$, $E = E_R$ and $I = I_R$

$$\therefore \quad E_R = E_S \cosh(\gamma l) - I_S Z_0 \sinh(\gamma l)$$

and
$$I_R = I_S \cosh(\gamma l) - \frac{E_S}{Z_0} \sinh(\gamma l)$$

Using $E_R / Z_R = Z_R$ we get,

$$Z_R = \frac{E_R}{I_R} = \frac{Z_0 [E_s \cosh \gamma l - I_S Z_0 \sinh \gamma l]}{[I_S Z_0 \cosh \gamma l - E_S \sinh \gamma l]}$$

But $Z_R = Z_0$

$$Z_0 = \frac{Z_0 [E_S \cosh \gamma l - I_S Z_0 \sinh \gamma l]}{[I_S Z_0 \cosh \gamma l - E_S \sinh l]} \quad \text{... (5)}$$

Solving we get,

$$\frac{E_S}{I_S} = Z_0 = Z_S \quad \text{... (6)}$$

Thus input impedance for a finite line terminated in Z_0 is also Z_0.

1.13.2 Finite Line Open Circuited at Distant End

The line of length 'l' is open circuited at a distance $x = l$ i.e. $I_R = 0$.

$$\therefore \quad 0 = I_S \cosh(\gamma l) - \frac{E_S}{Z_0} \sinh(\gamma l)$$

$$\therefore \quad \frac{E_S}{I_S} = Z_0 \frac{\cosh(\gamma l)}{\sinh(\gamma l)} = Z_0 \coth(\gamma l)$$

But E_S / I_S is the input impedance so let us call this input impedance of open circuited line as Z_{OC}.

$$\therefore \quad Z_{OC} = Z_0 \coth(\gamma l) \quad \text{... (7)}$$

If $l \to \infty$, then $\coth(\gamma l) \to 1$ and thus input impedance Z_{OC} approaches to Z_0 for an infinite line, on open circuit.

1.13.3 Finite Line Short Circuited at Distant End

The line of length 'l' is short circuited at a distance $x = l$ i.e. $E_R = 0$.

$$0 = E_S \cosh(\gamma l) - I_S Z_0 \sinh(\gamma l)$$

$$\therefore \quad \frac{E_S}{I_S} = Z_0 \tanh(\gamma l)$$

But E_S / I_S is the input impedance, so let us call this input impedance of short circuited line as Z_{SC}.

$$\therefore \quad Z_{SC} = Z_0 \tanh(\gamma l) \quad \text{... (8)}$$

Multiplying (7) and (8),

$$Z_0^2 = Z_{SC} Z_{OC}$$

$$\therefore \quad Z_0 = \sqrt{Z_{OC} Z_{SC}} \quad \text{... (9)}$$

This equation is already proved.

Similarly, $$\frac{Z_{SC}}{Z_{OC}} = \tanh^2(\gamma l)$$

$$\therefore \quad \tanh(\gamma l) = \sqrt{\frac{Z_{SC}}{Z_{OC}}} \quad \text{... (10)}$$

This is also proved earlier.

1.13.4 Determination of α, β and Primary Constants

If Z_{SC} and Z_{OC} are known then let,

$$\tanh(\gamma l) = \sqrt{\frac{Z_{SC}}{Z_{OC}}} = A + jB$$

$$\therefore \quad \tanh[(\alpha + j\beta) l] = A + jB$$

$$\tanh[\alpha l + j\beta l] = A + jB \quad \text{... (11)}$$

$$\text{and} \quad \tanh[\alpha l - j\beta l] = A - jB \quad \text{... (12)}$$

Mathematically if identity is true for + j, it must be true for – j.

$$\therefore \quad \tanh(2\gamma l) = \frac{2A}{1 + A^2 + B^2}$$

$$\text{and} \quad \tan(2\beta l) = \frac{2B}{1 - (A^2 + B^2)}$$

From these equations α and β can be obtained.

Now $\gamma = \sqrt{(R+j\omega L)(Y+j\omega C)}$

and $Z_0 = \sqrt{\dfrac{R+j\omega L}{Y+j\omega C}}$

$\therefore$ $Z_0 \gamma = R+j\omega L$ ... (13)

and $\dfrac{\gamma}{Z_0} = G+j\omega C$ ... (14)

Hence knowing Z_0 and γ as complex quantities, all the four primary constants of the line can be determined.

1.14 Input and Transfer Impedance

It is derived in earlier section that the input impedance of a transmission line is given by,

$$Z_{in} = Z_S = \frac{Z_0\left[Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)\right]}{\left[Z_0 \cosh(\gamma l) + Z_R \sinh(\gamma l)\right]} \quad \text{... (1)}$$

$$Z_S = Z_0\left[\frac{1+\dfrac{Z_0}{Z_R}\tanh(\gamma l)}{\dfrac{Z_0}{Z_R}+\tanh(\gamma l)}\right] \quad \text{... (2)}$$

Let $Z_T = \dfrac{E_S}{I_R}$ = **Transfer impedance of a line**

Now $I_R = I_S \cosh(\gamma l) - \dfrac{E_S}{Z_0}\sinh(\gamma l)$

$\therefore$ $1 = \dfrac{I_S}{I_R}\cosh(\gamma l) - \dfrac{Z_T}{Z_0}\sinh(\gamma l)$ ... (3)

While $E_R = E_S \cosh(\gamma l) - I_S Z_0 \sinh(\gamma l)$

$\therefore$ $I_S = \dfrac{E_S \cosh(\gamma l) - E_R}{Z_0 \sinh(\gamma l)}$ ... (4)

Substituting in (3),

$$1 = \frac{\cosh(\gamma l)}{I_R}\left[\frac{E_S \cosh(\gamma l) - E_R}{Z_0 \sinh(\gamma l)}\right] - \frac{Z_T}{Z_0}\sinh(\gamma l)$$

$$\therefore \quad 1 = \left(\frac{E_S}{I_R}\right)\frac{\cosh^2(\gamma l)}{Z_0 \sinh(\gamma l)} - \left(\frac{E_R}{I_R}\right)\frac{\cosh(\gamma l)}{Z_0 \sinh(\gamma l)} - \frac{Z_T \sinh(\gamma l)}{Z_0}$$

$$\ldots \frac{E_R}{I_R} = Z_R$$

$$\therefore \quad 1 = \frac{Z_T \cosh^2(\gamma l) - Z_R \cosh(\gamma l) - Z_T \sinh^2(\gamma l)}{Z_0 \sinh(\gamma l)}$$

$$\therefore \quad Z_0 \sinh(\gamma l) = Z_T\left[\cosh^2(\gamma l) - \sinh^2(\gamma l)\right] - Z_R \cosh(\gamma l)$$

$$\therefore \quad Z_0 \sinh(\gamma l) = Z_T - Z_R \cosh(\gamma l)$$

$$\therefore \quad Z_T = Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l) \quad \ldots (5)$$

In terms of exponential coefficients this can be expressed as,

$$Z_T = Z_R\left[\frac{e^{\gamma l} + e^{-\gamma l}}{2}\right] + Z_0\left[\frac{e^{\gamma l} - e^{-\gamma l}}{2}\right] \quad \ldots (6)$$

This is the required transfer impedance of a transmission line, terminated in an impedance Z_R.

Example 1.7 : *An open wire line has a characteristic impedance of $700\angle -12°\ \Omega$ and a propagation constant $\gamma = 0.012 + j\,0.058$ when 2V are applied to the sending end and a current of 4 mA flows in. What will be the current at the distant end which is 50 km away from the sending end ?*

Solutoin : $Z_0 = 700\ \angle -12°\Omega$ and $\gamma = 0.012 + j\ 0.058$

According to general solution,

$$I = I_S \cosh(\gamma x) - \frac{E_S}{Z_0}\sinh(\gamma x)$$

Now $\quad x = 50$ km, $E_S = 2\angle 0°$ and $I_S = 4$ mA

$$I = 4\times 10^{-3}\cosh[(0.012 + j\,0.058)\,50]$$

$$-\frac{2\angle 0°}{700\angle -12°}\sinh[(0.012 + j\,0.058)\,50]$$

$$= 4\times 10^{3}\cosh[0.6 + j\,2.9] - 0.0028\angle +12°\sinh[0.6 + j\,2.9]$$

Now $\quad \sinh[a \pm j\,b] = \sinh(a)\cos(b) \pm j\cosh(a)\sin(b)$

and $\quad \cosh[a \pm j\,b] = \cosh(a)\cos(b) \pm j\sinh(a)\sin(b)$

$$I = 4\times10^{-3}\{\cosh(0.6)\cos(2.9)+j\sinh(0.6)\sin(2.9)\}$$
$$-[0.0027+j\,0.00058]\{\sinh(0.6)\cos(2.9)+j\cosh(0.6)\sin(2.9)\}$$

... use radian mode

$$I = 4\times10^{-3}\{-1.151+j\,0.1523\}-(0.0027+j\,0.00058)\{-0.6181+j\,0.2836\}$$
$$= 4\times10^{-3}\{1.161\angle 3.01\ \text{rad}\}-(0.0028\angle 12°)\,(0.68\angle 2.711\ \text{rad})$$
$$= 0.00464\angle 172.473°-0.0019\angle 167.34°$$
$$= -0.0046+j\,0.0006+0.00185-j\,0.00041$$
$$= -0.00275+j\,0.00019$$
$$\mathbf{I = 0.00275\angle 176.04°\ A}$$

So current at 50 km distance is **2.75 mA.**

Example 1.8 : *A line 20 km long has the following constants :*

$Z_0 = 600\angle 0°\Omega$, $\alpha = 0.1$ *nepers/km,* $\beta = 0.05$ *rad/km*

Find the received current when 20 mA are sent into one end and receiving end is short circuited.

Solution : For a line short circuited at the receiving end,

$$\frac{E_S}{I_S} = Z_0\tanh(\gamma l)$$

$$\therefore \quad E_S = I_S Z_0\tanh(\gamma l) \quad \text{... (1)}$$

And
$$I = I_S\cosh(\gamma x)-\frac{E_S}{Z_0}\sinh(\gamma x)$$

and
$$x = l = 20 \text{ km at the receiving end}$$

$$\therefore \quad I = I_S\cosh(\gamma l)-\frac{I_S Z_0\tanh(\gamma l)}{Z_0}\sinh(\gamma l) \quad \text{... using (1)}$$

$$= I_S\cosh(\gamma l)-I_S\frac{\sinh^2(\gamma l)}{\cosh(\gamma l)} = I_S\frac{[\cosh^2(\gamma l)-\sinh^2(\gamma l)]}{\cosh(\gamma l)}$$

$$= \frac{I_S}{\cosh(\gamma l)}$$

Now
$$I_S = 20\text{ mA},\quad \gamma = 0.1+j\,0.05$$

$$\therefore \quad I = \frac{20\times10^{-3}}{\cosh[(0.1+j\,0.05)\times 20]} = \frac{20\times10^{-3}}{\cosh[2+j\,1]}$$

$$= \frac{20\times10^{-3}}{\cosh 2 \cos 1 + j \sinh 2 \sin 1} \quad \text{... use radian mode}$$

$$= \frac{20\times10^{-3}}{2.0327 + j\,3.05189} = \frac{20\times10^{-3}}{3.667\angle 0.9832 \text{ rad}}$$

$$= 0.00545\angle -0.9832 \text{ rad} = 5.45 \ \angle -56.33^\circ \text{mA}$$

Thus received voltage is zero and received current 5.45 mA lags the sent current by 56.33°.

1.15 Conditions for Minimum Attenuation

The value α is derived earlier in terms of primary constant of the line as,

$$\alpha = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}+(RG-\omega^2LC)\right\}}$$

Hence attenuation constant depends on the four primary constants alongwith the frequency under consideration. Thus to find the conditions for minimum attenuation it is necessary to vary these constants in turn.

1.15.1 Variable L

Consider L to be variable while R, C and G are the constants for the frequency under consideration. Hence for minimum attenuation, differentiation of α with respect to L must be zero.

$$\therefore \quad \frac{d\alpha}{dL} = 0$$

Rewriting expression of α in different form to obtain its differentiation,

$$\frac{d\alpha}{dL} = \frac{d}{dL}\left\{\frac{1}{2}\left[\left\{(R^2+\omega^2L^2)(G^2+\omega^2C^2)\right\}^{\frac{1}{2}}+RG-\omega^2LC\right]\right\}^{\frac{1}{2}}$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left[\left\{(R^2+\omega^2L^2)(G^2+\omega^2C^2)\right\}^{\frac{1}{2}}+RG-\omega^2LC\right]\right\}^{\frac{1}{2}-1}$$

$$\times\frac{1}{2}\left\{\left[\frac{1}{2}\left\{(R^2+\omega^2L^2)(G^2+\omega^2C^2)\right\}^{\frac{1}{2}-1}\right]\left[2\omega^2L(G^2+\omega^2C^2)\right]-\omega^2C\right\}$$

$$= \frac{1}{2}\frac{\frac{1}{2}\left\{\frac{1}{2}\frac{2\omega^2 L(G^2+\omega^2C^2)}{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}}-\omega^2C\right\}}{\sqrt{\frac{1}{2}\left[\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}+RG-\omega^2LC\right]}} = 0$$

The denominator can not be ∞ hence to satisfy this equation,

$$\frac{\omega^2 L(G^2+\omega^2C^2)}{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}}-\omega^2C = 0$$

$$\frac{L(G^2+\omega^2C^2)}{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}} = C$$

$$\therefore \quad L\sqrt{\frac{G^2+\omega^2C^2}{R^2+\omega^2L^2}} = C$$

$$\therefore \quad L\sqrt{(G^2+\omega^2C^2)} = C\sqrt{(R^2+\omega^2L^2)}$$

$$L^2(G^2+\omega^2C^2) = C^2(R^2+\omega^2L^2)$$

$$\therefore \quad L^2G^2+\omega^2L^2C^2 = C^2R^2+\omega^2L^2C^2$$

$$\therefore \quad L^2 = \frac{C^2R^2}{G^2}$$

$$\therefore \quad \boxed{L = \frac{CR}{G}} \qquad \ldots (1)$$

Thus when L is variable then the attenuation will be minimum when,

$$\boxed{L = \frac{CR}{G}\ \text{H/km}}$$

In practice L is kept less than this value and hence the attenuation can be reduced by artificially increasing L. This leads to the concept of loading of line. Hence this result is very important.

1.15.2 Variable C

Consider C to be variable while R, L and G are constants. Hence differentiating α with respect to C and equating it to zero, condition of C for minimum attenuation can be obtained.

$$\therefore \quad \frac{d\alpha}{dC} = 0$$

$$\therefore \quad \frac{d\alpha}{dC} = \frac{d}{dC}\left\{\frac{1}{2}\left[\left\{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)\right\}^{\frac{1}{2}}+RG-\omega^2LC\right]\right\}^{\frac{1}{2}}$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left[\left\{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)\right\}^{\frac{1}{2}}+RG-\omega^2LC\right]\right\}^{\frac{1}{2}-1}$$

$$\times\frac{1}{2}\left\{\left[\frac{1}{2}\left\{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)\right\}^{\frac{1}{2}-1}\right]\left[2\omega^2C\left(R^2+\omega^2L^2\right)\right]-\omega^2L\right\}$$

$$= \frac{1}{2}\frac{\frac{1}{2}\left\{\frac{1}{2}\frac{2\omega^2C\left(R^2+\omega^2L^2\right)}{\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}}-\omega^2L\right\}}{\sqrt{\frac{1}{2}\left[\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}+RG-\omega^2LC\right]}} = 0$$

Hence to satisfy this,

$$\frac{\omega^2C\left(R^2+\omega^2L^2\right)}{\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}}-\omega^2L = 0$$

$$\therefore \quad C\sqrt{R^2+\omega^2L^2} = L\sqrt{G^2+\omega^2C^2}$$

$$\therefore \quad C^2\left(R^2+\omega^2L^2\right) = L^2\left(G^2+\omega^2C^2\right)$$

$$\therefore \quad C^2R^2+C^2\omega^2L^2 = L^2G^2+L^2\omega^2C^2$$

$$\therefore \quad C^2 = \frac{L^2G^2}{R^2}$$

$$\therefore \quad \boxed{C = \frac{LG}{R} \quad \text{F/km}} \qquad \ldots (2)$$

In practice, C is normally larger than the value required for the minimum attenuation.

1.15.3 R and G for Minimum Attenuation

When R or G is variable, then by differentiating α and equating it to zero, no minima can be found out mathematically. So theoretically no values of R and G can be obtained for minimum attenuation.

But practically it can be seen that when R = 0 there are no losses along the line while when G = 0 there is no leakage thus in all when R and G are zero, the attenuation is zero. This can be observed from the expression for α at R = 0 and G = 0.

Hence for minimum attenuation, practically R and G values must be kept as small as possible.

1.16 Waveform Distortion

When the received signal is not the exact replica of the transmitted signal then the signal is said to be distorted. There exists some kind of distortion in the signal. There are three types of distortions present in the transmitted wave along the transmission line.

1. Due to variation of characteristic impedance Z_0 with frequency.
2. Frequency distortion due to the variation of attenuation constant α with frequency.
3. Phase distortion due to the variation of phase constant β with frequency.

1.16.1 Distortion due to Z_0 Varying with Frequency

The characteristic impedance Z_0 of the line varies with the frequency while the line is terminated in an impedance which does not vary with frequency in similar fashion as that of Z_0. This causes the distortion. The power is absorbed at certain frequencies while it gets reflected for certain frequencies. So there exists the **selective power absorption**, due to this type of distortion.

It is known that,

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{R}{G}\frac{\left(1+j\omega\frac{L}{R}\right)}{\left(1+j\omega\frac{C}{G}\right)}}$$

If for the line, the condition LG = CR is satisfied then $\frac{L}{R}=\frac{C}{G}$ and hence

$$\left(1+j\omega\frac{L}{R}\right) = \left(1+j\omega\frac{C}{G}\right)$$

$$Z_0 = \sqrt{\frac{R}{G}}\angle 0° = \sqrt{\frac{L}{C}}\angle 0°\ \Omega$$

For such a line Z_0 does not vary with frequency ω and it is **purely resistive** in nature.

Such a line can be easily and correctly terminated in an impedance which matches with Z_0 at all the frequencies. For such a line, $Z_R = \sqrt{\frac{R}{G}}$ or $\sqrt{\frac{L}{C}}$. This eliminates the distortion and hence selective power absorption.

1.16.2 Frequency Distortion

It is known that,

$$\alpha = \sqrt{\frac{1}{2}\left[\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}+RG-\omega^2LC\right]}$$

The attenuation constant α is a function of frequency. Hence the different frequencies transmitted along the line will be attenuated to the different extent. For example a voice signal consists of many frequencies. And all these frequencies will not be attenuated equally along the transmission line. Hence received signal will not be exact replica of the input signal at the sending end. Such a distortion is called a **frequency distortion.** Such a distortion is very serious and important for audio signals but not much important for video signals. Thus in high frequency radio broadcasting such frequency distortion is elliminated by use of **equalizers.** The frequency and phase characteristics of such equalizers are inverse to those of the line. Thus nullifying the distortion, making the overall frequency response, uniform in nature.

1.16.3 Phase Distortion

It is known that for a line,

$$\beta = \sqrt{\frac{1}{2}\left[\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}-[RG-\omega^2LC]\right]}$$

The phase constant β also varies with frequency. Now the velocity v is given by,

$$v = \frac{\omega}{\beta}$$

Thus the velocity of propagation of waves also varies with frequency. Hence some waves will reach receiving end very fast while some waves will get delayed than the others. Hence all frequencies will not have same transmission time. Thus the output wave at the receiving end will not be exact replica of the input wave at the sending end. This type of distortion is called **phase distortion** or **delay distortion.** It is not much important for the audio signals due to the characteristics of the human ears. But such a distortion is very serious in case of video and picture transmission. The remedy for this is to use co-axial cables for the picture transmission of television and video signals.

A line in which the distortions are elliminated by satisfying certain conditions is called distortionless line. Let us derive the condition for a distortionless line.

1.17 Distortionless Line

A line in which there is no phase or frequency distortion and also it is correctly terminated, is called a **distortionless line.**

To derive the condition for distortionless line consider,

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\therefore \quad \gamma^2 = (R+j\omega L)(G+j\omega C)$$

$$\therefore \quad \gamma^2 = \left(RG-\omega^2 LC\right)+j\omega C(RC+LG) \qquad \text{... (1)}$$

It is known that for minimum attenuation $L=\frac{CR}{G}$ i.e. LG = CR. Substituting this condition in equation (1) we get,

$$\gamma^2 = RG-\omega^2 LC+j\,2\omega RC$$

$$\text{But} \quad RC = LG=\sqrt{RCLG}$$

$$\therefore \quad \gamma^2 = RG-\omega^2 LC+j2\omega\sqrt{RCLG}$$

$$\therefore \quad \gamma^2 = \left(\sqrt{RG}+j\omega\sqrt{LC}\right)^2$$

$$\therefore \quad \gamma = \sqrt{RG}+j\omega\sqrt{LC} \qquad \text{... (2)}$$

$$\text{But} \quad \gamma = \alpha+j\beta$$

$$\therefore \quad \boxed{\alpha = \sqrt{RG}} \qquad \text{... (3)}$$

$$\text{and} \quad \boxed{\beta = \omega\sqrt{LC}} \qquad \text{... (4)}$$

It can be seen from the equation (3) that α does not vary with frequency which elliminates the frequency distortion.

$$\text{Now} \quad \beta = \omega\sqrt{LC} \qquad \text{... for the condition LG = CR}$$

$$\text{Now} \quad v = \frac{\omega}{\beta}=\frac{\omega}{\omega\sqrt{LC}}=\frac{1}{\sqrt{LC}} \text{ km/sec} \qquad \text{... (5)}$$

Thus for the condition LG = CR, the velocity becomes independent of frequency. This eliminates the phase distortion.

It is already proved that for RC = LG, the Z_0 becomes resistive and line can be correctly terminated to eliminate distortion due to Z_0 varying with frequency. Thus all the distortions are eliminated for a condition,

$$\mathbf{RC = LG} \quad \text{i.e.} \quad \frac{R}{G}=\frac{L}{C} \qquad \text{... (6)}$$

This is the required condition for a **distortionless line**. For such a line, received signal is exact replica of the signal at the sending end, though it is delayed by constant propagation time and its amplitude reduces.

Another important observation is that the condition for a distortionless line is identical to the condition for a minimum attenuation with L or C varied.

Thus if the primary line constants do not mutually naturally satisfy the condition of equation (6), then this condition will have to be satisfied artificially by increasing L or decreasing C. When this is done artificially, the line is said to be **loaded line** and the process of artificially achieving the condition is called **loading of a line.**

However if the frequency of operation is very high, then though the condition of distortionless line is not satisfied then $j\omega L >> R$ and $j\omega C >> G$ due to high ω. Hence R and G can be neglected. Thus Z_0 becomes equal to $\sqrt{L/C}$ which is automatically real and resitive and line can be perfectly terminated. Hence for very high frequency operation like radio frequencies though distortionless condition is not satisfied, the line is automatically distortionless and hence loading is not essential. Practically for the line, R/G is higher than L/C and usually L is increased artificially to match the distortionless condition.

1.18 Telephone Cable

The ordinary telephone cable is an underground cable which consists of wires insulated with paper and twisted in pair. For the audio frequency range, the inductance L and the conductance G of such a cable is negligibly small and hence can be neglected. Hence impedance and admittance of such a cable becomes,

$$Z = R \quad \text{... (1)}$$

and $$Y = j\omega C \quad \text{... (2)}$$

Now $$\gamma = \sqrt{YZ} = \sqrt{j\omega RC}$$

$\therefore$ $$\gamma = \sqrt{\frac{j2\omega RC}{2}} \quad \text{... (3)}$$

Now $$\sqrt{j} = \sqrt{1\angle 90^\circ} = 1\angle 45^\circ$$

$\therefore$ $$\sqrt{2}\sqrt{j} = \sqrt{2}\angle 45^\circ = (1 + j1)$$

$\therefore$ $$\sqrt{2j} = 1 + j1 \quad \text{... (4)}$$

$\therefore$ $$\gamma = (1 + j1)\sqrt{\frac{\omega RC}{2}}$$

$\therefore$ $$\gamma = \sqrt{\frac{\omega RC}{2}} + j\sqrt{\frac{\omega RC}{2}} = \alpha + j\beta$$

$$\therefore \quad \boxed{\alpha = \sqrt{\frac{\omega RC}{2}}} \quad \text{and} \quad \boxed{\beta = \sqrt{\frac{\omega RC}{2}}} \qquad \ldots (5)$$

$$\therefore \quad \boxed{v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega RC}{2}}} = \sqrt{\frac{2\omega}{RC}}} \qquad \ldots (6)$$

Both α and velocity v are the functions of frequency ω. Hence for high frequencies, there is large attenuation. And the velocity v is also high at high frequencies. Hence waves travel very fast than the lower frequencies, when frequency is high. Thus in the telephone cable both phase and frequency distortions are dominant.

1.19 Loading of Lines

It is seen earlier that if the primary constants of a line, mutually satisfy the relationship RC = LG then the distortionless transmission results.

For a practical line, R/G is always more than L/C and hence the signal is distorted. Thus the preventive remedy is to make the condition $\frac{R}{G} = \frac{L}{C}$ satisfy artificially.

To satisfy the condition, it is necessary to reduce R/G or increase L/C. Let us consider all the possibilities. To reduce R/G, it is necessary to decrease R or increase G. The resistance R can be decreased by increasing the area of cross-section i.e. diameter of the conductors. This increases the size and cost of the line. Hence this possibility is uneconomical.

To increase G, it is necessary to use poor insulators. To get poor insulator is easy and economical but from the receiving end point of view, increase in G is very much uneconomical. When G is increased, the leakage of the signal will increase, though it becomes distortionless. So quality improves but quantity decreases. Thus increase in G is quality at the cost of quantity. The signal at receiving end must activate the receivers. But if leakage is more, then received signal becomes so weak that amplifiers are required at the intermediate stages. This makes the design complicated. Hence advice of increasing G to reduce ratio R/G is 'penny wise pound foolish' advice. It is the worst advice and hence this possibility is ruled out in practice.

Now to increase L/C, it is necessary to increase L or decrease C. If C is to be reduced, then the seperation between the lines will be more. Thus the brackets which were carrying previously more wires will now carry very less number of wires due to increased separation. Hence more number of brackets are required. Taller towers and posts are required and also number of towers and posts per unit length of the line

will be increased. Thus for the same strength of the line, the line will become very much costlier due to decrease in C. Hence this possibility is also ruled out.

Thus the only alternative left is to increase L. This is opted in practice. The process of increasing the inductance L of a line artificially is called **loading of a line.** And such a line is called **loaded line.**

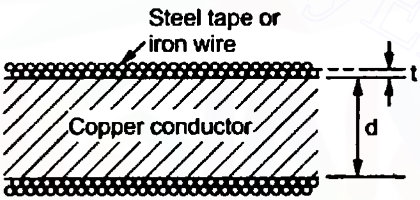
There are two methods of loading a line which are,

1. **Continuous loading** which is also called **Krarup loading** or **Heavyside loading.**
2. **Lump loading** which is also called **Pupin loading** or **Coil loading.**

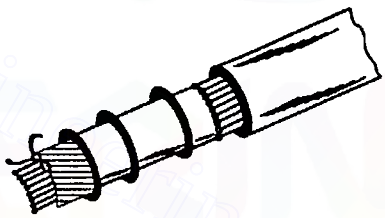
Let us discuss in detail, these two types of loading methods.

1.20 Continuous Loading

In this method of loading, to increase the inductance, on each conductor the tapes of magnetic material having high permeability such as permalloy or μ-metal are wound. This is shown in the Fig. 1.23 (a) while the loaded cable is shown in the Fig. 1.23 (b).



(a) Continuous loading



(b) Continuously loaded cable

Fig. 1.23

The increase in the inductance for a continuously loaded line is,

$$L \approx \frac{\mu}{\frac{d}{nt}+1} \text{ mH}$$

where

μ = Permeability of surrounding material

d = Diameter of copper conductor

t = Thickness per layer of tape or iron wire

n = Number of layers

1.20.1 Propagation Constant of Continuously Loaded Cable

For the continuously loaded cable it can be assumed that its G = 0 and L is increased such that $\omega L >> R$.

Now $$Z = R + j\omega L \quad \text{and} \quad Y = j\omega C \quad \text{as } G = 0$$

Thus $$\gamma = \sqrt{YZ} = \sqrt{(R+j\omega L)(j\omega C)}$$

$$= \sqrt{\sqrt{R^2 + \omega^2 L^2}\ \angle \frac{\pi}{2} - \tan^{-1}\frac{R}{\omega L}\ \ \omega C \angle \frac{\pi}{2}}$$

The angle of Z which is $\tan^{-1}\frac{\omega L}{R}$ is written as $\frac{\pi}{2} - \tan^{-1}\frac{R}{\omega L}$

$$\gamma = \sqrt{\omega L \sqrt{1 + \frac{R^2}{\omega^2 L^2}}\ \omega C \angle \pi - \tan^{-1}\frac{R}{\omega L}}$$

Neglecting $R^2/\omega^2 L^2$ as $\omega L >> R$,

$$\gamma = \omega\sqrt{LC}\ \angle \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}$$

Angle becomes half out of the square root sign.

Let $$\theta = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}$$

$\therefore$ $$\cos\theta = \cos\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right) = \sin\left(\frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right)$$

But for a small angle,

$\therefore$ $$\sin\theta = \tan\theta = \theta \quad \text{i.e.} \quad \tan^{-1}\theta = \theta$$

$\therefore$ $$\tan^{-1}\frac{R}{\omega L} = \frac{R}{\omega L} \quad \text{and} \quad \sin\left(\frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right) = \frac{R}{2\omega L}$$

$\therefore$ $$\cos\theta = \frac{R}{2\omega L}$$

and $$\sin\theta = \sin\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right) \approx \sin\frac{\pi}{2} = 1 \qquad \text{... as } \omega L >> R$$

$$\therefore \qquad \gamma = \omega\sqrt{LC}\ \angle\theta = \omega\sqrt{LC}\,[\cos\theta + j\sin\theta]$$

As $\cos\theta + j\sin\theta = \sqrt{\cos^2\theta + \sin^2\theta}\ \angle\tan^{-1}[\tan\theta] = 1\angle\theta$

$$\therefore \qquad \gamma = \omega\sqrt{LC}\cos\theta + j\,\omega\sqrt{LC}\sin\theta$$

$$\therefore \qquad \gamma = \omega\sqrt{LC}\,\frac{R}{2\omega L} + j\,\omega\sqrt{LC} = \alpha + j\beta$$

$$\therefore \qquad \boxed{\alpha = \frac{R}{2}\sqrt{\frac{C}{L}}} \quad \text{and} \quad \boxed{\beta = \omega\sqrt{LC}}$$

and
$$\boxed{v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}}$$

Thus the attenuation factor α is not the function of frequency while velocity v is also independent of frequency. And hence continuously loaded cable is distortionless.

1.20.2 Advantages

The advantages of the continuous loading are,

1. The attenuation to the signal is independent of the frequency and it is same to all the frequencies.
2. The attenuation can be reduced by increasing L, provided that R is not increased greatly.
3. The increase in the inductance upto 100 mH per unit length of line is possible.

1.20.3 Disadvantages

The disadvantages of the continuous loading are,

1. The method is very costly.
2. Existing lines can not be modified by this method. Hence total replacement of the existing cables by the new cables wound with magnetic tapes is required. This is again costly and uneconomical.
3. Extreme precision care must be taken while manufacturing continuously loaded cable, otherwise it becomes irregular.
4. The size is increased. Thus the capacitance increases. Hence the benefit obtained by increase in L is partly nullified.
5. All along the conductor, there will be huge mass of iron. Thus for a.c. signals there will be large eddy current and hysteresis losses. The eddy current losses increase directly with square of frequency while the hysteresis losses increase directly with the frequency. Hence overall this puts the upper limit to increase inductance.

6. The maximum value by which inductance can be increased is fixed as 100 mH per unit length of line.

Thus this method of loading is not used for the landlines but are preferred for the submarine cables. For underwater circuits lumped loading is difficult to use. It is not necessary to load the submarine cable continuously while the sections of loaded cable separated by the sections of unloaded cable can be used. This reduces the cost, still enjoying the advantages of continuous loading. This is called **patch loading.**

1.21 Lumped Loading

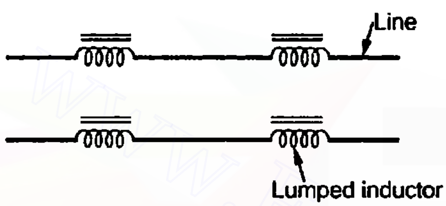


Fig. 1.24 Lumped loading

In this type of loading, the inductors are introduced in lumps at the uniform distances, in the line. Such inductors are called **lumped inductors.** The inductors are introduced in both the limbs to keep the line as balanced circuit. The lumped inductors are in the form of coils called loading coils. The method is shown in the Fig. 1.24.

The lumped loading is preferred for the open wire lines and cables for the transmission improvement. The loading coil design is very much important in this method. The core of the coils is usually toroidal in shape and made of permalloy. This type of core produces the coil of high inductance, having small dimensions, very low eddy current losses and negligible external field which restricts the interference with neighbouring circuits.

The loading coil is wound of the largest guage of wire consistent with small size. Each winding is divided into equal parts, so that exactly half the inductance can be inserted into each leg of the circuit. These are built into steel pots which are made in several standard sizes to accomodate one or more coils. The pots protect the coils from external magnetic fields, weather and mechanical damage. The Fig. 1.25 shows the construction of loading coils. While installing the coils, the care must be taken so that the circuit balance is maintained. No winding is reversed. If winding is reversed, it will neutralize the inductance of other winding reducing the overall inductance.

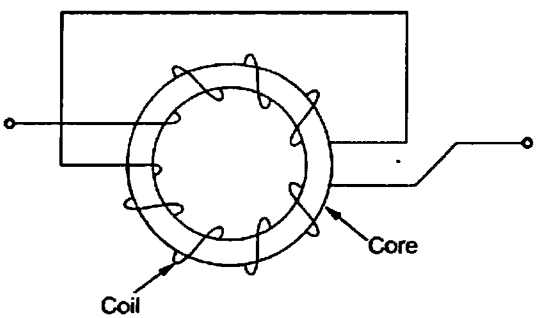


Fig. 1.25 Winding of single loading coil

In case of lumped loading the line behaves properly provided spacing is uniform and loading is balanced, upto a certain frequency called **cut-off frequency** of the line. Upto this frequency, the added inductance behaves as if it is distributed uniformly along the line. But above this cut-off frequency the attenuation constant increases rapidly. The line acts as low pass filter. The graph of α against the frequency called the **attenuation frequency characteristics** of the line is shown in the Fig. 1.26. It can be seen that for continuous loading, the attenuation is independent of frequency while for lumped loading it increases rapidly after the cut-off frequency.

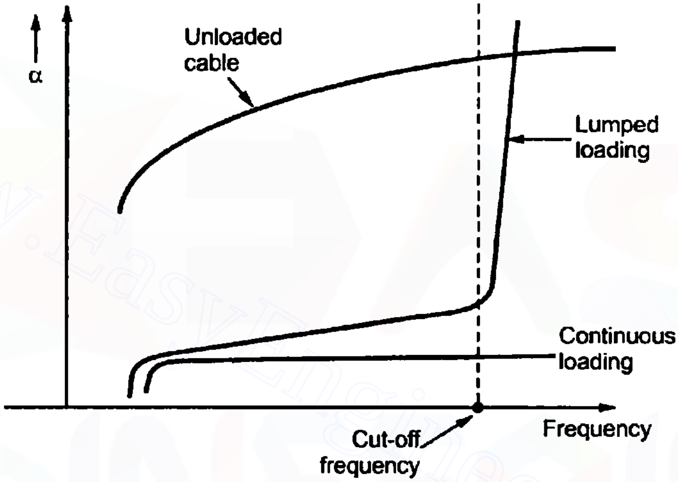


Fig. 1.26 Attenuation frequency characteristics

If the loading section distance is d then keeping inductance L_S of the loading coil constant, the cut-off frequency is found to be proportional to the $1 / \sqrt{d}$. Hence to get the higher cut-off frequency, small lumped inductances must be used at smaller distances.

1.21.1 Campbell's Equation

This equation gives the analysis of the performance of a loaded line.

Let Z_0 = Characteristic impedance of the line before loading

γ = Propagation constant of the line before loading

Z_L = Impedance of the loading coil

N = Coil spacing i.e. distance in km between loading coils

γ' = Propagation constant of the line after loading

The analysis can be done by considering a symmetrical section from one loading to the next loading coil. The T equivalent section of the line is used for the analysis.

The Fig. 1.27 (a) shows the unloaded line and corresponding T equivalent while the Fig. 1.27 (b) shows the loaded line and corresponding T equivalent section.

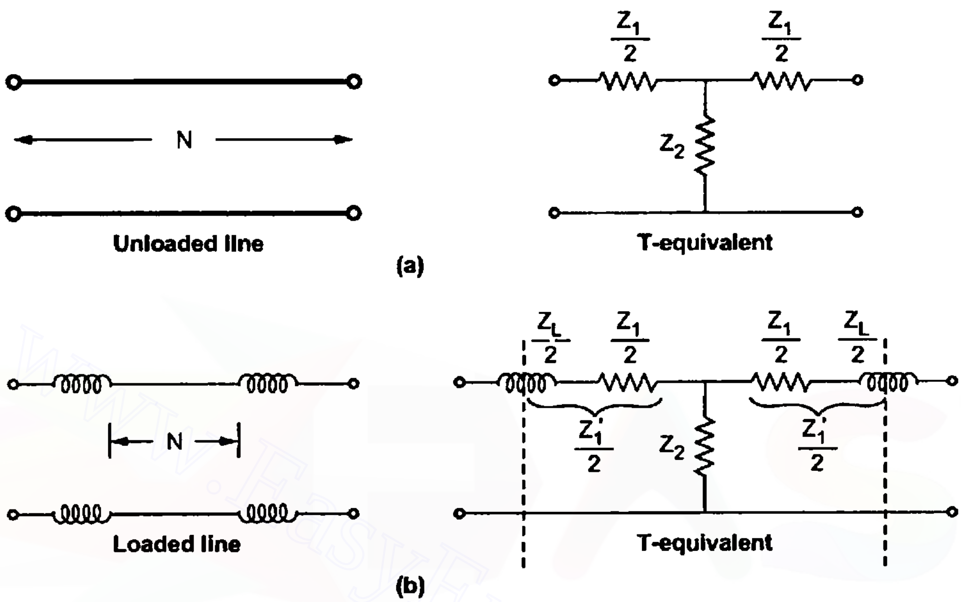


Fig. 1.27

Consider equivalent T section before loading the line. In the section 1.8 it is derived that, for a line of length N,

$$\sinh(N\gamma) = \frac{Z_0}{Z_2} \qquad \text{... (1)}$$

and
$$\cosh(N\gamma) = 1+\frac{Z_1}{2Z_2} \qquad \text{... (2)}$$

When the loading section is added, the equivalent series arm of the loaded section becomes,

$$\frac{Z_1'}{2} = \frac{Z_L}{2}+\frac{Z_1}{2} \qquad \text{... (3)}$$

The shunt arm of the equivalent T section remains unchanged as $Z_2 = Z_0 / \sinh(N\gamma)$.

Now from (2),

$$\frac{Z_1}{2} = Z_2\left[\cosh(N\gamma)-1\right] = \frac{Z_0}{\sinh(N\gamma)}\left[\cosh(N\gamma)-1\right] \qquad \text{... (4)}$$

Using (4) in (3),

$$\frac{Z_1'}{2} = \frac{Z_L}{2} + \frac{Z_0}{\sinh(N\gamma)}[\cosh(N\gamma) - 1] \quad \text{... (5)}$$

Now γ' is the new propagation constant after loading,

$$\therefore \quad \cosh(N\gamma') = 1 + \frac{Z_1'}{2Z_2} = 1 + \frac{(Z_1'/2)}{Z_2}$$

$$= 1 + \frac{\left\{\frac{Z_L}{2} + \frac{Z_0}{\sinh(N\gamma)}[\cosh(N\gamma) - 1]\right\}}{\left(\frac{Z_0}{\sinh(N\gamma)}\right)}$$

$$= 1 + \frac{\sinh(N\gamma)}{2Z_0}\left\{Z_L + \frac{2Z_0}{\sinh(N\gamma)}[\cosh(N\gamma) - 1]\right\}$$

$$= 1 + \frac{Z_L \sinh(N\gamma)}{2Z_0} + \cosh(N\gamma) - 1$$

$$\therefore \quad \cosh(N\gamma') = \cosh(N\gamma) + \frac{Z_L}{2Z_0}\sinh(N\gamma) \quad \text{... (6)}$$

This expression is known as the Campbell's equation for a loaded line. It gives the expression for the propagation constant of loaded line, interms of the propagation constant of a loaded line.

For a cable, Z_2 i.e. shunt arm is essentially capacitive. The cable capacitance and lumped inductance appear similar to the circuit of low pass filter. The cut-off frequency of the low pass filter is given by,

$$f_c = \frac{1}{\pi\sqrt{LC}} \quad \text{... (7)}$$

where L is the total inductance per unit length which is addition of inductance per unit length of line and that of the loading coil added to the line.

The attenuation reduces below f_c, due to the loading effect. But due to the filter action, it increases rapidly after f_c is crossed. Thus f_c puts an upper limit for the successful transmission over the lines.

In practice R and L are to some extent are the functions of frequency hence truely distortionless line is not possible. But loaded cables perform much better than the unloaded cables.

1.21.2 Advantages

The advantages of lumped loading are,

1. There is no practical limit to the value by which the inductance can be increased.
2. The cost involved is small.
3. With this method, the existing lines can be tackled and modified.
4. Hysteresis and eddy current losses are small.

1.21.3 Disadvantages

The only disadvantage of this method is its action like low pass filter. The attenuation increases considerably after the cut-off frequency. The cut-off frequency must be at the top of voice frequency. Hence fractional loading is used. Whatever distortion results due to fractional loading is corrected using equalizers. The care must be taken while installing the lumped inductors so as to maintain the exact balancing of the circuit.

1.21.4 Practical Formulae for Z_0 and γ for Loaded Underground Cable

For the loaded underground cable $\omega L >> R$ and $\omega C >> G$.

1. Characteristic impedance Z_0 :

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{\omega L\left(j+\frac{R}{\omega L}\right)}{\omega C\left(j+\frac{G}{\omega C}\right)}}$$

$$= \sqrt{\frac{Lj}{Cj}} = \sqrt{\frac{L\angle 90^\circ}{C\angle 90^\circ}} \quad \text{... neglecting } \frac{R}{\omega L} \text{ and } \frac{G}{\omega C}$$

$$\therefore \quad Z_0 = \sqrt{\frac{L}{C}}\angle 0^\circ \quad \text{... (1)}$$

This condition is permissible for heavy loading and for higher audio frequencies.

2. Propagation constant γ :

$$\gamma = \sqrt{ZY} = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{j\omega L\left(1+\frac{R}{j\omega L}\right)j\omega C\left(1+\frac{G}{j\omega C}\right)}$$

$$= j\omega\sqrt{LC}\sqrt{\left(1+\frac{R}{j\omega L}\right)\left(1+\frac{G}{j\omega C}\right)}$$

Expanding square root by Binomial theorem,

$$\gamma = j\omega\sqrt{LC}\left[1+\frac{R}{2j\omega L}+\ldots\right]\left[1+\frac{G}{2j\omega C}+\ldots\right]$$

The remaining terms can be neglected as,

$$\left|\frac{R}{j\omega L}\right| = \frac{R}{\omega L} << 1 \quad \text{and} \quad \left|\frac{G}{j\omega C}\right| = \frac{G}{\omega C} << 1$$

$$\therefore \quad \gamma = j\,\omega\sqrt{LC}\left[1+\frac{R}{2j\omega L}+\frac{G}{2j\omega C}\right]$$

$$\therefore \quad \gamma = \frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}+j\omega\sqrt{LC} = \alpha + j\beta$$

$$\therefore \quad \alpha = \frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}} \quad \text{nepers/km} \qquad \ldots (2)$$

and $\beta = \omega\sqrt{LC}$ rad/km ... (3)

Thus α is independent of frequency.

For loaded cables generally G is very small and can be neglected hence,

$$\alpha \approx \frac{R}{2}\sqrt{\frac{C}{L}} \approx \frac{R}{2\,|Z_0|} \qquad \ldots (4)$$

While

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \qquad \ldots (5)$$

Thus the velocity is also independent of frequency. In practice though the loading is only a small fraction of the value required to make RC = LG, the line behaves approximately as distortionless line.

Example 1.9 *A particular underground cable has following primary constants :*

$R = 44\ \Omega$ / km, $G = 1$ μmho / km, $L = 0.001$ H / km, $C = 0.065$ μF / km

It is loaded with 88 mH loading coils of resistance 3.7 Ω with 1.5 km spacing. Find the approximate values of Z_0, α and β for the cable at 1600 Hz. Also find the cut-off frequency.

Solution : Let us find the values of constants including loading coils.

Inductance per unit length of coil added $= \frac{88}{1.5} = 58.667$ mH/km

$\therefore$ L′ = total inductance/km $= 0.001 + 58.667 \times 10^{-3} = 0.05967$ H/km

Resistance per unit length of coil added $= \frac{3.7}{1.5} = 2.4667 \ \Omega/\text{km}$

$\therefore$ R' = total resistance/km = 44 + 2.4667 = 46.4667 Ω/km

The approximate values for the cable are,

$$Z_0 = \sqrt{\frac{L'}{C}} = \sqrt{\frac{0.05967}{0.065 \times 10^{-6}}} = 958.123\,\Omega$$

$$\alpha = \frac{R'}{2}\sqrt{\frac{C}{L'}} + \frac{G}{2}\sqrt{\frac{L'}{C}}$$

$$= \frac{46.4667}{2}\sqrt{\frac{0.065 \times 10^{-6}}{0.05967}}$$

$$+ \frac{1 \times 10^{-6}}{2}\sqrt{\frac{0.05967}{0.065 \times 10^{-6}}} = 0.0247 \ \text{nepers/km}$$

$$\beta = \omega\sqrt{L'C} = 1600 \times 2\pi\sqrt{0.05967 \times 0.065 \times 10^{-6}}$$

$$= 0.626 \ \text{rad/km}$$

The cut-off frequency is given by,

$$f_c = \frac{1}{\pi\sqrt{L'C}} = \frac{1}{\pi\sqrt{0.05967 \times 0.065 \times 10^{-6}}}$$

$$= \mathbf{5.1111 \ kHz}$$

➡ **Example 1.10 :** *For a cable it is decided to provide lumped loading. The primary constants of the cable are,*

R = 40 Ω / km, L = 1 mH / km, G = 1 μmho / km, C = 0.05 μF / km

Find the new value of inductance required to achieve the distortionless condition. By what factor, the inductance is required to be raised ?

Solution : The distortionless condition is,

$$RC = L'G$$

$$L' = \frac{RC}{G} = \frac{40 \times 0.05 \times 10^{-6}}{1 \times 10^{-6}}$$

$$= 2 \ \text{H/km}$$

The factor by which inductance to be increased is,

$$\text{factor} = \frac{L'}{L} = \frac{2}{1 \times 10^{-3}} = \mathbf{2000}$$

1.22 Reflection on a Line not Terminated in Z_0

Uptill now the current and voltage relationships are derived for the lines which are terminated in Z_0. But if a line is not terminated in Z_0 or it is joined to some impedance having value other than Z_0 then part of the wave is reflected back from the distant end or from the point of discontinuity. Thus reflection phenomenon exists for a line which is not terminated in Z_0. Such a reflection is maximum when the line is on open circuit i.e. $Z_R = \infty$ or short circuit i.e. $Z_R = 0$. The reflection is zero when $Z_R = Z_0$.

From the general solution of a line we can write,

$$E = \frac{E_R(Z_R + Z_0)}{2\,Z_R}\left[e^{\sqrt{ZY}\,s} + \frac{Z_R - Z_0}{Z_R + Z_0}\,e^{-\sqrt{ZY}\,s}\right] \quad \text{... (1)}$$

and
$$I = \frac{I_R(Z_R + Z_0)}{2\,Z_0}\left[e^{\sqrt{ZY}\,s} - \frac{Z_R - Z_0}{Z_R + Z_0}\,e^{-\sqrt{ZY}\,s}\right] \quad \text{... (2)}$$

The s is the distance measured from the receiving end and treated positive. E_R and I_R are the voltage and current at the receiving end. Z_R is the impedance at the receiving end in which the line is terminated.

The value of Z_R is not equal to Z_0 of the line.

Now $$\sqrt{ZY} = \gamma$$

And when Z_R is not equal to Z_0, each E and I consists of 2 parts,

1. One part varying exponentially with positive s
2. One part varying exponentially with negative s

$$E = \frac{E_R(Z_R + Z_0)}{2\,Z_R}\,e^{\gamma s} + \frac{E_R(Z_R + Z_0)}{2\,Z_R}\,\frac{(Z_R - Z_0)}{(Z_R + Z_0)}\,e^{-\gamma s}$$

$$E = \frac{E_R(Z_R + Z_0)}{2\,Z_R}\,e^{\gamma s} + \frac{E_R(Z_R - Z_0)}{2\,Z_R}\,e^{-\gamma s} \quad \text{... (3)}$$

and
$$I = \frac{I_R(Z_R + Z_0)}{2\,Z_0}\,e^{\gamma s} - \frac{I_R(Z_R - Z_0)}{2\,Z_0}\,e^{-\gamma s} \quad \text{... (4)}$$

The first component of E or I which varies exponentially with +s is called incident wave which flows **from the sending end to the receiving end.**

$$E_1 = \frac{E_R(Z_R + Z_0)}{2\,Z_R}\,e^{\gamma s} = \text{incident voltage wave}$$

and
$$I_1 = +\frac{I_R(Z_R + Z_0)}{2\,Z_0}\,e^{\gamma s} = \text{Incident current wave}$$

It can be noted that s is measured from receiving end. Thus s is minimum $(s = 0)$ at the receiving end and maximum $(s = l)$ at the sending end. Thus as incident wave travels from the sending end to the receiving end, its amplitude decreases. Such a wave which flows from sending end to the receiving end, with decreasing amplitude is the **incident wave.**

The second component of E or I varies exponentially with – s. It flows from the receiving end towards the sending end. When s is minimum (s = 0) at the receiving end, its amplitude is maximum while due to negative index, when s is maximum $(s = l)$ at the sending end, its amplitude is minimum. Thus such a wave which flows from the receiving end towards the sending end, with decreasing amplitude is called **reflected wave.**

$$\therefore \quad E_2 = \frac{E_R(Z_R - Z_0)}{2Z_R}\, e^{-\gamma s} = \text{Reflected voltage wave}$$

and

$$I_2 = -\frac{I_R(Z_R - Z_0)}{2Z_0}\, e^{-\gamma s} = \text{Reflected current wave}$$

Thus the total instantaneous voltage or current at any point on the line is the phasor sum of voltage or current of the incident and reflected waves.

Consider open circuit line with $Z_R = \infty$. Then the incident component of voltage becomes,

$$E_1 = \frac{E_R\left(1 + \frac{Z_0}{Z_R}\right)}{2}\, e^{\gamma s} = \frac{E_R}{2}\, e^{\gamma s}$$

This is the wave which progresses from sending end to the receiving end with decreasing amplitude.

The reflected component of voltage becomes,

$$E_2 = \frac{E_R\left(1 + \frac{Z_0}{Z_R}\right)}{2}\,\frac{\left(1 - \frac{Z_0}{Z_R}\right)}{\left(1 + \frac{Z_0}{Z_R}\right)}\, e^{-\gamma s} = \frac{E_R}{2}\, e^{-\gamma s}$$

The amplitude of this wave varies with $e^{-\gamma s}$ and hence its amplitude decreases as it travels from receiving end towards the sending end. Its initial value is equal to the incident voltage at the load on open circuit.

At s = 0 i.e. at the receiving end, on open circuit $E_1 = \frac{E_R}{2}$ while $E_2 = \frac{E_R}{2}$. Thus initial value of the reflected wave is equal to incident voltage at the load on open

circuit. This is shown in the Fig. 1.28. The solid curve is for incident wave while the dotted curve is for reflected wave.

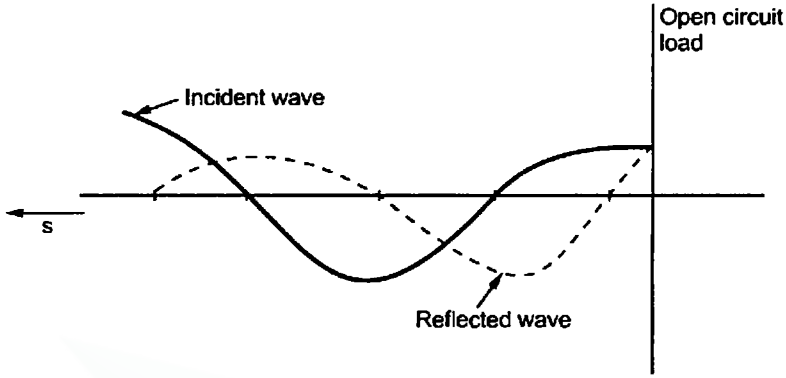


Fig. 1.28 Instantaneous voltage waves for open circuit line

It can be seen from the equations (3) and (4) that the only difference between the curves for voltage and current for open circuit line is the reversed phase of the reflected current wave. The waves of instantaneous current are shown in the Fig. 1.29.

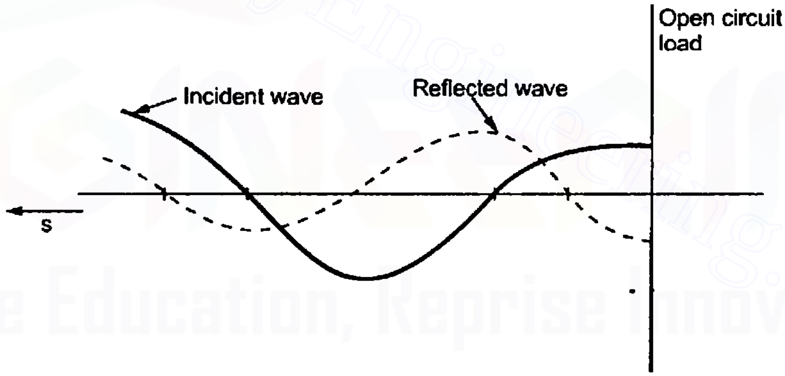


Fig. 1.29 Instantaneous current waves for open circuit line

The two current waves are equal and of opposite phases at the open circuited receiving end. Thus addition of instantaneous currents, at the open circuited receiving end is always zero as required by open circuit condition of the line.

The term $\frac{Z_R - Z_0}{Z_R + Z_0}$ decides the relative phase angles between incident and reflected waves. Thus magnitudes and phase angles of Z_R and Z_0 are important to determine the phase angle between the two waves.

If $Z_R = Z_0$ it can be seen that the reflected wave is absent and there is no reflection phenomenon at all. Such a line is uniform line and there is no discontinuity existing to send the reflected wave back along the line. The infinite line with $s = \infty$ also behaves in the similar fashion. The waves travel smoothly along the line and the energy is absorbed in the load Z_0 without any reflected wave. Such a finite line terminated in Z_0, without having any reflection is called a **smooth line.**

1.22.1 Reflection Phenomenon

The quantity which is actually transmitted along the line is not the current or voltage but the energy. Such energy is transmitted through electric and magnetic fields set up along the line.

The energy conveyed in the electric field depends on the voltage E and given by,

$$W_e = \frac{1}{2} C E^2 \ J/m^3 \quad \dots (5)$$

The energy conveyed in the magnetic field depends on the current I and given by,

$$W_m = \frac{1}{2} L I^2 \ J/m^3 \quad \dots (6)$$

For an ideal line which is terminated in Z_0, the ratio of E and I is fixed along the line which is Z_0.

$$\therefore \quad Z_0 = \frac{E}{I} \quad \dots \text{for line terminated in } Z_0$$

For such a line, R = G = 0 and the value of Z_0 is given by,

$$Z_0 = \sqrt{\frac{L}{C}} \quad \dots \text{for line terminated in } Z_0$$

Now $\quad E = I Z_0$

and $\quad Z_0^2 = \frac{L}{C}$ hence $C = \frac{L}{Z_0^2}$

$$\therefore \quad W_e = \frac{C E^2}{2} = \frac{1}{2} \times \frac{L}{Z_0^2} \times (I Z_0)^2$$

$$= \frac{1}{2} L I^2 = W_m$$

$$W_e = W_m \quad \dots \text{for ideal line}$$

Thus for an ideal line terminated in Z_0, at all the points along the line, electric field energy is equal to the magnetic field energy.

When line is not terminated in Z_0 then

$$Z_R = \frac{E_R}{I_R}$$

This ratio is practically required for which the redistribution of energy between electric and magnetic field takes place. This redistribution of energy creates the reflected wave, back along the line.

When the line is open circuit then $I_R = 0$ thus the magnetic field energy must become zero. The magnetic field energy carried can not be dissipated but it gets added to the electric field energy causing increased voltage to appear. This increased voltage is the cause for the reflected current wave, back on the line.

When the line is short circuited then $E_R = 0$ thus the electric field energy must become zero. Again the electric field energy carried, can not be dissipated but gets added to the magnetic field energy. This increased energy in the magnetic field set up a reflected voltage wave down the line.

1.22.2 Disadvantages of Reflection

The reflection is undesirable because of following disadvantages,

1. If the attenuation is not large then the reflected wave appears as echo at the sending end.
2. There is reduction in efficiency.
3. The part of the received energy is rejected by the load hence output reduces.
4. If the generator impedance at the sending end is not Z_0 then reflected wave is reflected again from the sending end and becomes a new incident wave. The energy is transmitted back and forth till all the energy get dissipated in the line losses.

Hence it is necessary that the line must be terminated in Z_0 to avoid the reflection.

1.22.3 Reflection Coefficient

The ratio of the amplitudes of the reflected and incident voltage waves at the receiving end of the line is called the **reflection coefficient.** It is denoted by K

$$K = \frac{\text{reflected voltage at load}}{\text{incident voltage at load}}$$

The reflected voltage at load is component E_2 at the receiving end i.e. with $s = 0$.

$$\therefore \quad E_2\big|_{s=0} = \frac{E_R\,(Z_R - Z_0)}{2\,Z_R} \qquad \text{... (7)}$$

The incident voltage at load is component E_1 at the receiving end i.e. with s = 0.

$$\therefore \quad E_1\big|_{s=0} = \frac{E_R(Z_R + Z_0)}{2 Z_R} \quad \text{... (8)}$$

$$\therefore \quad K = \frac{E_R(Z_R - Z_0)/2Z_R}{E_R(Z_R + Z_0)/2Z_R}$$

$$\therefore \quad K = \frac{Z_R - Z_0}{Z_R + Z_0} \quad \text{... (9)}$$

It is a measure of the mismatch between the load impedance Z_R and the characteristic impedance Z_0 of the line.

The following observations can be made with respect to the reflection coefficient K,

1. When $Z_R = Z_0$, K = 0 and there is no reflection.

2. When $Z_R = 0$ i.e. when the line is short circuited,

$$K = \frac{0 - Z_0}{0 + Z_0} = -1 = 1\angle{+180°}$$

Reflection is maximum.

3. When $Z_R = \infty$ i.e. when the line is open circuited,

$$K = \frac{1 - \frac{Z_0}{Z_R}}{1 + \frac{Z_0}{Z_R}} = +1 = 1\angle 0°$$

Reflection is maximum.

4. K ranges in magnitude from 0 to 1 and its phase angle ranges from 0° to 180°.

5. The sign of K and hence the polarity of the reflected wave is dependent on the angles and magnitudes of Z_0 and Z_R.

The reflection coefficient is very important while studying the Radio Frequency (RF) transmission lines.

1.22.4 Input Impedance Interms of Z_0 and K

The input impedance at the sending end is given by,

$$Z_S = \frac{E_S}{I_S} = \frac{Z_0\left[Z_R \cosh(\gamma l) + Z_0 \sinh(\gamma l)\right]}{\left[Z_0 \cosh(\gamma l) + Z_R \sinh(\gamma l)\right]}$$

$$\therefore \quad Z_S = \frac{Z_0\left[Z_R + Z_0 \tanh(\gamma l)\right]}{\left[Z_0 + Z_R \tanh(\gamma l)\right]}$$

But $$\tanh(\gamma l) = \frac{e^{\gamma l} - e^{-\gamma l}}{e^{\gamma l} + e^{-\gamma l}}$$

$$\therefore \quad Z_S = \frac{Z_0\left[Z_R + Z_0\left(\dfrac{e^{\gamma l} - e^{-\gamma l}}{e^{\gamma l} + e^{-\gamma l}}\right)\right]}{\left[Z_0 + Z_R\left(\dfrac{e^{\gamma l} - e^{-\gamma l}}{e^{\gamma l} + e^{-\gamma l}}\right)\right]}$$

$$= Z_0\left\{\frac{(e^{\gamma l} + e^{-\gamma l})Z_R + Z_0(e^{\gamma l} - e^{-\gamma l})}{Z_0(e^{\gamma l} + e^{-\gamma l}) + Z_R(e^{\gamma l} - e^{-\gamma l})}\right\}$$

$$= Z_0\left\{\frac{e^{\gamma l}[Z_R + Z_0] + e^{-\gamma l}[Z_R - Z_0]}{e^{\gamma l}[Z_R + Z_0] + e^{-\gamma l}[Z_0 - Z_R]}\right\}$$

$$= Z_0\left\{\frac{e^{\gamma l} + \dfrac{Z_R - Z_0}{Z_R + Z_0}e^{-\gamma l}}{e^{\gamma l} - \dfrac{Z_R - Z_0}{Z_R + Z_0}e^{-\gamma l}}\right\}$$

$$\therefore \quad Z_S = Z_0\left\{\frac{e^{\gamma l} + Ke^{-\gamma l}}{e^{\gamma l} - Ke^{-\gamma l}}\right\} \qquad \text{... (10)}$$

1.23 Reflection Loss and Reflection Factor

It is seen that when the line is terminated in Z_0 then there is no reflection. But under mismatch condition, the ratio of voltage to current gets disturbed. The part of the energy is rejected and reflected by the load. Thus energy delivered to the load under mismatch condition is always less than the energy which would be delivered to the load under matched impedance condition. This is because of a loss called reflection loss. The reflection loss is determined from the ratio of current which actually flows under mismatch condition in the load to that which would flow if the impedances are matched at the terminals of load.

The **reflection loss** is defined as the number of nepers or decibels by which the current in the load under image matched conditions would exceed the current actually flowing in the load.

So if I'_2 is the load current under image matching condition and I_2 is the actual load current under image mismatch condition then the reflection loss in nepers is given by,

$$\text{Reflection loss} = ln\left[\frac{|I'_2|}{|I_2|}\right] \text{ nepers} \qquad \text{... (1(a))}$$

$$\text{Reflection loss} = 20 \log \left[\frac{|I_2'|}{|I_2|}\right] \text{ dB} \qquad \text{... (1(b))}$$

Thus reflection loss is actually **mismatching loss.**

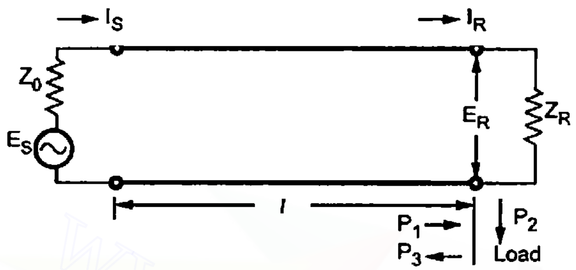


Fig. 1.30

The reflection loss can be measured in terms of the power absorbed by the load and received at the load terminals as power and current are directly related.

Consider a line of length l. The sending end voltage is E_S with an impedance of Z_0 while the line is terminated in an impedance of Z_R. Then E_R is the receiving end voltage. The line is shown in the Fig. 1.30.

Then as shown in the Fig. 1.30,

P_1 = Power at the receiving end due to incident wave

P_2 = Power absorbed by the load

P_3 = Power reflected back down the line

$$\therefore \quad P_1 = P_2 + P_3 \qquad \text{... (3)}$$

Now $\quad P \propto I^2 \quad$ i.e. $\; I \propto P^{\frac{1}{2}}$

$$\therefore \quad \text{Reflection loss} = 20 \log \left[\frac{|I_2'|}{|I_2|}\right]$$

$$= 20 \log \left[\frac{P_1^{\frac{1}{2}}}{P_2^{\frac{1}{2}}}\right] = 20 \log \left[\frac{P_1}{P_2}\right]^{\frac{1}{2}}$$

$$\therefore \quad \text{Reflection loss} = 10 \log \frac{P_1}{P_2} \text{ dB} = 20 \log \frac{1}{|k|} \qquad \text{... (4)}$$

The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0}$$

If E_R and I_R are the values of voltage and current at the receiving end due to incident wave then the values of voltage and current at the receiving end due to the reflected wave are $K\,E_R$ and $K\,I_R$.

So

$$P_1 = \text{Received power} = E_R\, I_R$$

$$P_2 = \text{Power absorbed by load} = P_1 - P_3$$

$$P_3 = \text{Reflected power} = (|K|\,E_R)\,(|K|\,I_R) = |K|^2\, P_1$$

$$P_2 = P_1 - |K|^2\, P_1 = (1-|K|^2)\, P_1 \qquad \dots (5)$$

Z_R and Z_0 are complex hence K is also complex but only magnitude of K is to be considered. The power calculated is the apparent power in volt amp and not the true power in watts.

Hence the reflection loss can be obtained as,

$$\text{Reflection loss} = 10 \log \frac{P_1}{P_2} = 10 \log \left\{\frac{P_1}{P_1\,[1-|K|^2]}\right\}$$

$$= 10 \log \left\{\frac{1}{1-|K|^2}\right\}$$

Now

$$\frac{1}{1-K^2} = \frac{1}{1-\left[\frac{Z_R - Z_0}{Z_R + Z_0}\right]^2} = \frac{(Z_R + Z_0)^2}{(Z_R + Z_0)^2 - (Z_R - Z_0)^2}$$

$$= \frac{(Z_R + Z_0)^2}{Z_R^2 + Z_0^2 + 2\, Z_R\, Z_0 - Z_R^2 - Z_0^2 + 2\, Z_R\, Z_0}$$

$$= \frac{(Z_R + Z_0)^2}{4\, Z_R\, Z_0}$$

$$\therefore \quad \frac{1}{1-|K|^2} = \frac{|Z_R + Z_0|^2}{|4\, Z_R\, Z_0|}$$

$$\therefore \quad \text{Reflection loss} = 10 \log \left\{\frac{|Z_R + Z_0|^2}{|4\, Z_R\, Z_0|}\right\} = 10 \log \left[\frac{|Z_R + Z_0|}{\left|2\sqrt{Z_R\, Z_0}\right|}\right]^2$$

$$\therefore \quad \text{Reflection loss} = 20 \log \frac{|Z_R + Z_0|}{\left|2\sqrt{Z_R\, Z_0}\right|}\ \text{dB} = 20 \log \frac{1}{|K|} \qquad \dots (6)$$

The ratio which indicates the change in current in the load due to reflection at the mismatched junction is called **reflection factor**. It is denoted by K and defined by,

$$K = \text{reflection factor} = \left|\frac{2\sqrt{Z_R\, Z_0}}{Z_R + Z_0}\right| \quad \text{... (7)}$$

The reflection loss is inversely proportional to the reflection factor.

1.23.1 Return Loss

The return loss is defined as,

$$\text{Return loss} = 10 \log \frac{P_1}{P_3} \text{ dB} \quad \text{... (8)}$$

It indicates the ratio of the power at the receiving end due to incident wave to the power reflected by the load.

$$\text{Return loss} = 10 \log \frac{P_1}{|K|^2 P_1} = 10 \log \frac{1}{|K|^2}$$

$$= 10 \log \left[\frac{1}{|K|}\right]^2 = 20 \log \left[\frac{1}{|K|}\right]$$

$$\therefore \quad \text{Return loss} = 20 \log \left|\frac{Z_R + Z_0}{Z_R - Z_0}\right| \text{ dB} \quad \text{... (9)}$$

This is also called Singing point.

1.24 Insertion Loss

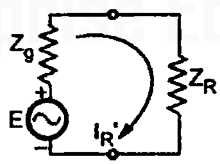


Fig. 1.31

The word insertion loss clears the fact that this is a loss which occurs due to the insertion of a network or a line in between the source and the load.

Suppose a generator of an impedance Z_g is connected to a load of impedance Z_R, as shown in the Fig. 1.31.

The generator is directly connected to the impedance Z_R. The current flowing through load is,

$$I_R' = \frac{E}{Z_g + Z_R}$$

Now suppose a transmission line is inserted between the load and the generator as shown in the Fig. 1.32.

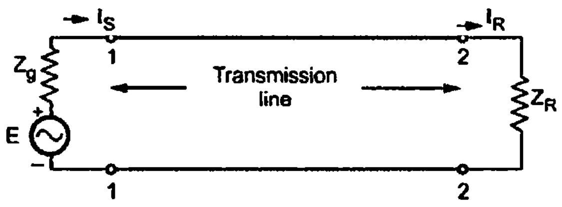


Fig. 1.32

It becomes a line terminated in an impedance Z_R. Now there are various losses due to insertion of line between the load and the generator. If input impedance Z_S is not equal to Z_g then reflection loss occurs at the terminals 1-1. If Z_R is not equal to Z_0 then second reflection loss occurs at the terminals 2-2. The attenuation loss occurs all along the line. The overall effect of insertion of a line is to change the current through the load and hence power delivered to the load as compared to current and power when load was directly connected to generator. The loss which causes such a change, due to insertion of a network or a line in between the load and the generator is called insertion loss. The insertion loss is the effect of several individual losses.

Thus **insertion** loss of a line or a network is defined as the number of nepers or decibels by which the current in the load is changed by the insertion of a line or a network in between the load and the source.

In some cases there can be increase in the load current due to insertion. Such an increase indicates a negative loss and in that case there exists an insertion gain rather than a loss.

1.24.1 Expression for Insertion Loss

Consider the circuit shown earlier in the Fig. 1.31 with a load current of

$$I'_R = \frac{E}{Z_g + Z_R} \qquad \text{... (1)}$$

The line is inserted between the load and the generator.

Let Z_S be the input impedance of a line which is different than Z_g hence,

$$I_S = \frac{E}{Z_g + Z_S} \qquad \text{... (2)}$$

It is known for the line that input impedance is given by,

$$Z_S = Z_0\left[\frac{e^{\gamma l}+Ke^{-\gamma l}}{e^{\gamma l}-Ke^{-\gamma l}}\right] \quad \text{... (3)}$$

Substituting in (2),

$$I_S = \frac{E}{Z_g + Z_0\left(\dfrac{e^{\gamma l}+Ke^{-\gamma l}}{e^{\gamma l}-Ke^{-\gamma l}}\right)}$$

$$\therefore \quad I_S = \frac{E\left(e^{\gamma l}-Ke^{-\gamma l}\right)}{Z_g\left(e^{\gamma l}-Ke^{-\gamma l}\right)+Z_0\left(e^{\gamma l}+Ke^{-\gamma l}\right)} \quad \text{... (4)}$$

We want the current through the load i.e. I_R due to the insertion of line. This can be obtained from the relation between I_S and I_R.

$$I_S = \frac{I_R\,(Z_R+Z_0)}{2\,Z_0}\left(e^{\gamma l}-Ke^{-\gamma l}\right) \quad \text{... (5)}$$

The equation is obtained from the general solution of a line substituting s = l i.e. $I = I_S$ at the sending end.

$$\therefore \quad I_R = \frac{2\,Z_0\,I_S}{(Z_R+Z_0)\left[e^{\gamma l}-Ke^{-\gamma l}\right]}$$

$$= \frac{\dfrac{2\,Z_0\,E\left(e^{\gamma l}-Ke^{-\gamma l}\right)}{Z_g\left(e^{\gamma l}-Ke^{-\gamma l}\right)+Z_0\left(e^{\gamma l}+Ke^{-\gamma l}\right)}}{(Z_R+Z_0)\left(e^{\gamma l}-Ke^{-\gamma l}\right)} \quad \text{... using (4)}$$

$$= \frac{2\,Z_0 E}{(Z_R+Z_0)\left[Z_0\left(e^{\gamma l}+Ke^{-\gamma l}\right)+Z_g\left(e^{\gamma l}-Ke^{-\gamma l}\right)\right]} \quad \text{... (6)}$$

Now introduce the value of reflection coefficient,

$$K = \frac{Z_R-Z_0}{Z_R+Z_0} \quad \text{... (7)}$$

$$I_R = \frac{2\,Z_0\,E}{(Z_R+Z_0)\left\{Z_0\left(e^{\gamma l}+\dfrac{(Z_R-Z_0)}{(Z_R+Z_0)}e^{-\gamma l}\right)+Z_g\left(e^{\gamma l}-\dfrac{(Z_R-Z_0)}{(Z_R+Z_0)}e^{-\gamma l}\right)\right\}}$$

$$= \frac{2\,Z_0\,E}{(Z_R+Z_0)\left\{Z_0\,e^{\gamma l}+\left[\dfrac{Z_R-Z_0}{Z_R+Z_0}\right]Z_0\,e^{-\gamma l}+Z_g\,e^{\gamma l}-\left[\dfrac{Z_R-Z_0}{Z_R+Z_0}\right]Z_g\,e^{-\gamma l}\right\}}$$

$$I_R = \frac{2 Z_0 E}{(Z_R + Z_0)(Z_0 + Z_g) e^{\gamma l} + (Z_R - Z_0)(Z_0 - Z_g) e^{-\gamma l}} \quad \text{... (8)}$$

The insertion loss is the ratio of currents in the load without insertion and with insertion.

$$\therefore \quad \frac{I'_R}{I_R} = \frac{\dfrac{E}{Z_g + Z_R}}{\dfrac{2 Z_0 E}{(Z_R + Z_0)(Z_0 + Z_g) e^{\gamma l} + (Z_R - Z_0)(Z_0 - Z_g) e^{-\gamma l}}}$$

$$\therefore \quad \frac{I'_R}{I_R} = \frac{(Z_R + Z_0)(Z_0 + Z_g) e^{\gamma l} + (Z_R - Z_0)(Z_0 - Z_g) e^{-\gamma l}}{2 Z_0 (Z_g + Z_R)} \quad \text{... (9)}$$

The current ratio is made up of two parts, one which is continuously increasing with line length and other decreasing with line length. The length of line is usually very large hence $e^{-\gamma l} \to 0$. Hence the second term in the numerator can be neglected compared to first.

$$\therefore \quad \frac{I'_R}{I_R} = \frac{(Z_R + Z_0)(Z_0 + Z_g) e^{\gamma l}}{2 Z_0 (Z_g + Z_R)} \quad \text{... (10)}$$

$$= \frac{(Z_R + Z_0)(Z_0 + Z_g) e^{\alpha l} e^{j \beta l}}{2 Z_0 (Z_g + Z_R)} \quad \text{... use } \gamma = \alpha + j\beta$$

But insertion loss is to be calculated as a function of ratio of **current magnitudes** and hence the term $e^{j\beta l}$ which gives phase angle can be neglected.

$$\therefore \quad \left|\frac{I'_R}{I_R}\right| = \frac{|Z_R + Z_0| |Z_0 + Z_g| e^{\alpha l}}{2 |Z_0| |Z_g + Z_R|} \quad \text{... (11)}$$

Now multiply numerator and denominator by $2\sqrt{Z_g Z_R}$,

$$\therefore \quad \left|\frac{I'_R}{I_R}\right| = \frac{2\sqrt{Z_g Z_R} |Z_R + Z_0| |Z_0 + Z_g| e^{\alpha l}}{4\sqrt{Z_g Z_R} |Z_0| |Z_g + Z_R|}$$

Using $|Z_0| = \sqrt{|Z_0|} \sqrt{|Z_0|}$ we get,

$$\therefore \quad \left|\frac{I'_R}{I_R}\right| = \frac{|Z_g + Z_0|}{2\sqrt{Z_g Z_0}} \frac{|Z_R + Z_0|}{2\sqrt{Z_R Z_0}} \frac{2\sqrt{Z_g Z_R}}{|Z_g + Z_R|} e^{\alpha l} \quad \text{... (12)}$$

All the terms on right hand side indicate **reflection factors.**

Let $$k_S = \frac{2\sqrt{Z_g Z_0}}{|Z_g + Z_0|} = \text{Reflection factor at source side} \qquad \text{... (13)}$$

This is the reflection factor at the terminals 1-1 when the generator is mismatched at its junction with the line.

Then $$k_R = \frac{2\sqrt{Z_R Z_0}}{|Z_R + Z_0|} = \text{Reflection factor at load side.} \qquad \text{... (14)}$$

It is factor at the junction between line and load i.e. at the terminals 2-2.

while $$k_{SR} = \frac{2\sqrt{Z_g Z_R}}{|Z_g + Z_R|} = \text{Reflection factor for direct connection} \qquad \text{... (15)}$$

This is the reflection factor when the generator and load were directly connected.

The last term $e^{\alpha l}$ indicates the loss in the line.

$$\left|\frac{I'_R}{I_R}\right| = \frac{k_{SR}}{k_S\, k_R}\, e^{\alpha l} \qquad \text{... (16)}$$

The insertion loss is defined in nepers or decibels hence can be expressed as,

$$\text{insertion loss} = ln\left|\frac{I'_R}{I_R}\right| = \left[ln\frac{1}{k_S} + ln\frac{1}{k_R} - ln\frac{1}{k_{SR}} + \alpha\, l\right] \text{ nepers} \qquad \text{... (17(a))}$$

$$\text{insertion loss} = 20 \log\left|\frac{I'_R}{I_R}\right| = 20\left[\log\frac{1}{k_S} + \log\frac{1}{k_R} - \log\frac{1}{k_{SR}} + 0.4343\alpha\, l\right] \text{ dB} \qquad \text{... (17(b))}$$

The term corresponding to k_{SR} is negative. It is the loss if generator and load would have been directly connected. It is not related to insertion hence it is subtracted from the overall loss.

The expression can be applied to a single section of network putting $l = 1$ and proper value of α of the network.

Note that while calculating the insertion loss, **only magnitudes** of all the impedances are to be considered.

The insertion loss also can be expressed interms of ratio of powers delivered to the load under two conditions.

Let P'_R = Power delivered to the load when generator is directly connected to load

P_R = Power delivered to the load when a line or a network is inserted between generator and load

Then, $$\text{insertion loss} = 10 \log \frac{P'_R}{P_R} \text{ dB} \qquad \text{... (18)}$$

Example 1.11 : *A transmission line has $Z_0 = 700 \angle -13.4° \ \Omega$ is inserted between a generator of 200 Ω and a load of 400 Ω. The attenuation and phase constants of a line are,*

$$\alpha = 0.00712 \text{ nepers / km} \quad \text{and} \quad \beta = 0.0288 \text{ rad / km}$$

Calculate the insertion loss if line length is 200 km.

Solution : $Z_0 = 700 \angle -13.4° \ \Omega$, $Z_g = 200 \angle 0° \ \Omega$, $Z_R = 400 \angle 0° \ \Omega$

$$k_S = \frac{2\sqrt{Z_g Z_0}}{|Z_g + Z_0|} = \frac{2\sqrt{200 \times 700}}{|200 + j\,0 + 700 \angle -13.4°|} = \frac{748.3314}{|200 + j\,0 + 680.94 - j\,162.2|}$$

$$= \frac{748.3314}{|880.94 - j\,162.22|} = \frac{748.3314}{895.7514} = 0.83542$$

$$k_R = \frac{2\sqrt{Z_R Z_0}}{|Z_R + Z_0|} = \frac{2\sqrt{400 \times 700}}{|400 + j\,0 + 680.94 - j\,162.22|} = \frac{1058.3}{|1080.94 - j\,162.22|}$$

$$= \frac{1058.3}{1093.044} = 0.9682$$

$$k_{SR} = \frac{2\sqrt{Z_g Z_R}}{|Z_g + Z_R|} = \frac{2\sqrt{200 \times 400}}{|200 + j\,0 + 400 + j\,0|} = \frac{565.685}{600}$$

$$= 0.9428$$

$$\text{Insertion loss} = 20\left[\log\frac{1}{k_S} + \log\frac{1}{k_R} - \log\frac{1}{k_{SR}} + 0.4343 \times \alpha\, l\right] \text{dB}$$

$$= 20\left[\log\frac{1}{0.83542} + \log\frac{1}{0.9682} - \log\frac{1}{0.9428} + 0.4343 \times 0.00712 \times 200\right]$$

$$= \mathbf{13.7 \ \ dB}$$

Example 1.12 : *A transmission line has $Z_0 = 745 \angle -12° \Omega$ and is terminated in $Z_R = 100 \ \Omega$. Calculate the reflection loss and return loss in dB.*

Solution : The reflection factor,

$$k = \frac{2\sqrt{Z_R Z_0}}{|Z_R + Z_0|} = \frac{2\sqrt{100 \times 745}}{|100 + j\,0 + 728.72 - j\,154.894|} = \frac{545.8937}{|828.72 - j\,154.894|}$$

$$= \frac{545.8937}{843.0711} = 0.6475$$

$$\therefore \quad \text{Reflection loss} = 20 \log\frac{1}{|k|} = 20 \log\frac{1}{0.6475} = \mathbf{3.7751 \ \ dB}$$

and Return loss $= 20\ \log\left|\dfrac{Z_R + Z_0}{Z_R - Z_0}\right| = 20\ \log\left|\dfrac{100 + j\,0 + 728.72 - j\,154.894}{100 + j\,0 - 728.72 + j\,154.894}\right|$

$$= 20\ \log\left|\frac{828.72 - j\,154.894}{-\,628.72 + j\,154.894}\right| = 20\ \log\left|\frac{843.0711}{647.519}\right| = \mathbf{2.2922\ dB}$$

Examples with Solutions

Example 1.13 : *A voltage of 45 V is applied to a 10 km long field quad cable. The receiving end voltage is 7.868 V and it lags behind by 110.2°. Calculate the attenuation and phase constants of the cable, if it is properly terminated.*

Solution : $x = 10$ km, $E_S = 45$ V, $E_x = 7.868$ V, $x\,\beta = 110.2°$

Now $E_x = E_S\, e^{-\gamma x} = E_S\, e^{-\alpha x} \angle -\beta x$

$\beta x = 110.2°$

$\therefore \quad \beta = \dfrac{110.2}{10} = 11.02°$ per km $= 0.1923$ rad/km

and $E_x = E_S\, e^{-\alpha x}$

$\therefore \quad 7.868 = 45\ e^{-\alpha \times 10}$

$\therefore \quad e^{-10\alpha} = 0.1748$

$\therefore \quad -10\alpha = ln\,(0.1748) = -\,1.743$

$\therefore \quad \alpha =$ **0.1743 Nepers per km**

Example 1.14 : *A transmission line has the following primary constants measured per km,*

$R = 10.15\ \Omega$, $L = 3.93$ mH, $C = 0.00797\ \mu$F, $G = 0.29\ \mu$mho

Determine Z_0 and propagation constant at a frequency of 796 Hz. Also calculate the ratio of current at a point which is 100 km down the line to the current at the sending end if the line is terminated in its characteristic impedance.

Solution : $\omega = 2\pi f = 2\pi \times 796 = 5 \times 10^3$ rad/sec

$$R + j\omega L = 10.15 + j\,5 \times 10^3 \times 3.93 \times 10^{-3} = 10.15 + j\,19.65\ \Omega$$

$$= 22.1166\ \angle 62.68°\ \Omega$$

$$G + j\,\omega\,C = 0.28 \times 10^{-6} + j\,5 \times 10^3 \times 0.00797 \times 10^{-6}$$

$$= 0.29 \times 10^{-6} + j\,3.985 \times 10^{-5} = 3.985 \times 10^{-5}\ \angle 89.58°$$

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{22.1166\angle 62.68^\circ}{3.985\times 10^{-5}\ \angle 89.58^\circ}}$$

$$= \mathbf{744.97\ \angle -13.45^\circ\ \Omega}$$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{22.1166\angle 62.68^\circ \times 3.985\times 10^{-5}\ \angle 89.58^\circ}$$

$$= 0.02968\ \angle 76.13^\circ = 0.00711 + j\,0.0288$$

$$\alpha = 0.00711 \text{ nepers/km}$$

and $\beta = \mathbf{0.0288\ \ rad/km}$

As the line is properly terminated we can use,

$$I_x = I_S\, e^{-\gamma x} = I_S\, e^{-\alpha x} \angle -\beta x$$

$$\therefore \quad \frac{I_x}{I_S} = e^{-\alpha x} \angle -\beta x$$

Now $x = 100$ km

$$\therefore \quad \frac{I_x}{I_S} = e^{-0.00711\times 100} \angle -0.0288\times 100 \text{ rad}$$

$$= 0.49115\ \angle -2.88 \text{ rad} = \mathbf{0.49115\ \angle -165.024^\circ}$$

... $2.88 \text{ rad} = 2.88\times 57.3^\circ = 165.024^\circ$

Example 1.15 : *Impedance measurements made on a 0.25 km field quad cable at 1600 Hz under open and short circuit conditions gave the following results,*

$$Z_{OC} = 2460\angle -86.3^\circ\ \Omega, \quad Z_{SC} = 215\angle +14^\circ\ \Omega$$

Calculate Z_0, α, β and the line parameters R, L, G and C per unit length of line.

Solution : For a line of length l,

$$\tanh(\gamma l) = \sqrt{\frac{Z_{SC}}{Z_{OC}}}$$

and $Z_0 = \sqrt{Z_{SC}\, Z_{OC}}$

$$\therefore \quad Z_0 = \sqrt{2460\times 21.5\ \angle -86.3^\circ + 14^\circ}$$

$$= \mathbf{230\ \angle -36.15^\circ\ \Omega}$$

and $$\tanh(\gamma l) = \sqrt{\frac{21.5\ \angle 14^\circ}{2460\ \angle -86.3^\circ}} = \sqrt{0.00873\ \angle 14^\circ + 86.3^\circ}$$

$$= 0.0934\ \angle 50.15^\circ = 0.0598 + j\,0.0717$$

$$= A + j\,B$$

$\therefore \quad \tanh(2\alpha l) = \dfrac{2A}{1+A^2+B^2} = \dfrac{2\times 0.0598}{1+(0.0598)^2+(0.0717)^2} = 0.1185$

and $\quad \tan(2\beta l) = \dfrac{2B}{1-(A^2+B^2)} = \dfrac{2\times 0.0717}{1-[(0.0598)^2+(0.0717)^2]} = 0.1446$

$\therefore \quad 2\alpha l = \tanh^{-1} 0.1185$

$\therefore \quad \alpha = \dfrac{0.119}{2\times 0.25} = \mathbf{0.2381}$ **nepers/km**

and $\quad 2\beta l = \tan^{-1}(0.1446) = 0.1436$ ... use radian mode

$\therefore \quad \beta = \dfrac{0.1436}{2\times 0.25} = \mathbf{0.2872}$ **rad/km**

$\gamma = \alpha + j\beta = 0.2381 + j\,0.2872 = 0.373\angle 0.8785$ rad

$= 0.373\angle 50.343°$

Now $\quad Z_0\gamma = R + j\omega L$

$\therefore \quad R + j\omega L = 230\angle -36.15° \times 0.373\angle 50.343° = 85.79\angle 14.193°$

$= 83.171 + j\,21.034$

$\therefore \quad R = \mathbf{83.171}$ **Ω/km**

and $\quad \omega L = 21.034$

$\therefore \quad L = \dfrac{21.034}{\omega} = \dfrac{21.034}{2\pi\times 1600} = \mathbf{0.00209}$ **H/km**

And $\quad \dfrac{\gamma}{Z_0} = G + j\omega C$

$\therefore \quad \dfrac{0.373\angle 50.343°}{230\angle -36.15°} = G + j\omega C$

$\therefore \quad G + j\omega C = 0.00162\angle 86.493° = 9.9\times 10^{-5} + j\,1.61\times 10^{-3}$

$G = 9.9\times 10^{-5}$ **mho/km**

and $\quad \omega C = 1.61\times 10^{-3}$

$\therefore \quad C = \dfrac{1.61\times 10^{-3}}{2\pi\times 1600} = \mathbf{0.1608}\ \boldsymbol{\mu}$ **F/km**

Examle 1.16 : *Find the primary and secondary constants of the line 50 km long when Z_{OC} measured by a bridge at 700 Hz is $286\angle -40°\Omega$ and Z_{SC} is $1520\angle 16°\Omega$.*

Solution : $l = 50$ km, $\omega = 2\pi f = 2\pi \times 700 = 4398.2297$ rad/sec

$$\tanh(\gamma l) = \sqrt{\frac{Z_{SC}}{Z_{OC}}} = \sqrt{\frac{1520\angle 16^\circ}{286\angle -40^\circ}}$$

$$= 2.3053\angle 28^\circ = 2.0354 + j\,1.0822$$

Now $\tanh(\gamma l) = \dfrac{e^{2\gamma l}-1}{e^{2\gamma l}+1} = 2.0354 + j\,1.0822$

$\therefore$ $e^{2\gamma l} - 1 = (e^{2\gamma l}+1)(2.0354 + j\,1.0822)$

$\therefore$ $e^{2\gamma l} - 1 = e^{2\gamma l}[2.0354 + j\,1.0822] + 2.0354 + j\,1.0822$

$e^{2\gamma l}[1 - 2.0354 - j\,1.0822] = 1 + 2.0354 + j\,1.0822$

$\therefore$ $e^{2\gamma l} = \dfrac{3.0354 + j\,1.0822}{-1.0354 - j\,1.0822} = \dfrac{3.2225\angle 19.622^\circ}{1.4977\angle -133.733^\circ}$

$\therefore$ $e^{2\gamma l} = 2.1516\angle 153.355^\circ$

$$2\gamma l = ln\,[2.1516\angle 153.355^\circ]$$

But $ln[a\angle b] = ln\,a + j\,b$

$\therefore$ $2\gamma l = ln\,2.1516 + j\,153.355^\circ$

$$\gamma = \frac{1}{2l}[ln\,2.1516 + j\,153.355^\circ]$$

$\therefore$ $\gamma = 0.00766 + j\,1.53355^\circ = \alpha + j\beta$

$$\alpha = \mathbf{0.00766 \ \ Nepers/km}$$

and $\beta = 1.5335$ deg/km $= \dfrac{1.5335}{57.3}$ rad/km $=$ **0.02676 rad/km**

$$\gamma = 0.00766 + j\,0.02676 = 0.0278\angle 74.026^\circ$$

Note that while obtaining polar form of γ, β must be expressed in rad/km.

$$Z_0 = \sqrt{Z_{SC}\,Z_{OC}} = \sqrt{1520\angle 16^\circ \times 286\angle -40^\circ}$$

$$= \mathbf{659.333\ \angle -12^\circ\ \Omega}$$

Now $R + j\omega L = Z_0\,\gamma = 659.333\angle -12^\circ \times 0.0278\angle 74.026^\circ$

$$= 18.3294\angle 62.026^\circ = 8.5977 + j\,16.187$$

$\therefore$ $R = \mathbf{8.5977\ \ \Omega / km}$

$$\omega L = 16.187$$

$$\therefore \quad L = \frac{16.187}{4398.2297} = 3.6805 \quad \text{mH/km}$$

and
$$G + j\omega C = \frac{\gamma}{Z_0} = \frac{0.0278\angle 74.026^\circ}{659.333\angle -12^\circ} = 4.2163\times 10^{-5}\angle 86.026^\circ$$

$$= 2.922\times 10^{-6} + j\,4.206\times 10^{-5}$$

$$\therefore \quad G = 2.922 \quad \mu \text{ mho/km}$$

and
$$\omega C = 4.206\times 10^{-5}$$

$$\therefore \quad C = \frac{4.206\times 10^{-5}}{4398.2297} = 9.563\times 10^{-9} \quad \text{F/km}$$

Note : In this problem if the direct formula of $\tan(2\beta l)$ is used we get,

$$\tan(2\beta l) = \frac{2B}{1-[A^2+B^2]} \quad \text{where} \quad A = 2.0354$$

and
$$B = 1.0822$$

$$\therefore \quad \tan(2\beta l) = -0.5017$$

$$2\beta l = -0.465 \qquad \text{... use radian mode}$$

Now β can not be negative, it is due to behaviour of tan function. It is necessary to add 180° i.e. π rad to this angle to calculate β.

$$\therefore \quad 2\beta l = -0.465 + \pi = 2.676 \quad \text{rad}$$

$$\therefore \quad \beta = \frac{2.676}{2\times 50} = 0.02676 \quad \text{rad/km}$$

To avoid this confusion, it is recommended to use the basic procedure to calculate α and β from Z_{SC} and Z_{OC} without using direct formulae.

Example 1.17 : *A transmission line with a characteristic impedance of 50 Ω is connected to a 100 Ω resistive load. Calculate the voltage reflection coefficient at the load.*

Solution : $Z_0 = 50\angle 0^\circ\,\Omega$ and $Z_R = 100\angle 0^\circ\,\Omega$

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{100-50}{100+50} = \frac{50\angle 0^\circ}{150\angle 0^\circ}$$

$$= 0.333\angle 0^\circ$$

Example 1.18 : *A telephone transmission line 100 km long has $Z_0 = 685\angle -12^\circ\Omega$, $\alpha = 0.00497$ N/km, $\beta = 0.0352$ rad/km at 10 Hz. The line is terminated in $Z_R = 2000 + j\,0\,\Omega$ and is supplied by a generator with an e.m.f of 10 V r.m.s and $R_m = 700\Omega$. Calculate the values of I_S, Z_S and I_R, E_R, P_S and P_R.*

Solution : $l = 100$ km, $Z_g = R_m = 700 + j\,0\,\Omega$, $Z_R = 2000 + j\,0\,\Omega$

$$Z_0 = 685\angle -12°\,\Omega = 670.03 - j\,142.419\,\Omega$$

$$e^{\gamma l} = e^{\alpha l}\angle \beta l = e^{0.00497\times 100}\angle 0.0352\times 100 = 1.6437\angle 3.52 \text{ rad}$$

$$= 1.6437\angle +201.696° = -1.5272 - j\,0.607$$

$$e^{-\gamma l} = e^{-\alpha l}\angle -\beta l = e^{-0.00497\times 100}\angle -0.0352\times 100 = 0.6083\angle -3.52 \text{ rad}$$

$$= 0.6083\angle -201.696° = -0.5652 + j\,0.2248$$

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{2000 + j\,0 - 670.03 + j\,142.419}{2000 + j\,0 + 670.03 - j\,142.419} = \frac{1329.97 + j\,142.419}{2670.03 - j\,142.419}$$

$$= \frac{1337.573\angle 6.112°}{2673.825\angle -3.05°} = 0.5\angle 9.162°$$

$$K\,e^{-\gamma l} = 0.5\angle 9.162° \times 0.6083\angle -201.696° = 0.30415\angle -192.534°$$

$$= -0.2969 + j\,0.066$$

$$e^{\gamma l} + K\,e^{-\gamma l} = -1.5272 - j\,0.607 - 0.2969 + j\,0.066 = -1.8241 - j\,0.541 = 1.9026\angle -163.48°$$

$$e^{\gamma l} - K\,e^{-\gamma l} = -1.5272 - j\,0.607 + 0.2969 - j\,0.066 = -1.2303 - j\,0.673 = 1.4023\angle -151.32°$$

$$Z_S = Z_0\left\{\frac{e^{\gamma l} + K e^{-\gamma l}}{e^{\gamma l} - K e^{-\gamma l}}\right\} = \frac{685\angle -12° \times 1.9026\angle -163.48°}{1.4023\angle -151.32°}$$

$$= 929.388\angle -24.16° = \mathbf{847.98 - j\,380.385\ \Omega}$$

$$I_S = \frac{E_S}{Z_S + Z_g} = \frac{10\angle 0°}{847.98 - j\,380.385 + 700 + j\,0}$$

$$= \frac{10\angle 0°}{1594.031\angle -13.8°} = \mathbf{6.2734\times 10^{-3}\angle +13.8°\ A}$$

Now
$$I_S = \frac{I_R (Z_R + Z_0)}{2 Z_0}\left[e^{\gamma l} - K\,e^{-\gamma l}\right]$$

... as line is not properly terminated

$$\therefore\ I_R = \frac{2 Z_0 I_S}{(Z_R + Z_0)\left[e^{\gamma l} - K\,e^{-\gamma l}\right]} = \frac{2\times 685\angle -12° \times 6.2734\times 10^{-3}\angle 13.8°}{2673.825\angle -3.05° \times 1.4023\angle -151.32°}$$

$$= \mathbf{2.2921\times 10^{-3}\angle +156.17°\ A}$$

$$E_R = I_R Z_R = 2.2921\times 10^{-3}\angle 156.17° \times 2000\angle 0°$$

$$= \mathbf{4.5842\angle 156.17°\ V}$$

$$P_S = E_S\, I_S \cos\left(E_S \,\hat{}\, I_S\right) = 10 \times 6.2734 \times 10^3 \cos(13.8°)$$

$$= \mathbf{0.06092\ W}$$

$$P_R = E_R\, I_R \cos\left(E_R \,\hat{}\, I_R\right) = 4.5842 \times 2.2921 \times 10^{-3} \times \cos(0°)$$

$$= \mathbf{0.0105\ W}$$

$$\%\ \eta = \frac{P_R}{P_S} \times 100 = \frac{0.0105}{0.06092} \times 100 = \mathbf{17.2479\ \%}$$

Example 1.19 : *A transmission line has following parameters per km*

$R = 15\ \Omega$, $C = 15\ \mu F$, $L = 1\ mH$, $G = 1\ \mu mho$

Find the additional inductance to give distortionless transmission. Calculate α and β for this inductance added transmission line.

Solution : For the distortionless line,

$$RC = L'G$$

$$L' = \frac{RC}{G} = \frac{15 \times 15 \times 10^{-6}}{1 \times 10^{-6}} = 225\ H$$

So additional inductance required is $225 - 1 \times 10^3 = 224.999$ H

For the loaded line,

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L'}} + \frac{G}{2}\sqrt{\frac{L'}{C}} = \frac{15}{2}\sqrt{\frac{15 \times 10^{-6}}{225}} + \frac{1 \times 10^{-6}}{2}\sqrt{\frac{225}{15 \times 10^{-6}}}$$

$$= \mathbf{0.00387\ \ N/km}$$

$$\beta = \omega\sqrt{L'C}$$

So assume $\omega = 6.283 \times 10^3$ rad/sec

$$\beta = 6.283 \times 10^3 \sqrt{225 \times 15 \times 10^{-6}} = \mathbf{365\ rad/km}$$

Example 1.20 : *A transmission line has the following per unit length parameters,*

$L = 0.1\ \mu H$, $R = 5\ \Omega$, $C = 300\ pF$, $G = 0.01\ mho$

Calculate the propagation constant and characteristic impedance of the line at 500MHz. Obtain the same parameters for the lossless line.

Solution : $\omega = 2\pi f = 2\pi \times 500 \times 10^{6}$ rad/sec

$$Z = R + j\omega L = 5 + j2\pi \times 500 \times 10^{6} \times 0.1 \times 10^{-6} = 5 + j\,314.1592 = 314.199\angle 89.088^{\circ}$$

$$Y = G + j\omega C = 0.01 + j2\pi \times 500 \times 10^{6} \times 300 \times 10^{-12} = 0.01 + j\,0.9424 = 0.9425\angle 89.39$$

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{314.199\angle 89.088^{\circ}}{0.9425\angle 89.39^{\circ}}} = \mathbf{18.2583\angle -0.151^{\circ}\Omega}$$

and
$$\gamma = \sqrt{YZ} = \sqrt{314.199\angle 89.088^{\circ} \times 0.9425\angle 89.39^{\circ}}$$
$$= 17.2085\angle 89.239^{\circ} = 0.2285 + j\,17.2069 = \alpha + j\beta$$
$$\alpha = \mathbf{0.2285\ \ N/km}$$
and
$$\beta = \mathbf{17.2069\ \ rad/km}$$

For the lossless line, $R = G = 0$

$$\alpha = 0$$
and
$$\beta = \omega\sqrt{LC} = 2\pi \times 500 \times 10^{6}\sqrt{0.1 \times 10^{-6} \times 300 \times 10^{-12}}$$
$$= \mathbf{17.2072\ rad/km}$$
and
$$Z_0 = \sqrt{\frac{L}{C}}\angle 0^{\circ} = \sqrt{\frac{0.1 \times 10^{-6}}{300 \times 10^{-12}}}\angle 0^{\circ}$$
$$= \mathbf{18.2574\ \angle 0^{\circ}\ \Omega}$$

Thus for a high frequency like 500 MHz, the line behaves almost as lossless line.

University Examples with Solutions

Example 1.21 *A generator of 1 V, 1 kHz supplies power to a 100 km open wire line terminated in 200 Ω resistance. The line parameters are,*

$R = 10\ \Omega / km$, $L = 3.8\ mH / km$, $G = 1 \times 10^{-6}$ mho / km, $C = 0.0085\ \mu F / km$

Calculate the input impedance, reflection coefficient, the input power, the output power and transmission efficiency. **(Dec.-2005, 16 Marks)**

Solution : The characteristic impedance is given by,

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{10 + j(2\pi \times 1 \times 10^{3})3.8 \times 10^{-3}}{1 \times 10^{-6} + j(2\pi \times 1 \times 10^{3})0.0085 \times 10^{-6}}}$$

$$= \sqrt{\frac{10 + j\,23.876}{1 \times 10^{-6} + j\,5.34 \times 10^{-5}}} = \sqrt{\frac{25.885\angle 67.27^{\circ}}{5.34 \times 10^{-5}\angle 88.92^{\circ}}}$$

$$= \sqrt{4.847 \times 10^5 \angle -21.65^\circ} = 696.204 \angle -10.825^\circ \ \Omega$$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{25.885 \times 5.34 \times 10^{-5} \angle 67.27^\circ + 88.92^\circ}$$

$$= 0.03717 \angle 78.095^\circ = 0.0076 + j\ 0.03637$$

$$\therefore \quad \alpha = 0.00766 \text{ nepers/km}$$

$$\text{and} \quad \beta = 0.03637 \text{ rad/km}$$

$$\therefore \quad K = \frac{Z_R - Z_0}{Z_R + Z_0} \quad \text{where } Z_R = 200 \angle 0^\circ \ \Omega \quad \text{given}$$

$$\therefore \quad K = \frac{200 \angle 0^\circ - 696.204 \angle -10.825^\circ}{200 \angle 0^\circ + 696.204 \angle -10.825^\circ}$$

$$\therefore \quad K = \frac{200 + j\,0 - 683.8153 + j\,130.754}{200 + j\,0 + 683.8153 - j\,130.754} = \frac{-483.8153 + j\,130.754}{883.8153 - j\,130.754}$$

$$= \frac{501.1724 \angle 164.876^\circ}{893.435 \angle -8.415^\circ}$$

$$\therefore \quad K = \mathbf{0.5609 \angle 173.291^\circ}$$

$$\text{Now} \quad e^{\gamma l} = e^{\alpha l} \angle \beta l \text{ rad} = e^{0.00766 \times 100} \angle 0.03637 \times 100 \text{ rad}$$

$$= 2.1511 \angle 3.637 \text{ rad} = 2.1511 \angle 208.4^\circ$$

$$Z_S = Z_0 \left\{ \frac{e^{\gamma l} + K e^{-\gamma l}}{e^{\gamma l} - K e^{-\gamma l}} \right\}$$

$$e^{-\gamma l} = e^{-\gamma l} \angle -\beta l \text{ rad} = e^{-0.00766 \times 100} \angle -0.03637 \times 100 \text{ rad}$$

$$= 0.4648 \angle -208.4^\circ$$

$$\therefore \quad Z_S = 696.204 \angle -10.825^\circ \left\{ \frac{2.1511 \angle 208.4^\circ + 0.5609 \angle 173.29^\circ \times 0.4648 \angle -208.4^\circ}{2.1511 \angle 208.4^\circ - 0.5609 \angle 173.29^\circ \times 0.4648 \angle -208.4^\circ} \right\}$$

$$= 696.204 \angle -10.85^\circ \left\{ \frac{2.1511 \angle 208.4^\circ + 0.2607 \angle -35.11^\circ}{2.1511 \angle 208.4^\circ - 0.2607 \angle -35.11^\circ} \right\}$$

$$= 696.204 \angle -10.85^\circ \left\{ \frac{-1.8922 - j\,1.023 + 0.2132 - j\,0.1499}{-1.8922 - j\,1.023 - 0.2132 + j\,0.1499} \right\}$$

$$= 696.204 \angle -10.85^\circ \left\{ \frac{-1.679 - j\,1.1729}{-2.1054 - j\,0.8731} \right\}$$

$$= 696.204 \angle -10.85^\circ \left\{ \frac{2.0481 \angle -145.062^\circ}{2.2792 \angle -157.476^\circ} \right\}$$

$$= 625.6122 \angle +1.564^\circ \ \Omega$$

It is known that,

$$I = \frac{I_R (Z_R + Z_0)}{2 Z_0}\left[e^{\gamma s} - \frac{Z_R - Z_0}{Z_R + Z_0} e^{-\gamma s}\right]$$

At sending end, s = *l* as s is measured from the receiving end.

$$\therefore \quad I_S = \frac{I_R (Z_R + Z_0)}{2 Z_0}\left[e^{\gamma l} - K e^{-\gamma l}\right]$$

Now $\quad Z_S = \frac{E_S}{I_S}$ and $E_S = 1\angle 0°\,V$

$$\therefore \quad I_S = \frac{E_S}{Z_S} = \frac{1\angle 0°}{625.6122\angle 1.564°} = 1.5984\times 10^{-3}\angle -1.1564°\ A$$

And $\quad Z_R + Z_0 = 200\angle 0° + 696.204\angle -10.825° = 893.435\angle -8.415°$

$$K e^{-\gamma l} = 0.5609\angle 173.291° \times 0.4648\angle -208.4° = 0.2607\angle -35.11°$$

$$\therefore\ 1.5984\times 10^{-3}\angle -1.1564° = \frac{I_R \times 893.435\angle -8.415°}{2\times 696.204\angle -10.825°}\left[2.1511\angle 208.4° - 0.2607\angle -35.11°\right]$$

$$1.5984\times 10^{-3}\angle -1.1564° = I_R\ 0.6416\angle +2.41°\left[2.2792\angle -157.476°\right]$$

$$\therefore \quad I_R = 1.0932\times 10^{-3}\angle +153.909°\ A$$

$$\therefore \quad E_R = I_R Z_R = 1.0932\times 10^{-3}\angle 153.909° \times 200\angle 0°$$

$$\therefore \quad E_R = 0.2186\angle +153.909°\ V$$

So $\quad P_S = E_S I_S \cos\left(E_S \hat{\ } I_S\right)$

$$= 1\angle 0° \times 1.5984\times 10^{-3} \times \cos(-1.1564°)$$

$$= \mathbf{1.598\times 10^{-3}\ W}$$

and $\quad P_R = E_R I_R \cos\left(E_R \hat{\ } I_R\right)$

$$= 0.2186\times 1.0932\times 10^{-3}\times \cos[0°]$$

$$= \mathbf{2.3897\times 10^{-4}\ W}$$

or $\quad P_R = I_R^2 \times R = \left(1.0932\times 10^{-3}\right)^2 \times 200 = 2.3897\times 10^{-4}\ W$

$$\therefore \quad \%\eta = \frac{P_R}{P_S}\times 100 = \frac{2.3897\times 10^{-4}}{1.598\times 10^{-3}}\times 100$$

$$= \mathbf{14.954\%}$$

Example 1.22 : *A generator of 1 volt 1000 Hz supplies power to a 1000 km open wire line terminated in Z_0 having the following parameters :* **(May-2005, 16 Marks)**

R = 10.4 ohms / km

L = 0.00367 H / km

G = 0.8 × 10^{-6} ohms / km

C = 0.00835 micro farad / km

Calculate the power delivered at the receiving end.

Solution : The characteristic impedance is given by

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{10.4+j(2\times\pi\times 1000)\times 3.67\times 10^{-3}}{0.8\times 10^{-6}+j(2\times\pi\times 1000)\times 0.00835\times 10^{-6}}}$$

$$= \sqrt{\frac{10.4+j23.0592}{0.8\times 10^{-6}+j5.2464\times 10^{-5}}} = \sqrt{\frac{25.2959\angle 65.72^\circ}{5.247\times 10^{-5}\angle 89.12^\circ}}$$

$$= \sqrt{4.821\times 10^5\angle -23.4^\circ} = 694.3342\angle -11.7^\circ\ \Omega$$

The propagation constant is given by,

$$\gamma = \sqrt{R+j\omega L)(G+j\omega c)}$$

$$\therefore \quad \gamma = \sqrt{(25.2929\angle 65.72^\circ)(5.247\times 10^{-5}\angle 89.12^\circ)}$$

$$\therefore \quad \gamma = \sqrt{1.3272\times 10^{-3}\angle 154.84} = 0.03643\ \angle 77.42^\circ$$

$$\therefore \quad \gamma = \alpha + j\beta = 0.007934 + j\ 0.03555$$

$$\therefore \quad \gamma = 0.007934 \text{ nepers/km}$$

$$\beta = 0.03555 \text{ rad/km}$$

For 1000 km long transmission line, the propagation constant can be written as,

$$e^{-\gamma l} = e^{-(\alpha+j\beta)l} = e^{-\alpha l}\angle -\beta l$$

$$= e^{-(0.007934)(1000)}\angle -(0.03553)(1000)$$

The transmission line is terminated in Z_0 at the load side. Also the line is connected to a generator of 1V. As internal impedance is not mentioned, we will assume generator to be ideal. The set up is as shown in the Fig. 1.33.

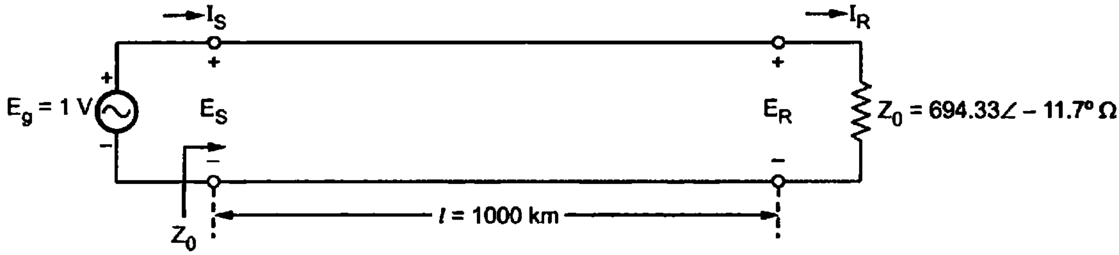


Fig. 1.33

At input side, $E_S = E_g = 1\angle 0°$ V

Thus $I_S = \dfrac{E_S}{Z_0} = \dfrac{1\angle 0°}{694.3342\angle -11.7°} = 1.4402 \times 10^{-3}\angle +11.7°$ A

Hence curent at the receiving end is given by

$$I_R = I_S \cdot e^{-\gamma l}$$

$$= [1.4402 \times 10^{-3}\angle 11.7°]\left[e^{-(0.007934)(1000)}\angle -(0.03553)(1000)\right]$$

$$= [1.4402 \times 10^{-3}\angle 11.7°]\,[3.5835 \times 10^{-4}\angle -35.53°]$$

But for angle 35.53 is expressed in radians. Its degree equivalent angle is 2035.72°.

Hence $I_R = [1.4402 \times 10^{-3}\angle 11.7°]\,[3.5835\angle 10^{-4}\angle -2035.72°]$

$$= [1.4402 \times 10^{-3}\angle 11.7°]\,[3.5835 \times 10^{-4}\angle -235.72]$$

$$= 0.5161 \times 10^{-6}\angle -224.02°\text{ A} = 0.5161\angle -224.02\ \mu\text{A}$$

The receiving end voltage is given by

$$E_R = I_R \cdot Z_0 = (0.5161 \times 10^{-6}\angle -224.02)\,(694.33\angle -11.7°)$$

$$= 0.3583 \times 10^{-3}\angle -235.72°\text{ V}$$

Hence received power is,

$$P_R = E_R\, I_R \cos(E_R \wedge I_R)$$

$$= 0.3583 \times 10^{-3} \times 0.5161 \times 10^{-6} \cdot \cos(11.7)$$

$$= \mathbf{0.181 \times 10^{-9}\ W}$$

$$= \mathbf{0.181\ nW}$$

Example 1.23 : *A transmission line 2 miles long operates at 10 kHz and has parameters R = 30 Ω/mile, C = 80 nF/mile, L = 2.2 mH/mile and G = 20 nV/mile*

Find the characteristic impedance, propagation constant, attenuation and phase shift per mile. **(May-2004, 8 Marks)**

Solution : For a transmission line of length l = 2 miles,

$$R = 30\ \Omega/\text{mile} \qquad L = 2.2\ \text{mH/mile}$$

$$G = 20\ \text{n℧/mile}, \qquad C = 80\ \text{nF/mile and}$$

$$f = 10\ \text{kHz}$$

The characteristic impedance Z_0 is given by,

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{30+j(2\times\pi\times 10\times 10^3)\times 2.2\times 10^{-3}}{20\times 10^{-9}+j\,(2\times\pi\times 10\times 10^3\times(80\times 10^{-9})}}$$

$$= \sqrt{\frac{30+j138.23}{20\times 10^{-9}+j5.0265\times 10^{-3}}}$$

$$= \sqrt{\frac{141.4479\angle 77.75^\circ}{5.0265\times 10^{-3}\angle 89.99^\circ}}$$

$$= \mathbf{167.7511\ \angle -6.12\ \Omega}$$

The propagation constant γ is given by,

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{(30+j2\times\pi\times 10\times 10^3\times 2.2\times 10^{-3})(20\times 10^{-9}+j2\times\pi\times 10\times 10^{-3}\times 80\times 10^{-9}}$$

$$= \sqrt{(141.4479\angle 77.75^\circ)\ (5.0265\times 10^{-3}\angle 89.99^\circ}$$

$$= \mathbf{0.8432\ \angle\ 83.87^\circ\ /km}$$

Representing γ in rectangular form, we get

$$\gamma = \alpha + j\beta = 0.09 + j\ 0.8383$$

Hence attenuation constant = α= **0.09 nepers/miles**

Phase constant β = **0.8383 rad/miles**

Example 1.24 : *Find the sending end impedance of the line having $Z_0 = 710\angle 14^\circ$, $r = 0.007 + j\ 0.028$/km, Z_R = 300 ohm, l = 100 km* **(May-2004, 6 Marks)**

Solution : The input impedance of a line having characteristic impedance Z_0 and terminated in Z_R is given by

$$Z_S = Z_0\left[\frac{Z_R\cosh\gamma l + Z_0\sinh\gamma l}{Z_0\cosh\gamma l + Z_R\sinh\gamma l}\right] \qquad \ldots(i)$$

Here $Z_0 = 710\ \angle 14^\circ\ \Omega$, l = 100 km

$Z_R = 300\ \Omega$, $\gamma = 0.007 + j\ 0.028$ /km

Calculating cosh γl and sinh γl separately.

$$\cosh \gamma l = \cosh(\alpha + j\beta) l = \cosh(\alpha l + j\beta l)$$

$$= \cosh[(0.007 \times 100) + j(0.028 \times 100)]$$

$$= \cosh[0.7 + j\,2.8]$$

From trigonometric results,

cosh (A + j β) = cosh A cos β + j sin A sin β and

sinh (A + j β) = sinh A cos β 1 j cosh A sin β

Hence

$$\cosh[0.7 + j\,2.8] = \cosh 0.7 \cos 2.8 + j \sin h\, 0.7 \sin 2.8$$

$$= -1.1826 + j\,0.2541 \qquad \ldots \text{(ii)}$$

Similarly,

$$\sinh \gamma l = (\sin \alpha + j\beta) l = \sin h\,(\alpha l + j\beta l)$$

$$= \sin h\,(0.7 + j\,2.8)$$

But using trigonometric result, we can write,

$$\sinh(0.7 + j\,2.8) = \sinh 0.7 \cos 2.8 + j \cosh 0.7 \sin 2.8$$

$$= -0.7147 + j\,0.42 \qquad \ldots \text{(iii)}$$

Substituting values of Z_0, Z_R sinh γl and cosh γl in equation (i) we get,

$$Z_S = 710\angle 14^\circ \left[\frac{300(-1.1826 + j0.2541) + 710\angle 14^\circ(-0.7147 + j0.42)}{710\angle 14^\circ(-1.1826 + j0.2541) + 300(-0.7147 + j0.42)}\right]$$

$$= 710\angle 14^\circ \left[\frac{(300\angle 0^\circ)(1.2095\angle 167.87^\circ) + (710\angle 14^\circ)(0.8289\angle 149.55^\circ)}{(710\angle 14^\circ)(1.2095\angle 167.87^\circ) + (300\angle 0^\circ)(0.8289\angle 149.55^\circ)}\right]$$

$$= 710\angle 14^\circ \left[\frac{(362.85\angle 167.87^\circ) + (588.519\angle 163.55^\circ)}{(858.745\angle 181.87^\circ) + (248.67\angle 149.55^\circ)}\right]$$

$$= 710\angle 14^\circ \left[\frac{(-354.748 + j76.2458) + (-564.4292 + j166.6559)}{(-858.2876 - j28.0224) + (-214.3713 + j126.0225)}\right]$$

$$= 710\angle 14^\circ \left[\frac{-919.1772 + j242.9017}{-1072.6589 + j98}\right]$$

$$= 710 \angle 14^\circ \left[\frac{950.7302\angle 165.19^\circ}{1077.1263\angle 174.77^\circ}\right]$$

$$= 626.68 \angle 4^\circ \Omega$$

Review Questions

1. *State the various types of transmission lines used in practice.*
2. *Which are the important parameters of a transmission line ?*
3. *What is the difference between lumped parameters and distributed parameters ?*
4. *State the important properties of the infinite line.*
5. *Prove that a finite line terminated in its characteristic impedance behaves as an infinite line.*
6. *Find the expression for* Z_0 *interms of T section equivalent parameters for finite line.*
7. *For a transmission line terminated in* Z_0 *prove that*

   i) $Z_0 = \sqrt{Z_{OC} Z_{SC}}$

   ii) $\tanh(\gamma l) = \sqrt{\frac{Z_{SC}}{Z_{OC}}}$
8. *Obtain the expression for current and voltage at any point along a line which is terminated in* Z_0.
9. *Define attenuation constant and phase constant.*
10. *Derive the relationship between* γ, Z_{OC} *and* Z_{SC}.
11. *Define wavelength of the line.*
12. *Obtain an expression for the phase velocity of a line.*
13. *What is a group velocity ?*
14. *Derive the relationship between* Z_0 *and primary constants of a line.*
15. *Derive the expression for* γ *interms of primary constants of a line.*
16. *Derive the expressions for* α *and* β *interms of primary constants of a line.*
17. *What are the practical considerations for an underground cable ?*
18. *Starting from fundamental, derive the expression for voltage and current at any point on line which is at a distance 's' from the receiving end interms of receiving end voltage and current.*
19. *Explain the physical significance of a general solution of a transmission line.*
20. *Derive the expressions for input impedance and transfer impedance interms of* Z_0, Z_R *and* γ.
21. *Derive the condition for minimum attenuation with,*

    i) L variable and ii) C variable
22. *What are the conditions of R and G for minimum attenuation ?*
23. *Which are the various types of distortions in a line ?*
24. *What is distortionless line ? Derive the condition for distortionless line.*
25. *Write a note on telephone cable.*
26. *What is loading of lines ? Which are the two types of loading ?*

27. *State the advantages, disadvantages and application of following types of loading*

    i) Continuous loading and ii) Lumped loading

28. *Derive the expressions for α and β for continuously loaded line.*

29. *What is patch loading ?*

30. *Derive the Campbell's equation.*

31. *What is the importance of cut-off frequency for the lumped loaded line.*

32. *Explain the reflection on a line not terminated in Z_0.*

33. *What are disadvantages of reflection ?*

34. *What is reflection coefficient ?*

35. *Write a note on*

    i) Reflection loss and reflection factor and

    ii) Insertion loss

36. *What is return loss ?*

37. *Derive the expression for the insertion loss of a line.*

38. *A line has the following primary line constants :*

    $R = 100\ \Omega / km$, $G = 1.5 \times 10^{-6}$ mho / km

    $L = 0.001$ H / km, $C = 0.062$ μF / km.

    Find Z_0 for the line **[Ans. : $507 \angle -43° \Omega$]**

39. *A sample of field quad cable has the following primary line constants,*

    $R = 78\ \Omega / km$, $G = 62 \times 10^{-6}$ mho / km

    $L = 1.75$ mH / km, $C = 0.0945$ μF / km

    Find the following at a frequency of 1600 Hz,

    (i) Z_0 *(ii)* α *(iii)* β *(iv)* λ

    (v) v and (vi) time for the wave to travel 100 km down the line.

    [Ans. : $290 \angle -36.5° \Omega$, 0.179 N/km, 0.209 rad/km, 30 km, 47840 km/sec, 2.09 msec]

40. *For a typical open wire telephone cable the primary constants are,*

    $R = 10\ \Omega/km$, $L = 0.0037$ H/km, $C = 0.0083\ \mu F/km$, $G = 0.4 \times 10^{-6}$ mho/km

    Determine Z_0 and the propagation constant at a frequency of 1 kHz.

    [Ans. : 683 – j 138 Ω, 0.0074 + j 0.0356 /km]

41. *An open wire line has $Z_0 = 730 \angle -11° \Omega$ at 1000 Hz and $\gamma = 0.012 + j\ 0.058$. When 2 volts are applied to the sending end, a current of 4 mA flows. What will be the current at the distant end 50 km away ?* **[Ans. : $2.8 \angle 176.3°$ mA]**

42. *A line 10 km long has the following constants :*

$Z_0 = 600\angle 0°\ \Omega$, $\alpha = 0.1$ *nepers/km*, $\beta = 0.05$ *rad / km*

Find the received current when 20 mA are sent into one end and receiving end is short circuited. By what angle received current lags with respect to current sent ?

[Ans. : 13.66 mA, 22.3° lagging]

43. *The following measurements are made on a 25 km line at a frequency of 796 Hz*

$Z_{SC} = 3220\ \angle -79.29°\ \Omega$, $Z_{OC} = 1301\ \angle 76.67°\ \Omega$

Determine the primary constants of the line.

[Ans. : 11.83 Ω / km, 0.035 H / km, 0.919 μmho / km, 0.00835 μF / km]

2

Transmission Line at Radio Frequencies

2.1 Introduction

In the previous chapter, we have discussed the theory of transmission line, definitions of line parameters, general solution of a transmission line and physical significance of the equation. In this chapter, we shall discuss the line at a radio and power frequencies. For the radio frequency line working at a frequency of the range of megahertz and more than that, the standard assumptions are different than that studied in the previous chapter. For the radio frequency line of either open wire type or coaxial line type, the standard assumptions made for the analysis of the performance of the line are as follows.

1) At very high frequency, the **skin effect** is considerable. Hence it is assumed that the currents may flow on the surface of conductor. Then the internal inductance becomes zero.

2) It is observed that due to the skin effect, resistance R increases with $\sqrt{f}$. But the line reactance ωL increases directly with frequency f. Hence the second assumption is $\omega L >> R$.

3) The third assumption is that the line at radio frequency is constructed such that the leakage conductance G may be considered zero.

There are two considerations for the analysis of the line performances. First consideration is that R is slightly small with respect to ωL while the second one is that R is completely negligible as compared with ωL. If R is neglected completely, then such a line is termed as **zero dissipation line.** This concept is useful when the line is used for transmission of power at a high frequency and the losses are neglected completely. While if R is small, then such a line is termed as **small dissipation line.** In the applications where line is considered as a circuit element or properties of resonance are involved, this concept of small dissipation line is very much useful. In this chapter we will discuss properties and different applications of the **zero dissipation line** only.

2.2 Line Constants for Zero Dissipation Line (Dissipationless Line)

In general, the characteristic impedance (Z_0) and propagation constant (γ) of a line are given by,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}} \quad \text{... (1)}$$

and

$$\gamma = \sqrt{ZY} = \sqrt{(R+j\omega L)(G+j\omega C)} \quad \text{... (2)}$$

According to the standard assumptions for line at a high frequency,

$j\omega L >> R$ and $j\omega C >> G$

$$\therefore \quad Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} \quad \text{... (3)}$$

As the value of characteristic impedance is real and resistive, it is represented by symbol R_0,

$$\therefore \quad \boxed{Z_0 = R_0 = \sqrt{\frac{L}{C}}} \quad \text{... (4)}$$

Similarly the propagation constant γ is given by,

$$\gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC}$$

$$\therefore \quad \gamma = 0 + j\,\omega\sqrt{LC} \quad \text{... (5)}$$

But

$$\gamma = \alpha + j\beta$$

Hence at high frequencies,

$$\boxed{\begin{aligned} \alpha &= 0 \text{ and} \\ \beta &= \omega\sqrt{LC} \quad \text{radian/m} \end{aligned}} \quad \text{... (6)}$$

Then the velocity of propagation is given by,

$$\boxed{v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \quad \text{m/sec}} \quad \text{... (7)}$$

From equation (7), the velocity of propagation for open wire dissipationless line, separated by air, is same as the velocity of light in space.

The distance corresponding to the phase shift of 2π radians is called **wavelength** (λ). For the dissipationless line wave length is given by,

$$\boxed{\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} \quad m} \qquad \text{... (8)}$$

Example 2.1 *A line with zero dissipation has R = 0.006 Ω/m, L = 2.5μH / m and C = 4.45pF/m. If the line is operated at 10 MHz find*

i) Ro ii) α iii) β iv) v, v) λ

Solution : Given $R = 0.006\ \Omega/m$, $L = 2.5 \times 10^{-6}$ H/m, $C = 4.45$ pF/ m, $f = 10$ MHz. At $f = 10MHz$, $\omega L = 2\pi fL = 2\times\pi\times 10\times 10^6 \times 2.5\times 10^{-6} = 15.708\ \Omega$. Hence $\omega L >> R$ at 10 MHz. So according to standard assumption for the dissipationless line, we can neglect R.

i) The characteristic impedance is given by,

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{2.5\times 10^{-6}}{4.45\times 10^{-12}}} = 749.53\ \Omega$$

ii) The propagation constant is given by,

$$\gamma = \alpha + j\beta = 0 + j\omega\sqrt{LC}$$

Hence

$$\gamma = \alpha + j\beta = 0 + j(2\times\pi\times 10\times 10^6)\sqrt{2.5\times 10^{-6}\times 4.45\times 10^{12}}$$

$$\therefore \quad \gamma = \alpha + j\beta = \mathbf{0 + j\ 0.2095\ per\ m}$$

$\therefore$ Attenuation constant $= \alpha = 0$

Phase constant $= \beta = \mathbf{0.2095\ rad/m}$

iii) The velocity of propagation is given by,

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2.5\times 10^{-6}\times 4.45\times 10^{-12}}} = \mathbf{2.998\times 10^8\ m/sec}$$

iv) The wavelength is given by,

$$\gamma = \frac{2\pi}{\beta} = \frac{2\pi}{0.2095} = \mathbf{29.9913\ m}$$

2.3 Voltages and Currents on Dissipationless Line

Consider a transmission line of length l and terminated in Z_R as shown in the Fig. 2.1.

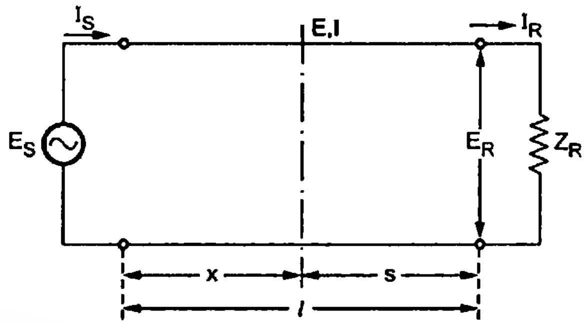


Fig. 2.1

In previous chapter we have obtained the expression for voltage E and current I at a distance x from sending end in terms of receiving end voltage E_R and receiving end current I_R.

The voltage E at a distance x from the sending end is given by,

$$E = E_R \cdot \cosh \gamma (l - x) + I_R \cdot Z_0 \sinh \gamma (l - x) \qquad \text{... (1)}$$

Putting $(l - x) = s$, equation (1) reduces to,

$$E = E_R \cosh \gamma s + I_R \cdot Z_0 \sinh \gamma s \qquad \text{... (2)}$$

But at very high frequencies,

$$Z_0 = R_0 \quad \text{and} \quad \gamma = j\beta$$

Hence equation (2) can be rewritten as,

$$E = E_R \cosh (j\beta s) + I_R \, R_0 \sinh (j\beta s)$$

$$\therefore \quad E = E_R \left[\frac{e^{j\beta s} + e^{-j\beta s}}{2}\right] + j\, I_R \, R_0 \left[\frac{e^{j\beta s} - e^{-j\beta s}}{2}\right]$$

$$\therefore \quad E = E_R \cos(\beta s) + j\, I_R \cdot R_0 \sin(\beta s) \qquad \text{... (3)}$$

Above equation represents a voltage in terms of receiving end voltage and current, at a point distance 's' away from receiving end.

Similarly for current at a point distance 's' away from receiving end is given by,

$$I = I_R \cos(\beta s) + j \frac{E_R}{R_0} \sin(\beta s) \qquad \text{... (4)}$$

But $\quad \beta = \dfrac{2\pi}{\lambda}$

Hence equations (3) and (4) reduce to,

$$E = E_R \cos\frac{2\pi s}{\lambda} + j\, I_R \cdot R_0 \sin\frac{2\pi s}{\lambda} \qquad \dots (5)$$

and

$$I = I_R \cos\frac{2\pi s}{\lambda} + j\frac{E_R}{R_0} \sin\frac{2\pi s}{\lambda} \qquad \dots (6)$$

From equations (5) and (6) it is clear that the voltage and current distribution is the sum of cosine and sine distributions.

Let us consider different conditions at the receiving end.

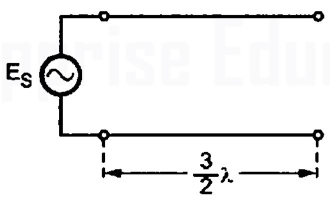
1) When line is open circuited at the receiving end, $I_R = 0$. Then the expressions for voltage and current at a point, distance 's' away from the receiving end are given by

$$E_{OC} = E_R \cos\frac{2\pi s}{\lambda} \qquad \dots (7)$$

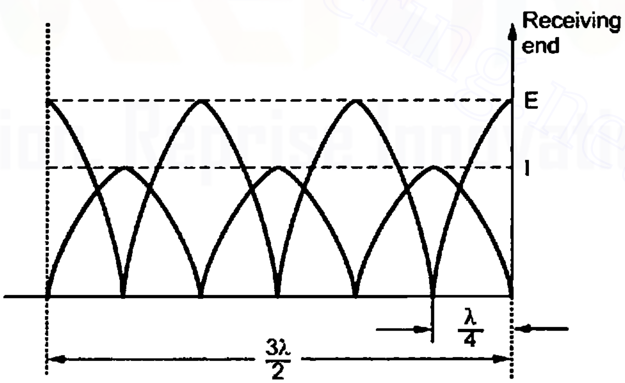
$$I_{OC} = j\frac{E_R}{R_0} \sin\frac{2\pi s}{\lambda} \qquad \dots (8)$$

From above equations it is clear that current and voltage are in quadrature.

The magnitudes of voltage and current distributions for an open circuited line $\frac{3}{2}$ wavelengths long are as shown in the Fig. 2.2 (b). At every $\frac{\lambda}{4}$ distance, voltage changes from maximum to minimum or vice versa, and so the current also.



(a) $\frac{3\lambda}{2}$ long line open circuited at the receiving end



(b) Voltage and current on a line open circuited at the receiving end

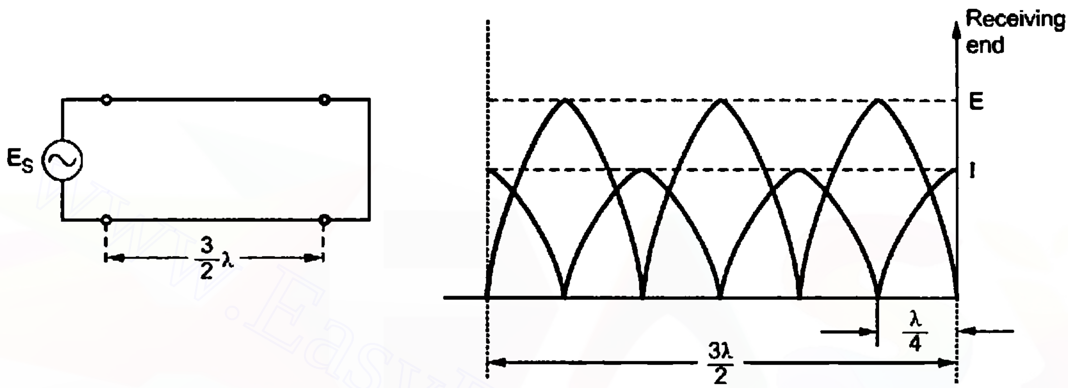
Fig. 2.2

2) When the line is short circuited at the receiving end, then $E_R = 0$. Then the expressions for voltage and current at a point, distance 's' away from the receiving end are given by

$$E_{SC} = j I_R \cdot R_0 \cdot \sin \frac{2\pi s}{\lambda} \quad \text{... (9)}$$

$$I_{SC} = I_R \cos \frac{2\pi s}{\lambda} \quad \text{... (10)}$$

Then again the magnitudes of voltage and current distributions for a short circuited line $\frac{3}{2}$ wavelengths long are as shown in the Fig. 2.3 (b).



(a) $\frac{3\lambda}{2}$ long line short circuited at the receiving end

(b) Voltage and current on a line short circuited at the receiving end

Fig. 2.3

3) When a line is terminated in an impedance $Z_R = R_0$, the reflection coefficient is given by

$$K = \frac{Z_R - R_0}{Z_R + R_0} = \frac{R_0 - R_0}{R_0 + R_0} = 0$$

That means the reflected wave is absent. Then the voltage and current on the line are given by

$$E = E_R \cdot e^{j\beta s} \quad \text{... (11)}$$

and

$$I = I_R \cdot e^{j\beta s} \quad \text{... (12)}$$

From equations (11) and (12) it is clear that both voltage and current have constant magnitude with zero attenuation ; only continuous varying phase angle along the line.

The magnitudes of voltage and current distributions are represented in the Fig. 2.4

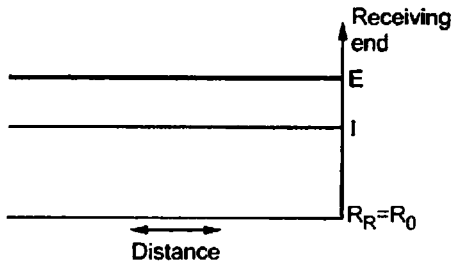
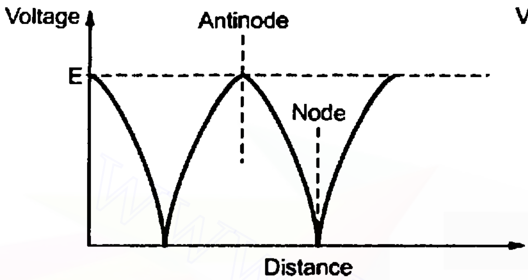


Fig. 2.4 Voltage and current on a line properly terminated in R_0 at the receiving end

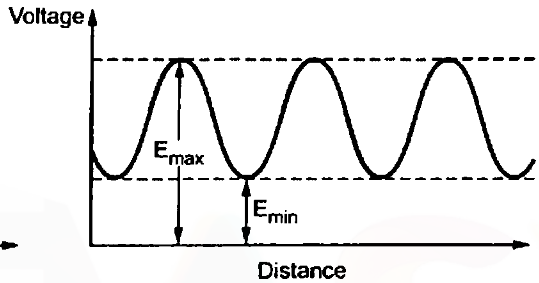
2.4 Standing Waves

According to the discussion in the previous section, if a line is either open circuited or short circuited at the receiving end, we get nodes and antinodes in voltage distribution as shown in the Fig. 2.5 (a).

If a line is terminated in a load other than R_0, the distribution of voltage at a point along the length of the line consists maximum and minimum values of voltage as shown in the Fig. 2.5 (b).



(a) Standing waves on open or shorted line



(b) Standing waves on a line terminated in a load not equal to R_0

Fig. 2.5 Standing waves on a line

We know that voltage and current are in quadrature, thus it is obvious that the magnitude of the current along the line would be same except for a $\lambda/4$ shift in a position of maxima and minima.

The points along the line where magnitude of voltage or current is zero are called **Nodes** while the points along the lines where magnitude of voltage or current is maximum are called **Antinodes** or **Loops.** The nodes and antinodes are as shown in the Fig. 2.5 (a).

When a line is terminated in R_0, the standing waves are absent, such a line is called **smooth line.**

2.5 Standing Wave Ratio (S)

The ratio of the maximum to minimum magnitudes of voltages or currents on a line having standing waves is called **standing wave ratio** and it is denoted by S.

The standing wave ratio (S) is given by,

$$\boxed{S = \frac{|E_{max}|}{|E_{min}|} = \frac{|I_{max}|}{|I_{min}|}} \quad \text{... (1)}$$

When line is not terminated properly, standing waves are produced. Then the total power absorption is not possible in such case. The standing wave ratio S is measured

using RF voltmeter across the line at a point. Then the ratio of E_{max} to E_{min} is referred as **voltage standing wave ratio (VSWR).** Similarly the ratio of I_{max} to I_{min} can be measured using **RF Ammeter in series** with the line at a point. Then such ratio is referred as **current standing wave ratio (ISWR).** But in practice, ISWR calculation is very impractical because for this one has to cut the line, insert RF ammeter and then rejoin the line. Hence practically only VSWR measurement is done. So it is understood that VSWR is nothing but SWR. Theoretically the value of S lies between 1 and ∞.

2.5.1 Relation between Standing Wave Ratio (S) and Magnitude of Reflection Coefficient (K)

The standing wave ratio bears a simple relationship with the magnitude of the reflection coefficient i.e. $|K|$.

Along the line, at a point, if the incident and reflected waves are in phase and added directly, we get voltage maxima at that point.

Let E^+ = Magnitude of the incident wave

E = Magnitude of the reflected wave

Then the magnitude of voltage maxima is given by,

$$|E_{max}| = |E^+| + |E^-| \quad \text{... (1)}$$

Similarly along the line, at a point, if the incident and reflected waves are out of phase and subtracted directly, we get voltage minima at that point.

The magnitude of voltage minima is given by,

$$|E_{min}| = |E^+| - |E^-| \quad \text{... (2)}$$

Then the standing wave ratio is given by,

$$S = \frac{|E_{max}|}{|E_{min}|} = \frac{|E^+| + |E^-|}{|E^+| - |E^-|}$$

$$\therefore \quad S = \frac{1 + \frac{|E^-|}{|E^+|}}{1 - \frac{|E^-|}{|E^+|}} \quad \text{... (3)}$$

But the ratio $\frac{|E^-|}{|E^+|}$ is nothing but the magnitude of the reflection coefficient,

$$|K| = \frac{|E^-|}{|E^+|}$$

Then,
$$\boxed{S = \frac{1+|K|}{1-|K|}} \qquad \dots (4)$$

or
$$\boxed{|K| = \frac{S-1}{S+1} = \frac{|E_{max}|-|E_{min}|}{|E_{max}|+|E_{min}|}} \qquad \dots (5)$$

From equations (4) and (5), it is possible to calculate value of S and |K| from measurements of maximum and minimum voltages on the line. Similar expressions may be obtained by considering maximum and minimum currents on the line.

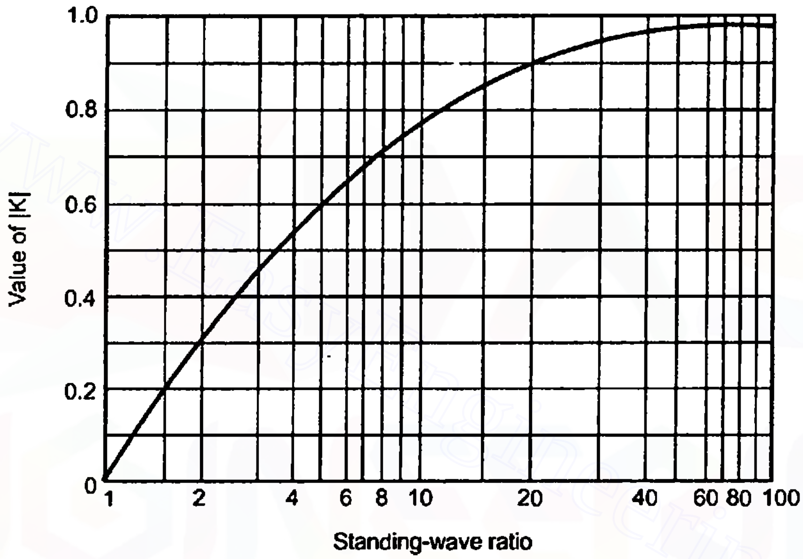


Fig. 2.6 Relation between standing wave ratio and magnitude of reflection coefficient

$$|K| = \frac{R_R - R_0}{R_R + R_0}$$

$$\therefore \qquad S = \frac{1+|K|}{1-|K|} = \frac{1+\left[\frac{R_R - R_0}{R_R + R_0}\right]}{1-\left[\frac{R_R - R_0}{R_R + R_0}\right]} \qquad \dots (6)$$

Then if $R_R > R_0$,

$$S = \frac{R_R}{R_0}$$

and if $R_R < R_0$

$$S = \frac{R_0}{R_R}$$

In general, for resistive load R_R,

$$S = \frac{1+|K|}{1-|K|} = \frac{1+\left[\frac{R_R - R_0}{R_R + R_0}\right]}{1-\left[\frac{R_R - R_0}{R_R + R_0}\right]} = \frac{R_R}{R_0} \quad \text{... (7)}$$

2.5.2 Relation between Standing Wave Ratio and Reflection Coefficient

The voltage at a point distance s away from the receiving end is given by,

$$E = E_R \cdot \cosh(j\beta s) + I_R \cdot Z_0 \sinh(j\beta s) \quad \text{... (1)}$$

$$\therefore \quad E = E_R \frac{e^{j\beta s}+e^{-j\beta s}}{2} + I_R \cdot Z_0 \frac{e^{j\beta s}-e^{-j\beta s}}{2}$$

$$\therefore \quad E = \frac{e^{j\beta s}}{2}[E_R + I_R \cdot Z_0] + \frac{e^{-j\beta s}}{2}[E_R - I_R \cdot Z_0]$$

$$\therefore \quad E = \frac{e^{j\beta s}}{2}[I_R \cdot Z_R + I_R \cdot Z_0] + \frac{e^{-j\beta s}}{2}[I_R \cdot Z_R - I_R \cdot Z_0]$$

$$\therefore \quad E = I_R \cdot \frac{e^{j\beta s}}{2}[(Z_R + Z_0) + e^{-j2\beta s}(Z_R - Z_0)]$$

$$\therefore \quad E = I_R \frac{(Z_R + Z_0)}{2} e^{j\beta s}\left[1 + \frac{Z_R - Z_0}{Z_R + Z_0} e^{-j2\beta s}\right] \quad \text{... (2)}$$

But $$K = \frac{Z_R - Z_0}{Z_R + Z_0} = |K| \angle\phi \text{ say as k is complex,}$$

$$\therefore \quad E = I_R \frac{Z_R + Z_0}{2} e^{j\beta s}[1 + |K| e^{j\phi} \cdot e^{-j2\beta s}]$$

$$\therefore \quad E = I_R \frac{Z_R + Z_0}{2} e^{j\beta s}[1 + |K| e^{j(\phi - 2\beta s)}]$$

$$\therefore \quad E = I_R \cdot \frac{Z_R + Z_0}{2} e^{j\beta s}[1\angle 0 + |K| \angle\phi - 2\beta s] \quad \text{... (3)}$$

In above equation (3), the first term represents voltage in the incident wave while the second term represents voltage in the reflected wave.

Thus the voltage E at any point is the vector sum of voltages in incident and reflected wave. This voltage will be maximum when both, the incident and reflected waves, are in phase. When both waves are in phase, their phase angles will be same.

Thus for E_{max},

$$0 = \phi - 2\beta s \quad \text{... (4)}$$

Then equation (3) is modified as,

$$E_{max} = I_R \frac{Z_R + Z_0}{2} e^{j\beta s} [1 \angle 0 + |K| \angle 0]$$

$$\therefore \quad E_{max} = I_R \cdot \frac{Z_R + Z_0}{2} e^{j\beta s} [1 + |K|] \quad \text{... (5)}$$

When the incident wave and reflected wave are out of phase, we get minimum voltage. Then the difference of angles of the two waves is π.

Thus for E_{min},

$$0 + \pi = \phi - 2\beta s \quad \text{... (6)}$$

Then equation (3) is modified as,

$$E_{min} = I_R \frac{Z_R + Z_0}{2} e^{j\beta s} [1 \angle 0 + |K| \angle \pi]$$

$$\therefore \quad E_{min} = I_R \frac{Z_R + Z_0}{2} e^{j\beta s} [1 \angle 0 - |K|]$$

$$\therefore \quad E_{min} = I_R \frac{Z_R + Z_0}{2} e^{j\beta s} [1 - |K|] \quad \text{... (7)}$$

Hence from equations (4) and (7), the standing wave ratio S can be determined as,

$$\boxed{S = \frac{E_{max}}{E_{min}} = \frac{1 + |K|}{1 - |K|}} \quad \text{... (8)}$$

Example 2.2 *A certain transmission line, working at radio frequencies, has following constants.*

L = 9 μH/m , C = 16 pF/m

The line is terminated in a resistive load of 1000 Ω. Find the reflection coefficient and standing wave ratio.

Solution :

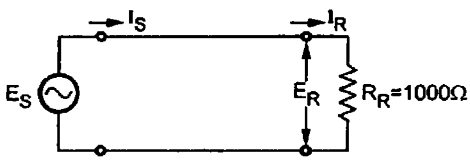


Fig. 2.7

The characteristic impedance of line is given by,

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{9 \times 10^{-6}}{16 \times 10^{-12}}} = 750\ \Omega \quad \text{... at R.F. only}$$

Hence reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{1000-750}{1000+750} = 0.1428$$

The standing wave ratio S is given by,

$$S = \frac{1+|K|}{1-|K|} = \frac{1+0.1428}{1-0.1428} = 1.3333$$

Example 2.3 *A certain R.F. transmission line is terminated in pure resistive load. The characteristic impedance of the line is 1200 Ω and the reflection coefficient was observed to be 0.2. Calculate the terminating load, which is less than characteristic impedance.*

Solution : At radio frequencies, $S = \frac{R_0}{R_R} = \frac{1200}{R_R}$ for resistive load

$$K = \frac{s-1}{s+1} = 0.2$$

$$\therefore \quad K = \frac{\frac{1200}{R_R}-1}{\frac{1200}{R_R}+1} = 0.2$$

Simplifying for R_R, we get

$$R_R = 800 \ \Omega$$

Example 2.4 *Calculate standing wave ratio and reflection coefficient on a line having* $Z_0 = 300\ \Omega$ *and terminated in* $Z_R = 300 + j\,400$.

Solution : The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{(300+j\,400)-(300)}{(300+j\,400)+(300)} = \frac{j\,400}{600+j\,400}$$

$$\therefore \quad K = \frac{400\angle 90^\circ}{721.11\angle 33.69^\circ}$$

$$\therefore \quad K = 0.5547 \angle 56.31^\circ$$

The standing wave ratio S is given by

$$S = \frac{1+|K|}{1-|K|}$$

$$= \frac{1+0.5547}{1-0.5547}$$

$$= 3.4913$$

2.6 Input Impedance of the Dissipationless Line

The expressions for sending end voltage and current at a distance s from receiving end for a line of length s are given by

$$E_S = E_R \cos\beta s + j I_R R_0 \sin\beta s$$

and

$$I_S = I_R \cos\beta s + j\frac{E_R}{R_0} \sin\beta s$$

Consider a line of length s and terminated in Z_R as shown in the Fig. 2.8.

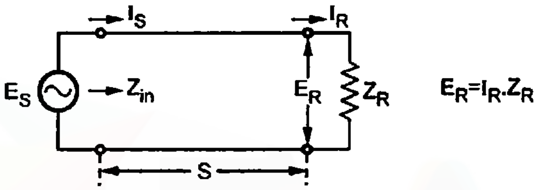


Fig. 2.8 A line of length s and terminated in Z_R

The input impedance of such a line is given by,

$$Z_{in} = \frac{E_S}{I_S} = \frac{E_R \cos\beta s + j I_R \cdot R_0 \sin\beta s}{I_R \cos\beta s + j\frac{E_R}{R_0}\sin\beta s}$$

$$= R_0\left[\frac{E_R \cos\beta s + j I_R \cdot R_0 \sin\beta s}{I_R\ R_0 \cos\beta s + j E_R \sin\beta s}\right]$$

$$= R_0\left[\frac{E_R + j I_R \cdot R_0 \tan\beta s}{I_R\ R_0 + j E_R \tan\beta s}\right]$$

$$= R_0\left[\frac{\frac{E_R}{I_R} + j R_0 \tan\beta s}{R_0 + j\frac{E_R}{I_R}\tan\beta s}\right]$$

$$\therefore \quad Z_{in} = R_0\left[\frac{Z_R + j R_0 \tan\beta s}{R_0 + j Z_R \tan\beta s}\right] \quad \text{... (1)}$$

From equation (1), it is clear that in general the input impedance Z_{in} is complex.

Another convenient form of the input impedance is obtained as follows.

$$Z_{in} = R_0\left[\frac{Z_R + j R_0 \tan\beta s}{R_0 + j Z_R \tan\beta s}\right]$$

$$= R_0 \left[\frac{Z_R \cos\beta s + j R_0 \sin\beta s}{R_0 \cos\beta s + j Z_R \sin\beta s} \right]$$

Writing numerator and denominator in exponential forms and rearranging,

$$Z_{in} = R_0 \left[\frac{1 + \frac{Z_R - Z_0}{Z_R + Z_0} e^{-j2\beta s}}{1 - \frac{Z_R - Z_0}{Z_R + Z_0} e^{-j2\beta s}} \right] \quad \text{... (2)}$$

But as we know,

$K = \dfrac{Z_R - Z_0}{Z_R + Z_0}$ where K is reflection coefficient and writing in modulus-angle form,

$$Z_{in} = R_0 \left[\frac{1 + |K| \angle\phi \cdot e^{-2\beta s}}{1 - |K| \angle\phi \cdot e^{-j2\beta s}} \right]$$

$$= R_0 \left[\frac{1 + |K| \underline{/\phi - 2\beta s}}{1 - |K| \underline{/\phi - 2\beta s}} \right] \quad \text{... } e^{-j2\beta s} = -[\cos(2\beta s) + j\sin(2\beta s)]$$

$$= 1 \angle -2\beta s$$

As Z_{in} is the input impedance measured at the sending end hence it is also termed as Z_S.

$$\therefore \quad Z_{in} = Z_S = R_0 \left[\frac{1 + |K| \underline{/\phi - 2\beta s}}{1 - |K| \underline{/\phi - 2\beta s}} \right] \quad \text{... (3)}$$

The input impedance will be maximum at a distances,

$$\therefore \quad \phi = 2\beta s \quad \text{or} \quad \phi - 2\beta s = 0,$$

$$\text{or} \quad s = \frac{\phi}{2\beta} \quad \text{... (4)}$$

Then

$$Z_{S(max)} = R_0 \left[\frac{1 + |K|}{1 - |K|} \right] = S R_0$$

where S is standing wave ratio, thus $Z_{S\,(max)}$ becomes resistive.

Along a line if we travel a distance of $\frac{\lambda}{4}$ from the point where impedance is maximum, we get a point of minimum impedance.

Hence input impedance will be minimum if

$$s = \frac{\phi}{2\beta} + \frac{\lambda}{4} \quad \text{... (5)}$$

$$\therefore \quad s = \frac{\phi}{2\beta} + \frac{1}{4}\left(\frac{2\pi}{\beta}\right) \quad \text{... } \lambda = \frac{2\pi}{\beta}$$

$$\therefore \quad s = \frac{\phi + \pi}{2\beta}$$

or

$$2\beta s = \phi + \pi \quad \text{... (6)}$$

Then

$$Z_{S(min)} = R_0 \left[\frac{1 + |K| \underline{/\phi - (\phi + \pi)}}{1 - |K| \underline{/\phi - (\phi + \pi)}}\right]$$

$$= R_0 \left[\frac{1 + |K| \angle -\pi}{1 - |K| \angle -\pi}\right]$$

$$= R_0 \left[\frac{1 - |K|}{1 + |K|}\right]$$

$$= \frac{R_0}{S}$$

where S is standing wave ratio. Thus $Z_{S(min)}$ is also resistive.

2.7 Input Impedance of Open and Short Circuited Lines

Consider equation (1) obtained in the section 2.6.

$$Z_S = Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan \beta s}{R_0 + j Z_R \tan \beta s}\right]$$

Let us find input impedance of a line, open circuited and short circuited at the receiving end separately.

2.7.1 Input Impedance of Short Circuited Line

If a line is short circuited at the receiving end,

then

$$Z_R = 0$$

Then the input impedance is given by,

$$\therefore \quad Z_{SC} = Z_S = R_0 \left[\frac{j\,R_0 \tan\beta s}{R_0}\right] = j\,R_0 \tan\beta s \qquad \text{... (1)}$$

But $\quad \beta = \dfrac{2\pi}{\lambda}$ then,

$$Z_S = j\,R_0 \tan\left(\frac{2\pi s}{\lambda}\right) \qquad \text{... (2)}$$

As Z_S is purely reactive, let it be denoted by X_S.

$$\therefore \quad Z_S = j\,X_S = j\,R_0 \tan\left(\frac{2\pi s}{\lambda}\right)$$

$$\therefore \quad \frac{X_S}{R_0} = \tan\left(\frac{2\pi s}{\lambda}\right) \qquad \text{... (3)}$$

Above ratio gives normalized value of reactance for a short circuited line.

Let us calculate the ratio $\dfrac{X_S}{R_0}$ for different values of s.

When $\quad s = 0, \quad \tan\left(\dfrac{2\pi\cdot 0}{\lambda}\right) = 0 \qquad \therefore \quad \dfrac{X_S}{R_0} = 0,$

$$s = \frac{\lambda}{4}, \quad \tan\left(\frac{2\pi\cdot\frac{\lambda}{4}}{\lambda}\right) = \tan\left(\frac{\pi}{2}\right) = \infty \qquad \therefore \quad \frac{X_S}{R_0} = \infty,$$

$$s = \frac{\lambda}{2}, \quad \tan\left(\frac{2\pi\cdot\frac{\lambda}{2}}{\lambda}\right) = \tan(\pi) = 0 \qquad \therefore \quad \frac{X_S}{R_0} = 0,$$

$$s = \frac{3\lambda}{4}, \quad \tan\left(\frac{2\pi\cdot\frac{3\lambda}{4}}{\lambda}\right) = \tan\left(\frac{3\pi}{2}\right) = \infty \qquad \therefore \quad \frac{X_S}{R_0} = \infty, \qquad \text{and}$$

$$s = \lambda, \quad \tan\left(\frac{2\pi\cdot\lambda}{\lambda}\right) = \tan(2\pi) = 0 \qquad \therefore \quad \frac{X_S}{R_0} = 0$$

The graph of variation of $\frac{X_S}{R_0}$ for a short circuited line with various lengths of lines is as shown in the Fig. 2.9.

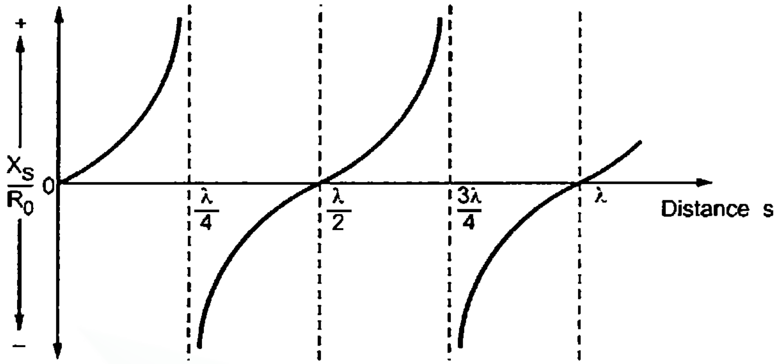


Fig. 2.9 Variation of $\left(\frac{X_S}{R_0}\right)$ for a shorted line with various lengths of lines

2.7.2 Input Impedance of Open Circuited Line

Rearranging the expression for input impedance as follows,

$$Z_S = R_0 \left[\frac{1 + j\frac{R_0}{Z_R} \tan\beta s}{\frac{R_0}{Z_R} + j\tan\beta s} \right]$$

When a line is open circuited at the receiving end,

$$Z_R = \infty$$

Then the input impedance is given by

$$Z_S = R_0 \left[\frac{1}{j\tan\beta s} \right] = \frac{-j R_0}{\tan\beta s} = -j R_0 \cot\beta s \qquad \dots (1)$$

But $$\beta = \frac{2\pi}{\lambda}$$

$$\therefore \quad Z_{OC} = Z_S = -j R_0 \cot\left(\frac{2\pi s}{\lambda}\right) \qquad \dots (2)$$

Again $Z_{OC} = Z_S$ is purely reactive, let it be denoted by X_S

$$\therefore \quad j X_S = -j R_0 \cot\left(\frac{2\pi s}{\lambda}\right)$$

$$\therefore \quad \frac{X_S}{R_0} = -\cot\left(\frac{2\pi s}{\lambda}\right) \qquad \dots (3)$$

Let us calculate the value of ratio $\frac{X_S}{R_0}$ for different values of s.

when $s = 0, \quad \cot\left(\frac{2\pi \cdot 0}{\lambda}\right) = \cot(0) = \infty, \quad \therefore \frac{X_S}{R_0} = -\infty$

$s = \frac{\lambda}{4}, \quad \cot\left(\frac{2\pi \cdot \lambda}{\lambda \; 4}\right) = \cot\left(\frac{\pi}{2}\right) = 0, \quad \therefore \frac{X_S}{R_0} = 0$

$s = \frac{\lambda}{2}, \quad \cot\left(\frac{2\pi}{\lambda} \cdot \frac{\lambda}{2}\right) = \cot(\pi) = \infty, \quad \therefore \frac{X_S}{R_0} = \infty$

$s = \frac{3\lambda}{4}, \quad \cot\left(\frac{2\pi}{\lambda} \cdot \frac{3\lambda}{4}\right) = \cot\left(\frac{3\pi}{2}\right) = 0, \quad \therefore \frac{X_S}{R_0} = 0$ and

$s = \lambda, \quad \cot\left(\frac{2\pi}{\lambda} \cdot \lambda\right) = \cot(2\pi) = \infty, \quad \therefore \frac{X_S}{R_0} = \infty$

The graph of variation of $\frac{X}{R_0}$ for an open circuited line with various lengths of lines is as shown in the Fig. 2.10.

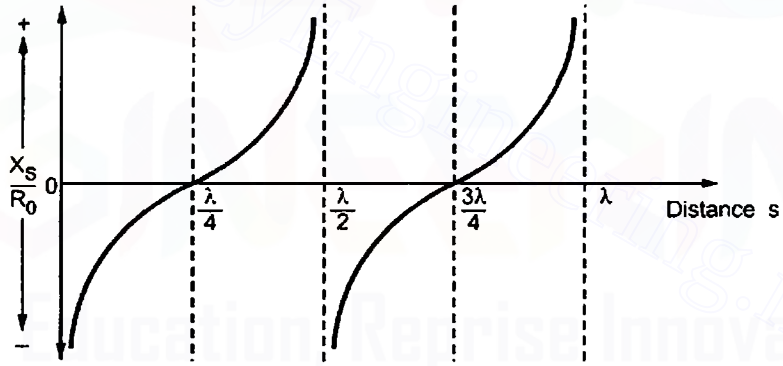


Fig. 2.10 Variation of $\left(\frac{X_S}{R_0}\right)$ for a shorted line with various lengths of lines

According to the discussion in the last two sections, it is clear that the input impedance of a line either open circuited or short circuited is pure reactive in nature. It is also observed that the value of reactance is repeated after every $s = \frac{\lambda}{2}$ period. For first quarter wavelength, short circuited line acts as an inductance while the open circuited line acts as a capacitance. After each quarter wavelength, the nature of reactances reverses. These curves as shown in Fig. 2.9 and Fig. 2.10 are for ideal dissipationless line. In practical line, zero or infinite impedances can not be achieved because of small resistive component indicating some power loss. Thus practically the values of impedances tend to maxima and minima.

When transmission line is open circuited at its end, the current is always zero and voltage is maximum at the open end. After every half wavelength ($\lambda / 2$) distance, these conditions of voltage and current will repeat themselves. When there is a voltage maxima, there is current minima and at voltage minima we get current maxima. This repeats at every $\frac{\lambda}{4}$ distance from open end. From this it is clear that the input impedance varies all along the length of a line. The nature of the input impedance is also varying such as low resistance, high resistance, inductive reactance or capacitive reactance. These characteristics are similar to those of resonant circuit. Hence mismatched lines are called **resonant lines.**

The variation of input impedance of an open circuited line is as shown in the Fig. 2.11 (a).

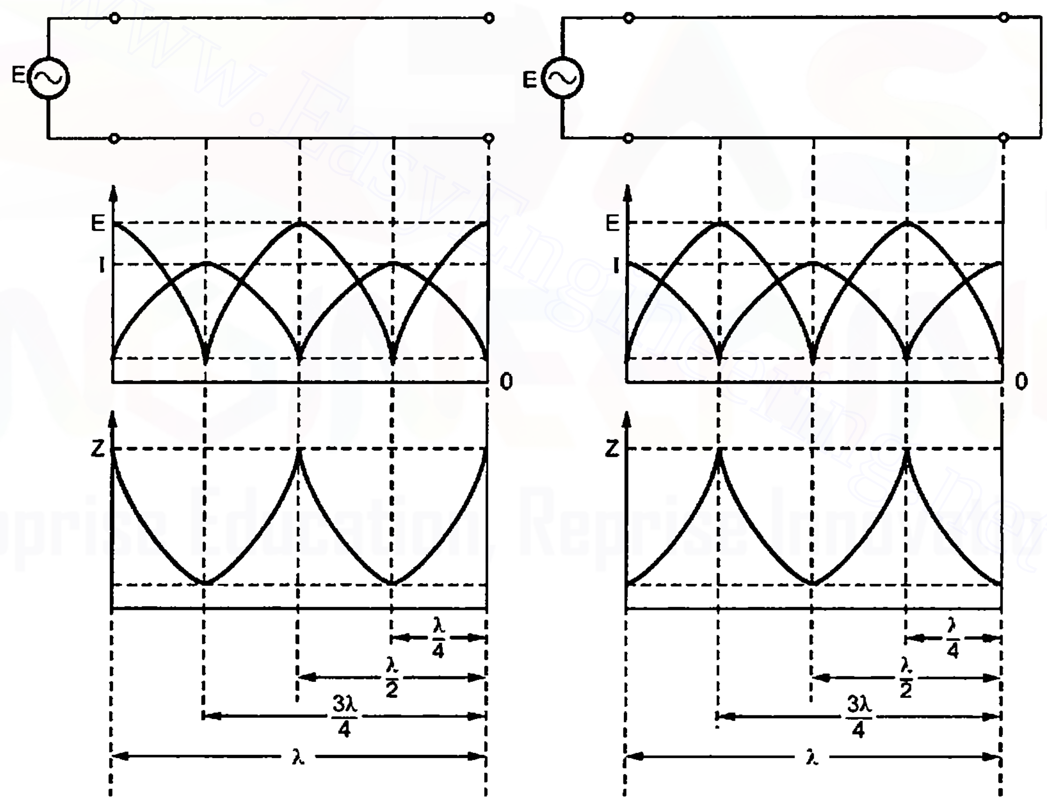


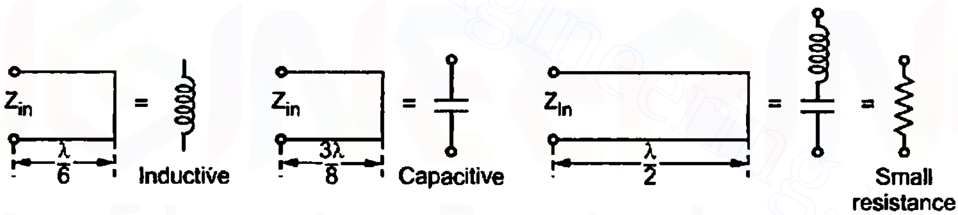
Fig. 2.11 Input impedance variation along open circuited and short circuited line

For open circuited line, as terminating end is open, the impedance is high. The line acts as parallel resonant circuit. But quarter wavelength (λ / 4) back, the input impedance is low resistance. The line acts as series resonant circuit. In between the two such points, the input impedance is capacitive. Between the quarter and half wavelength, the input impedance is inductive.

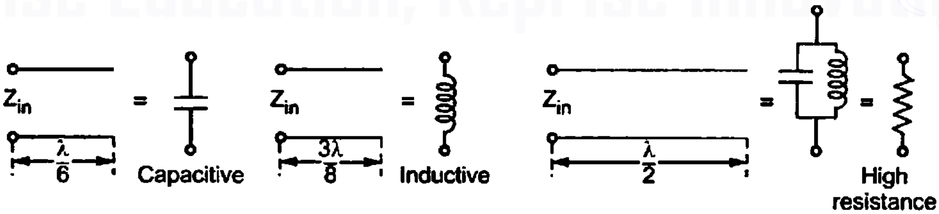
When a line is shorted at terminating end, the voltage is zero and current is maximum at that end. Similar to open circuited line, these voltage and current will repeat themselves at half wavelength intervals back from the short circuit. Refer Fig. 2.11 (b). The standing waves on short circuited lines are displaced by a distance equivalent to quarter of a wavelength compared to waves on the open circuited line.

At the short circuit end, the input impedance is low, thus line acts as a series resonant circuit. A quarter wavelength (λ / 4) distance back, the input impedance is high resistance, hence the line acts as parallel resonant circuit. In between two points, the input impedance is inductive. Between a quarter wavelength and half wavelength the input impedance is capacitive.

It is observed that for a given length back from the end, if open and short circuited lines are compared, then the reactances are opposite to one another. The following diagrams illustrate the input impedance presented to the generator by the different lengths of open and short circuited lines.



(a) Input impedance for different wavelengths of short circuited line



(b) Input impedance for different wavelengths of open circuited lines

Fig. 2.12

2.8 Power and Impedance Measurement on Line

The expressions for voltage and current on the dissipationless line are given by,

$$E = I_R \cdot \frac{Z_R + R_0}{2} (1 + |K| \angle \phi - 2\beta s) \quad \text{... (1)}$$

and

$$I = I_R \cdot \frac{Z_R + R_0}{2 R_0} (1 - |K| \angle \phi - 2\beta s) \quad \text{... (2)}$$

Now consider two phasors A and B representing voltage E and current I respectively as shown in the Fig. 2.13.

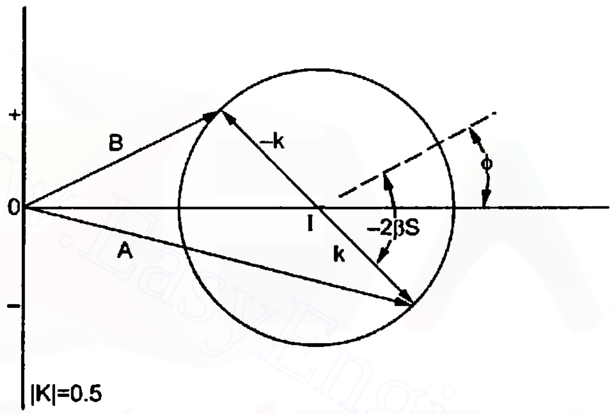


Fig. 2.13 Phasor representation of E and I

We have already studied that at a point where the voltage is maximum, the incident and reflected voltage waves are in phase. Hence, maximum voltage is given by,

$$\therefore \quad E_{max} = \frac{I_R |Z_R + R_0|}{2} (1 + |K|) \quad \text{... (3)}$$

Similarly for current, the incident and reflected current waves are in phase at the current maximum point. Hence, maximum current is given by,

$$I_{max} = \frac{I_R |Z_R + R_0|}{2 R_0} (1 + |K|) \quad \text{... (4)}$$

Hence,

$$\frac{E_{max}}{I_{max}} = R_0 \quad \text{... (5)}$$

Similar condition may be obtained for minimum values of current and voltage with only phase reversal of reflected wave. The phase reversal will be same for E_{min} as well as I_{min}.

Hence $$\frac{E_{min}}{I_{min}} = R_0 \quad \text{... (6)}$$

From Fig. 2.13 and the theory of standing wave discussed earlier, it is clear that a voltage maximum and current minimum occur at the same point on a line. Thus the impedance looking into line towards load is purely resistive. The expression for minimum value of current is given by

$$I_{min} = \frac{I_R |Z_R + R_0|}{2 R_0} (1-|K|) \quad \text{... (7)}$$

Dividing equation (3) by (7), we get,

$$\frac{E_{max}}{I_{min}} = R_0 \left(\frac{1+|K|}{1-|K|}\right) = S R_0 \quad \text{... (8)}$$

This impedance is the impedance in voltage loop represented as R_{max}

$$\therefore \quad \boxed{\frac{E_{max}}{I_{min}} = S\ R_0 = R_{max}} \quad \text{... (9)}$$

Similarly, the expression for minimum value of voltage is given by,

$$E_{min} = \frac{I_R |Z_R + R_0|}{2} (1-|K|) \quad \text{... (10)}$$

Dividing equation (10) by (4), we get,

$$\frac{E_{min}}{I_{max}} = R_0 \left(\frac{1-|K|}{1+|K|}\right) = \frac{R_0}{S} \quad \text{... (11)}$$

This impedance is the impedance in current loop represented by R_{min}

$$\therefore \quad \boxed{\frac{E_{min}}{I_{max}} = R_{min} = \frac{R_0}{S}} \quad \text{... (12)}$$

The effective power flowing into a resistance R_{max} is the power passing voltage loop at voltage E_{max}. This power is given by

$$P = \frac{E_{max}^2}{R_{max}} \quad \text{... (13)}$$

Since there is dissipation in the line same power must flow into the resistance R_{min} in the current loop at voltage E_{min}. This power is given by,

$$P = \frac{E_{min}^2}{R_{min}} \qquad \text{... (14)}$$

Multiplving equations (13) and (14),

$$P^2 = \frac{E_{max}^2 \cdot E_{min}^2}{R_{max} \cdot R_{min}} \qquad \text{... (15)}$$

If we substitute values of E_{max}, E_{min}, R_{max} and R_{min} from equations (1), (10), (9), (12) respectively, the power flow along the line is given by

$$\boxed{P = \frac{(|E_{max}| \cdot |E_{min}|)}{R_0}} \qquad \text{... (16)}$$

Similarly,

$$\boxed{P = (|I_{max}| \cdot |I_{min}|) \cdot R_0} \qquad \text{... (17)}$$

From above equations (16) and (17), power flow measurement on a line is possible. It is clear that the greatest or maximum power will be transmitted if the magnitudes of E_{max} and E_{min} are same. This is possible only when a line is smooth line i.e. without standing waves and terminated properly in R_0.

By carrying out standing wave measurements on the open wire line, an unknown load impedance value can be obtained. Generally for carrying such measurements some form of bridge circuits are used. It is necessary that R_0 i.e. characteristic impedance of a line must be known or calculated.

The impedance is minimum at a point where voltage is also minimum.

$$\therefore \qquad Z_S = R_{min} = \frac{R_0}{S}$$

At any point on the line, the input impedance is given by,

$$Z_S = R_0 \left[\frac{Z_R + j R_0 \tan\left(\frac{2\pi s}{\lambda}\right)}{R_0 + j Z_R \tan\left(\frac{2\pi s}{\lambda}\right)} \right] \qquad \text{... (18)}$$

At the point of voltage minimum such that point is distance s away from load, the input impedance is given by,

$$\frac{R_0}{S} = R_{min} = R_0 \left[\frac{Z_R + j R_0 \tan\left(\frac{2\pi s'}{\lambda}\right)}{R_0 + j Z_R \tan\left(\frac{2\pi s'}{\lambda}\right)} \right] \qquad \text{... (19)}$$

Solving equation (19) for Z_R,

$$\therefore \quad \left[R_0 + j Z_R \tan\left(\frac{2\pi s'}{\lambda}\right)\right] = S\left[Z_R + j R_0 \tan\left(\frac{2\pi s'}{\lambda}\right)\right]$$

$$\therefore \quad -S Z_R + j Z_R \tan\left(\frac{2\pi s'}{\lambda}\right) = -R_0 + j R_0 S \tan\left(\frac{2\pi s'}{\lambda}\right)$$

$$\therefore \quad -Z_R\left[S - j\tan\left(\frac{2\pi s'}{\lambda}\right)\right] = -R_0\left[1 - j S\tan\left(\frac{2\pi s'}{\lambda}\right)\right]$$

$$\therefore \quad Z_R = R_0\left[\frac{1 - j\tan\left(\frac{2\pi s'}{\lambda}\right)S}{S + j\tan\left(\frac{2\pi s'}{\lambda}\right)}\right]$$

$$\therefore \quad Z_R = R_0\left[\frac{1 - j S\tan\left(\frac{2\pi s'}{\lambda}\right)}{S + j\tan\left(\frac{2\pi s'}{\lambda}\right)}\right] \qquad \text{... (20)}$$

In this method, measurement of voltage minimum is preferred over that of voltage maximum as voltage minimum measurement is possible with greater accuracy.

2.9 The Eighth-Wave Line

Let λ be the wavelength of the transmitted frequency 'f'. Consider a transmission line of length $\frac{\lambda}{8}$ as shown in the Fig. 2.14.

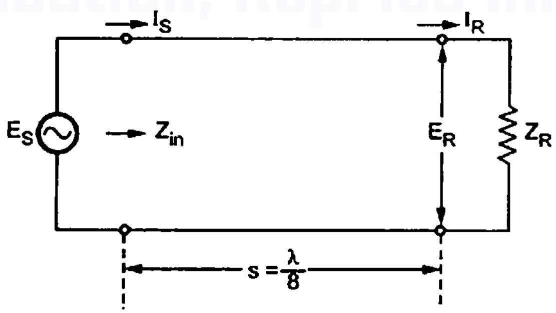


Fig. 2.14 The eighth-wave line

The generalised expression for the input impedance is given by,

$$Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan(\beta s)}{R_0 + j Z_R \tan(\beta s)} \right] \quad \text{... (1)}$$

But $\beta = \frac{2\pi}{\lambda}$

$$\therefore \quad Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan\left(\frac{2\pi}{\lambda} s\right)}{R_0 + j Z_R \tan\left(\frac{2\pi}{\lambda} s\right)} \right]$$

But for a line, $s = \frac{\lambda}{8}$,

$$\therefore \quad Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan\left(\frac{2\pi}{\lambda} \cdot \frac{\lambda}{8}\right)}{R_0 + j Z_R \tan\left(\frac{2\pi}{\lambda} \cdot \frac{\lambda}{8}\right)} \right]$$

$$\therefore \quad Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan\left(\frac{\pi}{4}\right)}{R_0 + j Z_R \tan\left(\frac{\pi}{4}\right)} \right]$$

$$\therefore \quad \boxed{Z_{in} = R_0 \left[\frac{Z_R + j R_0}{R_0 + j Z_R} \right]} \quad \text{... (2)}$$

If a line is terminated in pure resistance $Z_R = R_R$ then,

$$Z_{in} = R_0 \left[\frac{R_R + j R_0}{R_0 + j R_R} \right] \quad \text{... (3)}$$

From equation (3), it is clear that Z_{in} is a complex quantity.

Thus the magnitude of the input impedance is given by,

$$\boxed{|Z_{in}| = R_0 \left[\frac{\sqrt{R_R^2 + R_0^2}}{\sqrt{R_0^2 + R_R^2}} \right] = R_0} \quad \text{... (4)}$$

Thus, the eight-wave line is generally used to transform any resistance R_R to an impedance Z_{in} having its magnitude equal to the characteristic resistance R_0 of the line.

2.10 The Quarter-Wave Line - Impedance Matching

The generalised expression for the input impedance of the line is given by

$$Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan(\beta s)}{R_0 + j Z_R \tan(\beta s)} \right] \quad \text{... (1)}$$

Rearranging the terms in R.H.S. of the equation (1),

$$Z_{in} = R_0 \left[\frac{\dfrac{Z_R}{\tan(\beta s)} + j R_0}{\dfrac{R_0}{\tan(\beta s)} + j Z_R} \right]$$

But $\beta = \dfrac{2\pi}{\lambda}$,

$$\therefore \quad Z_{in} = R_0 \left[\frac{\dfrac{Z_R}{\tan\left(\dfrac{2\pi}{\lambda} s\right)} + j R_0}{\dfrac{R_0}{\tan\left(\dfrac{2\pi}{\lambda} s\right)} + j Z_R} \right] \qquad \ldots (2)$$

For quarter-wave line, $s = \dfrac{\lambda}{4}$

$$\therefore \quad Z_{in} = R_0 \left[\frac{\dfrac{Z_R}{\tan\left(\dfrac{2\pi}{\lambda} \cdot \dfrac{\lambda}{4}\right)} + j R_0}{\dfrac{R_0}{\tan\left(\dfrac{2\pi}{\lambda} \cdot \dfrac{\lambda}{4}\right)} + j Z_R} \right]$$

$$\therefore \quad Z_{in} = R_0 \left[\frac{\dfrac{Z_R}{\tan\left(\dfrac{\pi}{2}\right)} + j R_0}{\dfrac{R_0}{\tan\left(\dfrac{\pi}{2}\right)} + j Z_R} \right]$$

$$\therefore \quad Z_{in} = R_0 \left[\frac{j R_0}{j Z_R} \right]$$

$$\therefore \quad \boxed{Z_{in} = \frac{R_0^2}{Z_R}} \qquad \ldots (3)$$

Thus equation (3) is similar to the equation for impedance matching using transformer.

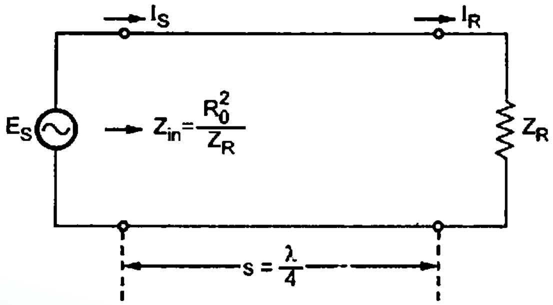


Fig. 2.15 The quarter-wave line

Thus a quarter wave line may be used as a transformer for impedacne matching of load Z_R with input impedance $Z_{in} = Z_R$.

For matching impedances Z_R and Z_{in}, the line with characteristic impedance R_0 may be selected such that condition given in equation (4) gets satisfied.

$$\boxed{R_0 = \left|\sqrt{Z_R \cdot Z_{in}}\right|} \quad \text{... (4)}$$

A quarter wave line can transform a low impedance into a high impedance and vice versa, thus it can be considered as an impedance inverter. Hence an open circuited $\frac{\lambda}{4}$ line gives zero input impedance while a short circuited $\frac{\lambda}{4}$ line gives infinite input impedance. Thus a short circuit quarter wave line behaves as an open circuit at the other end while an open circuit quarter wave line behaves as a short circuit at the other end.

The quarter wave matching section has number of applications. One of the important applications is the impedance transformation in coupling a transmission line to a resistive load such as an antenna. If the antenna resistance is R_A and the characteristic impedance of the line is R_0, then a quarter wave impedance matching section is designed such that its characteristic impedance R_0' transforms antenna resistance R_A to the characteristic impedance of line R_0 given by

$$R_0' = \sqrt{R_A \cdot R_0} \quad \text{... (5)}$$

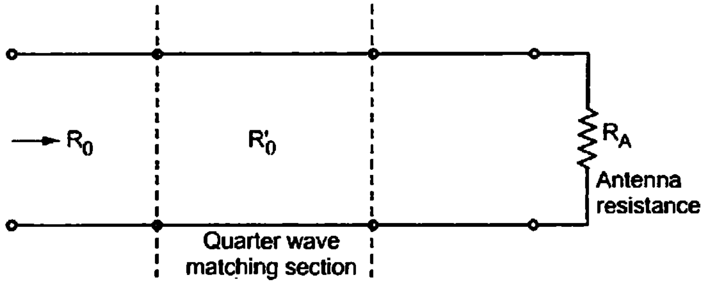


Fig. 2.16 Application of quarter wave line to couple line to the antenna

The value of the characteristic impedance R'_0 is the value which gives critical coupling condition between two impedances and hence maximum power transfer takes place from the line to the load. In case if the distance between the line and antenna is greater than the quarter wave section of the line, it is possible to achieve same transformation by using a line any odd number of quarter wave in length. (such as 3 times, 5 times, 7 times, ... $\lambda / 4$ length). But eventhough the coupling is achieved with line of odd multiple of $\frac{\lambda}{4}$ in length, the losses in the line increase with increase in length reducing the efficiency.

Another application of a quarter wave section is in the line with load which is not pure resistive. Under such condition, the impedance of the line at the points where voltage is minimum (E_{min}) or current is maximum (I_{max}), the resistive impedance of the line is either $S R_0$ or $\frac{R_0}{S}$. Thus for step down in impedance from value R_0, the characteristic impedance of the matching section R'_0 should be

$$R'_0 = \sqrt{R_0\left(\frac{R_0}{S}\right)} = R_0 \sqrt{\frac{1}{S}} \quad \text{... (5)}$$

For step up in impedance from value R_0, the characteristic impedance of the matching section should be,

$$R'_0 = \sqrt{R_0 (S R_0)} = R_0 \sqrt{S} \quad \text{... (6)}$$

Other than the applications discussed above, the quarter wave line may be used to provide mechanical support to the open wire line or centre conductor of a coaxial cable. Such a line used as mechanical support is shorted at ground as shown in the Fig. 2.17.

Please refer Fig. 2.17 on next page.

As quarter wave line is shorted at ground, its input impedance is very high. So the signal on line passes to the receiving end, without any loss due to this mechanical support. Thus the line acts as an **insulator** at this point. Hence such line is referred as **copper insulator.**

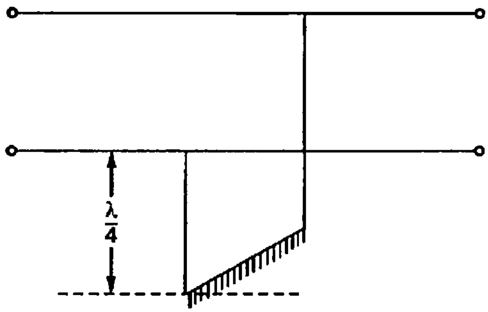


Fig. 2.17 A quarter wave line used as a mechanical support to an open wire line

2.11 The Half-Wave Line

The generalised expression for the input impedance of a line is given by,

$$Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan(\beta s)}{R_0 + j Z_R \tan(\beta s)} \right] \quad \text{... (1)}$$

But $\beta = \frac{2\pi}{\lambda}$,

$$\therefore \quad Z_{in} = R_0 \left[\frac{Z_R + j R_0 \cdot \tan\left(\frac{2\pi}{\lambda} s\right)}{R_0 + j Z_R \cdot \tan\left(\frac{2\pi}{\lambda} s\right)} \right] \quad \text{... (2)}$$

But for a half wave line, $s = \frac{\lambda}{2}$,

$$\therefore \quad Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan\left(\frac{2\pi}{\lambda} \cdot \frac{\lambda}{2}\right)}{R_0 + j Z_R \tan\left(\frac{2\pi}{\lambda} \cdot \frac{\lambda}{2}\right)} \right]$$

$$\therefore \quad Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan(\pi)}{R_0 + j Z_R \tan(\pi)} \right]$$

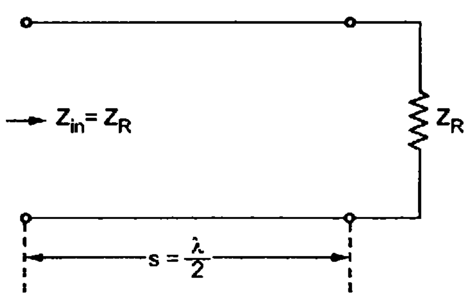


Fig. 2.18 A half wave line

$$\therefore \quad Z_{in} = R_0 \left[\frac{Z_R}{R_0} \right]$$

$$\therefore \quad \boxed{Z_{in} = Z_R} \quad \text{... (3)}$$

From equation (3), it is clear that a half-wave line repeats its terminating impedance. In other words, the half wave line may be considered as one to one transformer.

The main application of a half wave line is to connect a load to a source where both of them can't be made adjacent. In such a case, we may connect a parallel half wave line at load point as shown in the Fig. 2.19. We can then take suitable measurement as half wave line repeats its impedance.

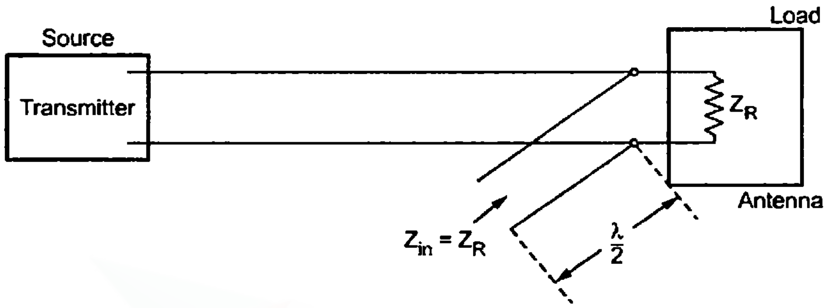


Fig. 2.19 Application of a half-wave line

2.12 Single Stub Matching on a Line

When a high frequency line is terminated in its characteristic impedance R_0, it is operated as a smooth line. Under such conditions reflections are absent, hence we get the maximum power delivered to the load and hence efficiency. But in practice, the loads such as antennas do not provide resistances equal to R_0 of the line. Hence it is necessary to add some impedance matching sections between the line and the load, such that load appears as a resistance R_0 to the line. As we have already discussed that the quarter wave line or the tapered line may be used as impedance matching sections.

The other method of achieving impedance matching is to use open or closed stub lines. In this method, a stub of suitable length is connected in parallel with the line at a certain distance from the load as shown in the Fig. 2.20. By using such stub, antiresonance is achieved providing impedance at resonance equal to R_0.

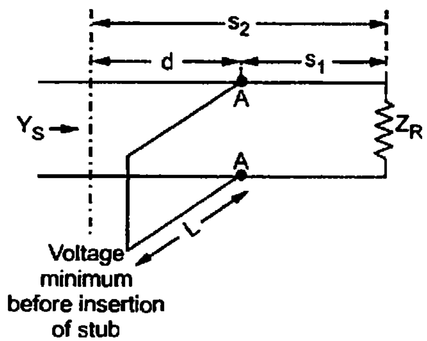


Fig. 2.20 Use of single stub for impedance matching

Because of paralleling of the element, it is convenient to work with admittances. Then the input admittance Y_S, looking towards the load from any point on line is given by

$$Y_S = G_0 \pm j B \qquad \text{... (1)}$$

This may be the admittance at point A before stub was connected. The point A is located such that at the point A, $G_0 = \frac{1}{R_0}$. Then at the point A, a short stub line is connected. This line is selected such that its input succeptance is $\mp B$. This stub is connected across the transmission line. Then the total admittance at input is given by,

$$Y_S = (G_0 \pm j B) + (\mp j B) = G_0 = \frac{1}{R_0} \qquad \text{... (2)}$$

$$\therefore \quad Z_S = R_0 \qquad \text{... (3)}$$

Thus the input impedance of the line at point looking towards load is

$$Z_S = \frac{1}{Y_S} = R_0$$

Thus, the line from the source to the point A is then terminated in R_0. It acts as a smooth line. The reflection and hence standing waves occur in between the portion of line from point A to the load. But by making this distance less than the wavelength, the losses can be minimised.

For the impedance matching using single stub it is very much essential to know the exact point at which the stub is to be connected to the line and also the length of the stub. For this two independent measurements must be made on the line. It is easy to measure standing wave ratio S and the voltage minimum nearest to the load. The measurement on the line is made for the voltage minimum because it is observed that it can be measured accurately rather than the voltage maximum.

In the last chapter, we have already obtained the expression for the input impedance as,

$$Z_{in} = Z_S = R_0 \left[\frac{1 + |K| \angle \phi - 2\beta s}{1 - |K| \angle \phi - 2\beta s} \right] \qquad \text{... (1)}$$

Hence the input admittance is given by,

$$Y_S = \frac{1}{Z_S} = \frac{1}{R_0} \left[\frac{1 - |K| \angle \phi - 2\beta s}{1 + |K| \angle \phi - 2\beta s} \right] \qquad \text{... (2)}$$

Writing terms in rectangular co-ordinates,

$$Y_S = G_S + j B_S = \frac{1}{R_0}\left\{\frac{1-[|K|\cos(\phi-2\beta s)+j|K|\sin(\phi-2\beta s)]}{1+[|K|\cos(\phi-2\beta s)+j|K|\sin(\phi-2\beta s)]}\right\}$$

$$\therefore \quad G_S + j B_S = \left[\frac{1-|K|\cos(\phi-2\beta s)-j|K|\sin(\phi-2\beta s)}{1+|K|\cos(\phi-2\beta s)-j|K|\sin(\phi-2\beta s)}\right]\cdot G_0$$

$$\therefore \quad G_S + j B_S = G_0\left[\frac{1-|K|^2-2j|K|\sin(\phi-2\beta s)}{1+|K|^2+2|K|\cos(\phi-2\beta s)}\right]$$

$$\therefore \quad \frac{G_S + j B_S}{G_0} = \frac{(1-|K|)^2 - j[2|K|\sin(\phi-2\beta s)]}{\left(1+|K|^2\right)+2|K|\cos(\phi-2\beta s)}$$

$$\therefore \quad \frac{G_S}{G_0} + j\frac{B_S}{G_0} = \frac{1-|K|^2}{1+|K|^2+2|K|\cos(\phi-2\beta s)} - j\,\frac{2|K|\sin(\phi-2\beta s)}{1+|K|^2+2|K|\cos(\phi-2\beta s)}$$

$$\therefore \quad \frac{G_S}{G_0} = \frac{1-|K|^2}{1+|K|^2+2|K|\cos(\phi-2\beta s)} \qquad \text{... (A)}$$

$$\frac{B_S}{G_0} = \frac{2|K|\sin(\phi-2\beta s)}{1+|K|^2+2|K|\cos(\phi-2\beta s)} \qquad \text{... (B)}$$

From equation (A) and (B), we can get the two important parameters related to the stub length and point at which the stub is to be connected. The Fig. 2.21 shows the plots of $\frac{G_S}{G_0}$ and $\frac{B_S}{G_0}$ for arbitrary value of $|K|$ say 0.5.

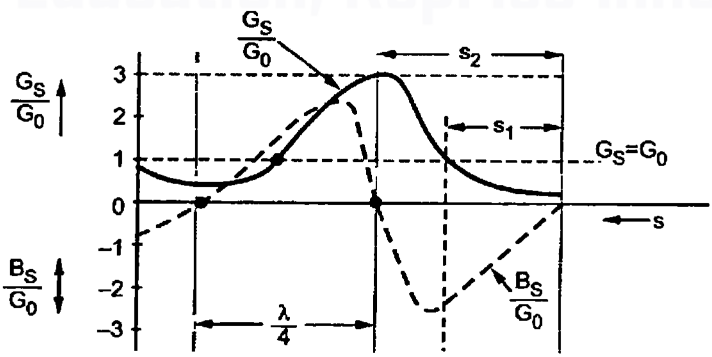


Fig. 2.21 Admittance conditions on line

From above plot and expressions (A) and (B), it is observed that the value of $\frac{G_S}{G_0}$ is maximum if the cosine term in the expression (A) is negative i.e. – 1.

$$\therefore \quad \phi - 2\beta s_2 = -\pi$$

$$\therefore \quad s_2 = \frac{\phi + \pi}{2\beta} \qquad \text{... (3)}$$

At distance s_2, the maximum value of $\frac{G_S}{G_0}$ is given by

$$\left(\frac{G_S}{G_0}\right)_{max} = \frac{1-|K|^2}{1+|K|^2-2|K|} = \frac{(1+|K|)(1-|K|)}{(1-|K|)^2} = \frac{1+|K|}{1-|K|} = S \qquad \text{... (4)}$$

$$\therefore \quad \frac{G_S}{G_0} = \frac{1/R_S}{1/R_0} = \frac{R_0}{R_S} = S \quad \text{or} \quad R_S = \frac{R_0}{S} \qquad \text{... (5)}$$

Thus at point distance s_2 from load input impedance R_S is resistive and its value is minimum equal to $\frac{R_0}{S}$. So **this is the point of minimum voltage at distance s_2 from the load.**

At a distance s_1 from the load, $G_S = G_0$.

This is the point at which the stub is to be connected.

The value of $\frac{G_S}{G_0}$ is unity at this point. From equation (A)

$$\frac{G_S}{G_0} = 1 = \frac{1-|K|^2}{1+|K|^2+2|K|\cos(\phi - 2\beta s_1)}$$

$$\therefore \quad 1+|K|^2+2|K|\cos(\phi-2\beta s_1) = 1-|K|^2$$

$$\therefore \quad 2|K|\cos(\phi-2\beta s) = -2|K|^2$$

$$\therefore \quad \cos(\phi-2\beta s_1) = -|K|$$

$$\therefore \quad \phi - 2\beta s_1 = \cos^{-1}(-|K|)$$

But
$$\cos^{-1}(-|K|) = -\pi \pm \cos^{-1}(|K|)$$

Hence,

$$\phi - 2\beta s_1 = -\pi \pm \cos^{-1}(|K|)$$

$$\therefore \quad s_1 = \frac{\phi + \pi \mp \cos^{-1}(|K|)}{2\beta} \qquad \text{... (6)}$$

Thus the distance d from voltage minimum to the point of stub connection is given by,

$$d = s_2 - s_1$$

$$= \left(\frac{\phi+\pi}{2\beta}\right) - \left(\frac{\phi+\pi \mp \cos^{-1}(|K|)}{2\beta}\right)$$

$$= \pm \frac{\cos^{-1}(|K|)}{2\beta}$$

$$= \frac{\pm \cos^{-1}\left(\frac{S-1}{S+1}\right)}{2\left(\frac{2\pi}{\lambda}\right)}$$

$$\therefore \quad \boxed{d = \frac{\pm \cos^{-1}\left(\frac{S-1}{S+1}\right)}{\pi}\left(\frac{\lambda}{4}\right)} \quad \text{... (7)}$$

In general, the stub may be located at distance d measured in either direction from voltage minimum. But it is observed that for better performance the stub is placed on the load side of that minimum which is nearest to the load.

Let us calculate now, the input succeptance of the line at a distance s_1 using expression (B). From expression (B), if $s = s_1$,

$$\frac{B_S}{G_0} = \frac{-2|K|\sin(\phi - 2\beta s_1)}{1+|K|^2 - 2|K|\cos(\phi - 2\beta s_1)}$$

But $$s_1 = \frac{\phi+\pi \mp \cos^{-1}(|K|)}{2\beta} \quad \text{... from equation (6)}$$

$$\therefore \quad \cos(\phi - 2\beta s_1) = \cos\left[\phi - 2\beta\left(\frac{\phi+\pi \pm \cos^{-1}|K|}{2\beta}\right)\right]$$

$$= \cos(\pi \pm \cos^{-1}|K|)$$

$$= -|K|$$

$$\sin(\phi - 2\beta s_1) = \sin\left[\phi - 2\beta\left(\frac{\phi+\pi \pm \cos^{-1}|K|}{2\beta}\right)\right]$$

$$= \sin(\pi \pm \cos^{-1}|K|)$$

$$= \pm\sqrt{1-K^2}$$

Hence we can write,

$$\frac{B_S}{G_0} = \frac{\mp 2|K|\sqrt{1-|K|^2}}{1+|K|^2+2|K|(-|K|)} = \frac{\mp 2|K|\sqrt{1-|K|^2}}{1-|K|^2}$$

$$\therefore \quad B_S = G_0\left[\mp \frac{2|K|}{\sqrt{1-|K|^2}}\right] \quad \text{... (8)}$$

Equation (8) gives the succeptance of the line at the distance s_1 where stub is connected. To cancel this succeptance of the line, the succeptance of the stub should be,

$$B_{stub} = G_0\left[\frac{\pm 2|K|}{\sqrt{1-|K|^2}}\right] \quad \text{... (9)}$$

In general, the input impedance of the shorted line is given by

$$Z_{in} = Z_S = j R_0 \tan(\beta s)$$

The stub connected is also a transmission line short circuited and of the total length L.

$$\therefore \quad B_{stub} = G_0\left[\frac{\pm 2|K|}{\sqrt{1-|K|^2}}\right] = \frac{1}{R_0 \tan(\beta L)}$$

$$\therefore \quad G_0\left[\frac{\pm 2|K|}{\sqrt{1-|K|^2}}\right] = \frac{G_0}{\tan(\beta L)}$$

$$\therefore \quad \tan(\beta L) = \pm \frac{\sqrt{1-|K|^2}}{2|K|} \quad \text{... (10)}$$

Let us make assumption that R_0 of stub and original transmission line are same

$$\therefore \quad \beta L = \tan^{-1}\left(\frac{\sqrt{1-|K|^2}}{\pm 2|K|}\right) = \left(\frac{2\pi}{\lambda}\right) L$$

Hence the length of the stub is,

$$L = \frac{\lambda}{2\pi} \tan^{-1}\left[\pm \frac{\sqrt{1-|K|^2}}{2|K|}\right] \quad \text{... (11)}$$

As we measure standing wave ratio along the line, equation (11) can be expressed in terms of S.

$$\therefore \quad L = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{S}}{S-1}\right] \quad \text{... (12)}$$

This is the length of the stub which is short circuited to be placed 'd' meters forward to the load, from point at which voltage minimum exists before the connection of the stub. Then the stub succeptance cancels the succeptance of the line at point d, so that the line appears as a smooth line terminated in R_0 from generator to the point at which the stub is placed.

The stub can be placed d meters towards the source from the voltage minimum. The sign of the reactance then reverses with respect to the sign for the location nearer to the load.

Then the stub length is given by,

$$L' = \frac{\lambda}{2} - L \quad \text{... (13)}$$

Let us summarise the result. For a short circuited stub the point of location of the stub on the line and the length of the stub are given by

$$s_1 = \frac{\phi + \pi - \cos^{-1}(|K|)}{\pi} \cdot \frac{\lambda}{4} \; ; \; L = \frac{\lambda}{2\pi} \tan^{-1}\left[+\frac{\sqrt{1-|K|^2}}{2|K|}\right]$$

$$s_1' = \frac{\phi + \pi + \cos^{-1}(|K|)}{\pi} \cdot \frac{\lambda}{4} \; ; \; L = \frac{\lambda}{2\pi} \tan^{-1}\left[-\frac{\sqrt{1-|K|^2}}{2|K|}\right]$$

Example 2.5 *Determine length and location of a single short circuited stub to produce an impedance match on a transmission line with R_0 of 600 Ω and terminated in 1800 Ω.*

Solution : Given

$$R_0 = 600\ \Omega$$

$$Z_R = 1800\ \Omega$$

The reflection coefficient is given by,

$$K = \frac{Z_R - R_0}{Z_R + R_0} = \frac{1800-600}{1800+600} = \frac{1200}{2400} = 0.5 \angle 0°$$

Hence $\phi = 0, \quad |K| = 0.5$

Calculation for the location and length of stub :

Case (1) : $$s_1 = \frac{\phi + \pi - \cos^{-1}(|K|)}{2\beta}$$

But $$\beta = \frac{2\pi}{\lambda}$$

$$\therefore \quad s_1 = \frac{\phi + \pi - \cos^{-1}(|K|)}{2\left(\frac{2\pi}{\lambda}\right)}$$

$$= \frac{0 + \pi - \cos^{-1}(0.5)}{4\pi} \cdot \lambda$$

$$= \mathbf{0.1666\ \lambda}$$

$$\therefore \quad L = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-|K|^2}}{2\ |K|}\right]$$

$$= \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(0.5)^2}}{2(0.5)}\right]$$

$$= \mathbf{0.1135\ \lambda}$$

Case (2) : $$s_1 = \frac{\phi + \pi + \cos^{-1}(|K|)}{4\pi} \cdot \lambda$$

$$= \frac{0 + \pi + \cos^{-1}(0.5)}{4\pi} \cdot \lambda$$

$$= \mathbf{0.3333\ \lambda}$$

$$\therefore \quad L_1 = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-|K|^2}}{-2\ |K|}\right]$$

$$= \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(0.5)^2}}{-2(0.5)}\right]$$

$$= \frac{\lambda}{2\pi} \tan^{-1} (-0.866)$$

$$= \frac{\lambda}{2\pi} (\pi - \tan^{-1} 0.866)$$

$$= 0.386\ \lambda$$

Example 2.6 *Design a suitable transmission line for properly matching a load as shown in Fig. 2.22 (a), to a transmission line of $R_0 = 400\ \Omega$. Also carry out the design if load is changed to the condition as shown in the Fig. 2.22 (b)*

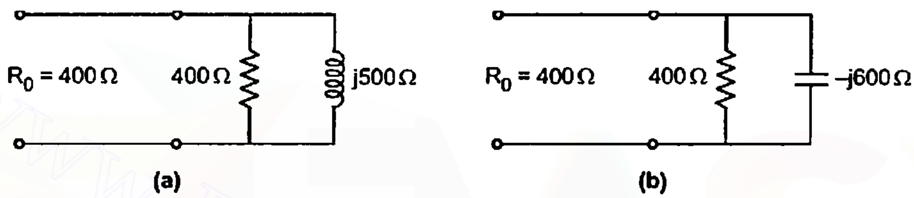


Fig. 2.22

Solution : Consider the circuit shown in Fig. 2.22 (a).

For proper termination, the reactance j500 Ω must be cancelled out by a transmission line impedance matching section of length s.

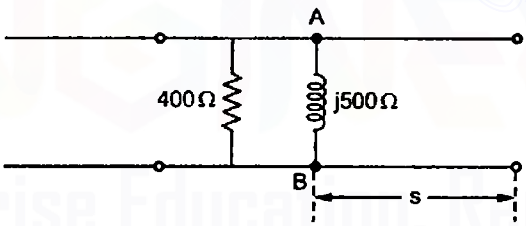


Fig. 2.23

The length of the open circuited line i.e. s must be selected such that it presents the reactance of – j500 Ω at A-B terminals. The input impedance for an open circuited line is given by,

$$Z_{OC} = -jX = -jR_0 \cot\left(\frac{2\pi s}{\lambda}\right)$$

$$\therefore \quad -j\,500 = -j\,400 \cot\left(\frac{2\pi s}{\lambda}\right)$$

$$\therefore \quad 1.25 = \cot\left(\frac{2\pi s}{\lambda}\right)$$

$$\therefore \quad \frac{2\pi s}{\lambda} = \cot^{-1} (1.25)$$

$\therefore \qquad s = 1.073\,\lambda$

Thus for proper termination of a line with $R_0 = 400\,\Omega$, an open circuited line of length $1.073\,\lambda$ may be used to cancell out $+j500\,\Omega$ reactance in the terminating impedance.

b) Refer Fig. 2.22 (b)

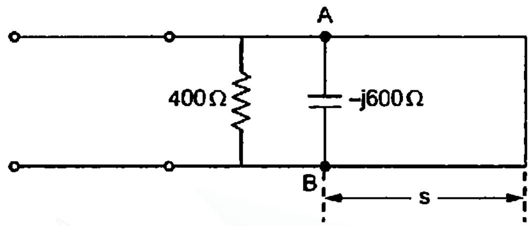


Fig. 2.24

For proper termination, the reactance $-j\,600\,\Omega$ must be cancelled out by a transmission line having its input impedance equal to $+j\,600\,\Omega$ as shown in the Fig. 2.24.

Choose a short circuited line of length s which cancells out $-j\,600\Omega$ load reactance.

The input impedance of the short circuited line is given by

$$Z_{SC} = j\,X = j\,R_0 \tan\left(\frac{2\pi s}{\lambda}\right)$$

$$\therefore \qquad j\,600 = j\,400 \tan\left(\frac{2\pi s}{\lambda}\right)$$

$$\therefore \qquad \frac{2\pi s}{\lambda} = \tan^{-1}\left(\frac{600}{400}\right)$$

$$\therefore \qquad s = 0.1564\,\lambda$$

Thus a line with $400\,\Omega$ characteristic impedance appears as a properly terminated line for the given load conditions by connecting a short circuit line of length $0.1564\,\lambda$.

2.13 Circle Diagram for Dissipationless Line

In general, the design of dissipationless line can be simplified significantly by drawing circle diagram which is useful in simplifying the impedance equation. This is also called impedance circle diagram which facilitates rapid calculations for the transmission line.

[A] Constant S-circles

Consider the expression for input impedance of a dissipationless line given by,

$$Z_S = Z_{in} = R_0\left[\frac{1+|K|\angle\phi - 2\beta s}{1-|K|\angle\phi - 2\beta s}\right] \qquad \text{... (1)}$$

The normalized impedance is given by,

$$\frac{Z_S}{R_0} = \frac{1+|K|\angle\phi - 2\beta s}{1-|K|\angle\phi - 2\beta s} \quad ...(2)$$

Equation(2) is valid for all the lines irrespective of their characteristic impedances. In general Z_s is complex in nature consisting real and imaginary terms. Let it be denoted by $r_a + jx_a$; where r_a and x_a are the resistance and reactance expressed per unit of R_0. Hence we can write,

$$\frac{Z_s}{R_0} = r_a + jx_a = \frac{1+|K|\angle\phi - 2\beta s}{1-|K|\angle\phi - 2\beta s} \quad ..(3)$$

Simplifying above equation, using componendo and dividendo method.

$$\frac{N+D}{N-D} = \frac{r_a + jx_a + 1}{r_a + jx_a - 1} = \frac{(1+|K|\underline{/\phi - 2\beta s}) + (1-|K|\underline{/\ \phi - 2\beta s})}{(1+|K|\underline{/\phi - 2\beta s}) - (1+|K|\underline{/\ \phi - 2\beta s})}$$

$$\therefore \quad \frac{(r_a+1)+jx_a}{(r_a-1)+jx_a} = \frac{2}{2\,|K|\underline{/\phi - 2\beta s}} = \frac{1}{|K|\underline{/\ \phi - 2\beta s}} \quad ...(4)$$

$$(r_a+1)+jx_a\left[|K|\underline{/\phi - 2\beta s}\right] = (r_a-1)+jx_a$$

Equating magnitudes on both the sides, we get,

$$\left[\sqrt{(r_a+1)^2 + x_a^2}\right]|K| = \sqrt{(r_a+1)^2 + x_a^2}$$

squaring both the sides, we get,

$$\left[(r_a+1)^2 + x_a^2\right]K^2 = \left[(r_a-1)^2 + x_a^2\right]$$

$$\therefore K^2\left[r_a^2 + 2r_a + 1 + x_a^2\right] = \left[r_a^2 - 2r_a + 1 + x_a^2\right]$$

$$\therefore \quad r_a^2(K^2-1) + 2r_a(K^2+1) + x_a^2(K^2-1) + (K^2-1) = 0 \quad ...(5)$$

Dividing by factor (K^2-1), we get,

$$r_a^2 + 2r_a\left(\frac{K^2+1}{K^2-1}\right) + x_a^2 + 1 = 0 \quad ...(6)$$

We have already derived expressions as $|K| = \frac{S-1}{S+1}$.

$$\therefore \quad \frac{|K|^2+1}{|K|^2-1} = \frac{\left(\frac{S-1}{S+1}\right)^2 + 1}{\left(\frac{S-1}{S+1}\right)^2 - 1} = \frac{S^2 - 2S + 1 + S^2 + 2S + 1}{S^2 - 2S + 1 - (S^2 + 2S + 1)} = \frac{2(S^2+1)}{-4S}$$

$$\therefore \qquad \frac{K^2+1}{K^2-1} = -\left(\frac{S^2+1}{2S}\right)$$

Substituting above value in equation (5), we get,

$$r_a^2 + 2r_a\left[-\frac{S^2+1}{2S}\right] + x_a^2 + 1 = 0$$

$$\therefore \qquad r_a^2 = r_a\left[\frac{S^2+1}{S}\right] + x_a^2 = -1 \qquad \text{... (7)}$$

To have complete square term on L.H.S., adding term $\left[\frac{S^2+1}{2S}\right]^2$ on both the sides, we get,

$$r_a^2 - r_a\left[\frac{S^2+1}{S}\right] + \left[\frac{S^2+1}{2S}\right]^2 + x_a^2 = \left[\frac{S^2+1}{2S}\right]^2 - 1$$

$$\therefore \qquad \boxed{\left[r_a\left(\frac{S^2+1}{2S}\right)\right]^2 + x_a^2 = \left(\frac{S^2-1}{2S}\right)^2} \qquad \text{... (8)}$$

Comparing above equation with equation of circle, whose center shifts c units from origin on positive x – axis given as,,

$$(x-c)^2 + y^2 = r^2$$

Equation (8), thus represents a family of circle with raidus equal to,

$$\boxed{r = \frac{S^2-1}{2S} = \frac{S-\frac{1}{S}}{2}} \qquad \text{... (9)}$$

ans with center c on the positive real axis given by,

$$\boxed{c = \frac{S^2+1}{2S} = \frac{S+\frac{1}{S}}{2}} \qquad \text{... (10)}$$

A family of circles drawn with radius r given by equation (9) and with center, shifted by c on positive real axis r_a given by equation (10) is as shown in the Fig.2.25.

Please refer Fig. 2.25 on next page.

Note that the magnitude of k ranges from 0 to 1 and thus value of S ranges from 1 to ∞. The radius and the location of center along the real axis both are functions of S, the family of circles may be drawn for different values of S.

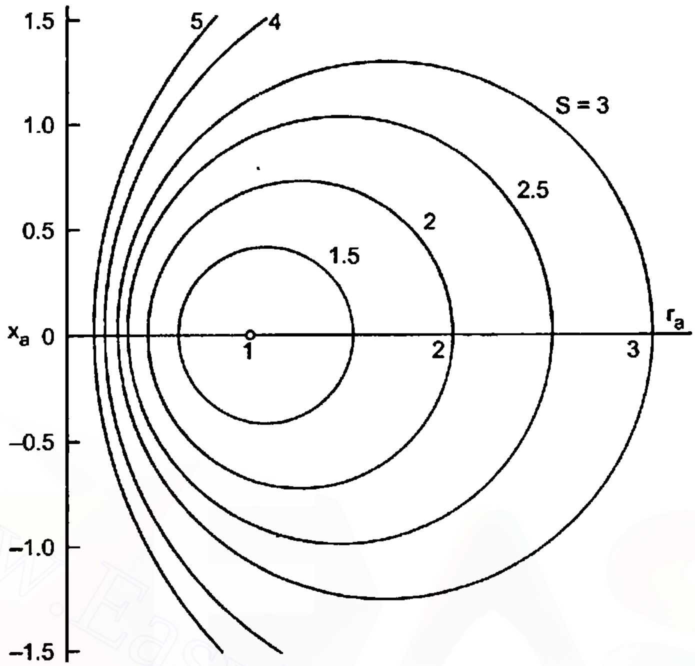


Fig. 2.25

The S-circle makes two intercepts along the r_a-axis (real axis); with one intercept located near origin while other far away from the origin. The intercept near origin is located at $\frac{1}{S}$, while that far away from origin is located at S.

For minimum value of S i.e. unity, the radius r =0, while the center is located on the positive real axis at (1,0). Thus for S=1, the circle is the point located at (1,0) itself. All the remaining circles for different values of S surround this point. When the line is open or short circuited then the value of S is maximum i.e. ∞. Thus for S = ∞, i.e. for extreme case, the imaginary axis x_a represents circle a because as value of S increase, the radius of the circle also increases. At the same time, the center shifts towards right along the positive real axis i.e. r_a axis.

Hence, from above discussion, we can conclude that, **for given $\frac{Z_S}{R_0}$ normalized impedance, a given constant S-circle represents all possible values of the resistive part r_a reactive part x_a. A line drawn from origin to the point on the S-circle indicates normalized impedance $\frac{Z_S}{R_0}$ with real and imaginary component i.e. r_a and jx_a respectively.**

The two intercepts made by the given S-circle indicate the location of voltage minima and voltage maxima. We have already studied that the input impedance at the voltage minima is $\frac{1}{S}$, while that at the voltage maxima is s. Thus the intercepts made near origin by different S-circles indicate the minima points. Thus, **the voltage minima points are located in between 0 and 1 along the real axis near the origin.** Similarly the intercepts, made far away from the origin, indicate the voltage maxima points. Thus, **the voltage maxima points are located between 1 and ∞. along the real axis, away from the origin.**

[B] Constant βs circles

Consider equation (4),

$$\frac{(r_a+1)+jx_a}{(r_a-1)+jx_a} = \frac{1}{|K|\angle\phi-2\beta s}$$

$$\therefore \qquad \frac{(r_a+1)+jx_a}{(r_a-1)+jx_a} = |K|\angle\phi-2\beta s$$

Rationalizing term on L.H.S. and rewriting equation,

we get,

$$\frac{r_a^2-1+x_a^2+j2x_a}{(r_a+1)^2+x_a^2} = |K|\angle\phi-2\beta s$$

$$\therefore \qquad \frac{(r_a^2-1)+x_a^2}{(r_a+1)^2+x_a^2}+j\frac{2x_a}{(r_a-1)^2+x_a^2} = |K|\angle\phi-2\beta s \qquad ...(11)$$

Here angle ϕ can be considered to be zero so as to start the βs scale at 0° Equating tangents of the angles on both the sides of the equation (11) we get

$$\tan\left[\tan^{-1}\left\{\frac{\dfrac{2x_a}{(r_a+1)^2+x_a^2}}{\dfrac{(r_a^2-1)^2+x_a^2}{(r_a+1)^2+x_a^2}}\right\}\right] = \tan(-2\beta s)$$

$$\therefore \qquad \frac{2x_a}{(r_a^2+1)-x_a^2} = \tan(-2\beta s)$$

$$\therefore \qquad r_a^2-1+x_a^2 = \frac{2x_a}{\tan(-2\beta s)}$$

$$\therefore \quad r_a^2 - 1 + x_a^2 = -\frac{2x_a}{\tan(2\beta s)} \qquad \ldots \tan(-\theta) = -\tan\theta$$

$$\therefore \quad r_a^2 - 1 + x_a^2 + \frac{2x_a}{\tan(2\beta s)} = 0$$

$$\therefore \quad r_a^2 + x_a^2 + \frac{2x_a}{\tan(2\beta s)} = 1 \qquad \ldots (12)$$

Adding term $\frac{1}{\tan^2(2\beta s)}$ on the both the sides to adjust for the perfect square on L.H.S., we get,

$$\therefore \quad r_a^2 + x_a^2 + \frac{2x_a}{\tan(2\beta s)} + \frac{1}{\tan^2(2\beta s)} = 1 + \frac{1}{\tan^2(2\beta s)}$$

$$\therefore \quad \boxed{r_a^2 + \left[x_a + \frac{1}{\tan(2\beta s)}\right]^2 = 1 + \frac{1}{\tan^2(2\beta s)} = \frac{1}{\sin^2 2\beta s}} \qquad \ldots (13)$$

Equation (13) represents equation of circle similar to the equation (8). For above circle, the radius is given by,

$$\therefore \quad \boxed{r = \frac{1}{\sin 2\beta s}} \qquad \ldots (14)$$

The center of these circles shift downwards on imaginary axis i.e. x_a axis given by,

$$\therefore \quad \boxed{C = \frac{1}{\tan 2\beta s}} \qquad \ldots (15)$$

The family of constant βs-circle is as shown in the Fig.2.26

Note that equaiton (13) indicates a family of circles with each circle passes through the point (1,0). All the constant βs are orthogonal to the constant S-circels.

When the value of βs lies in between 0 and $\frac{\pi}{2}$, the constant βs-circles lie in the positive region of the imaginary axis i.e. x_a axis. While when the value of βs lies in between $\frac{\pi}{2}$ and π, the constant βs-circles lie in the negative region of the imaginary axis i.e. x_a axis.

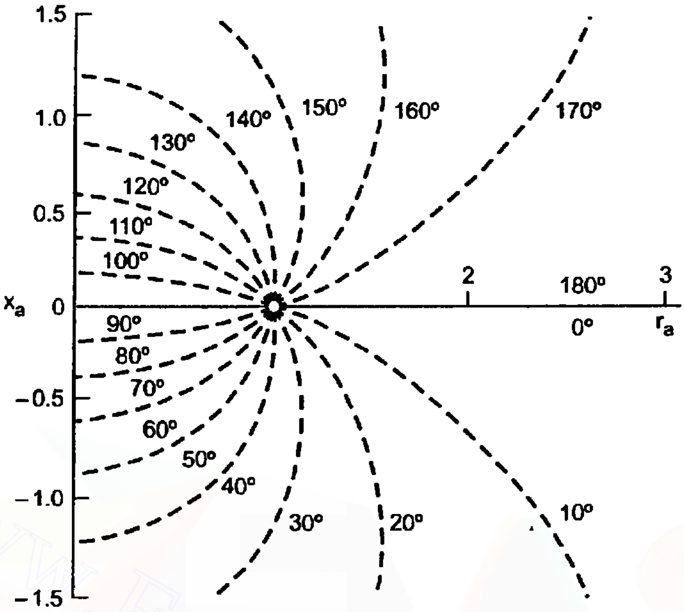


Fig. 2.26

By superpositioning the constant βs-circles on constant S-circles, we get final circle diagram useful to calculate input impedance rapidly.

The circle diagram is as shown in the Fig. 2.27

Please refer Fig. 2.27 on page.

In the circle diagram it is necessary to specify the direction of travel between load and generator. If the travel is in clockwise direction along the circle diagram, it indicates travel from load to the generator along the uniform transmission line. Similarly if the movement along the circle diagram is in anticlockwise direction, then the travel is from generator to load along the line.

Another important point is that using the circle diagram, along with per unit impedance i.e. normalized impedance, it is possible to represent normalized admittance also. The normalized admittance can be written as,

$$\frac{Y_S}{G_0} = \frac{1}{\left(\frac{Z_S}{R_0}\right)} = \frac{1}{r_a + jx_a} = \frac{r_a - jx_a}{r_a^2 + x_a^2}$$

$$\therefore \quad \frac{Y_S}{G_0} = \left[\frac{r_a}{r_a^2 + x_a^2} - j\left[\frac{x_a}{r_a^2 + x_a^2}\right]\right] = g_a - jb_a \qquad \text{... (16)}$$

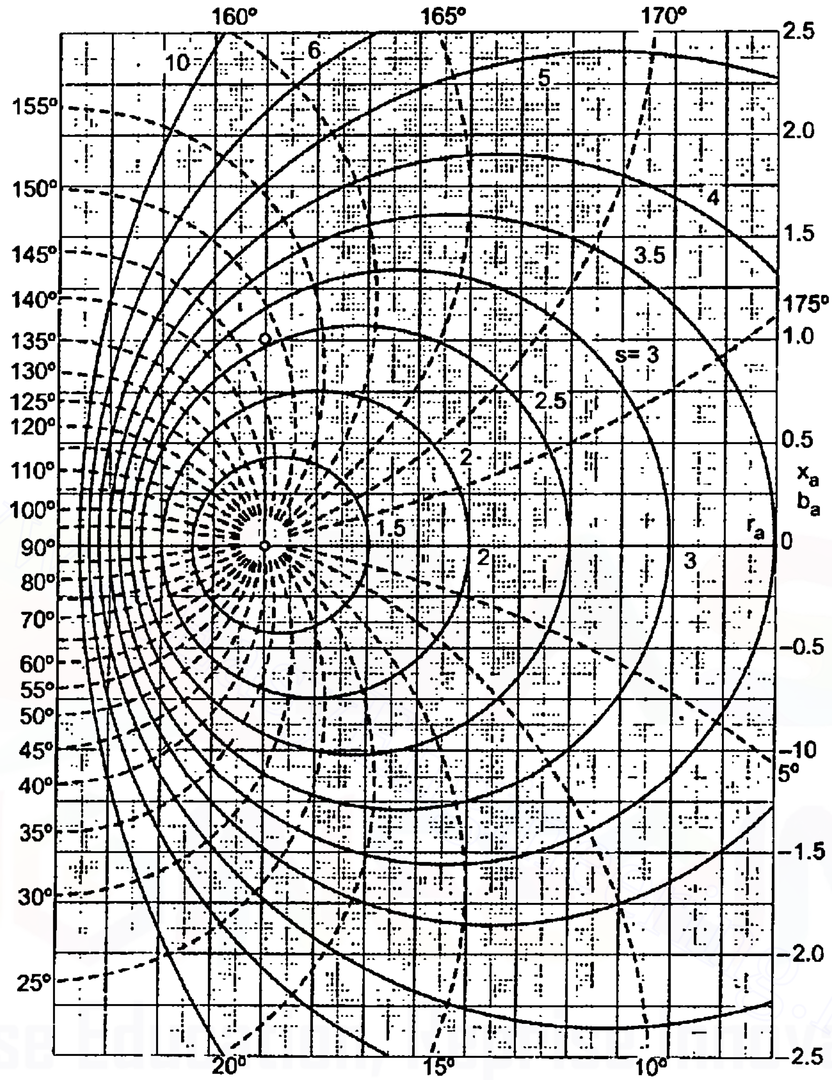


Fig. 2.27

From equation (16) it is clear that positive inductive reactance x_a becomes negative susceptance b_a.

From equation (1), we can write,

$$\frac{Y_S}{G_0} = g_a - jb_a = \frac{1-|K|\angle\phi-2\beta s}{1+|K|\angle\phi-2\beta s} \qquad \text{...(17)}$$

Equation (17) can be considered as the equation (1) itself with r_a replaced by g_a, x_a b_y $-$ b_a and $+|K|$ by $-|K|$. So we can similarly make changes in equation (8) so as to get following equation given by,

$$\left[g_a - \left(\frac{S^2+1}{2S}\right)\right]^2 + b_a^2 = \left(\frac{S^2-1}{2s}\right)^2 \quad \ldots (18)$$

Hence it is clear that the circle diagram used for impedance (z), resistance (r) reactance (x) can also be used for admittance(Y), conductance (g) and susceptance (b) by simply changing r_a scale to g_a. It is observed from equation (18) that the substitution of $-b_a$ for x_a is not necessary since the term in equation (18) is squared term i.e. b_a^2.

In normal circle diagram, the inductive reactance is plotted upwards from the real axis while that for capacitive reactance plot is downward form the real axis. When the same chart is used for the susceptances, the positive inductive reactance changes to negative susceptance and hence plotted downwards to the real axis. And the negative capacitive reactance chnages to positive susceptance, so it is plotted upwards to the real axis. Note that both the types of the susceptances are plotted with the same scale used for x_a.

2.14 The Smith Chart - The Smith Circle Diagram

The main drawback of the circle diagram is that the constant S-circles and constance βs-circles are not concentric which makes it difficult to interpolate these circles. Moreover the circle diagram can be used for the limited range of the impedance values with reasonable, practical chart size.

P.H. Smith of Bell Laboratories, developed a new chart, similar to the impedance of circle diagram chart, in which the drawbacks of the circle diagram were removed. The modified chart is named as the **Smith chart** after P.H. Smith and it is extensively used for the analysis of the transmission line problems.

The **Smith Chart** is a valuable graphical tool for solving radio frequency transmission line problems. Under the matched impedance condition the value of reflection coefficient is 0 and that of VSWR is 1. In almost all the transmission line problems, the main objective is to match the impedances of line to that with load.

2.14.1 Construction of the Smith Chart

Basic difference between the circle diagram and the Smith Chart is that the values of resistive and reactive components are represented in the rectangular form which are extended to infinity. But in the Smith Chart, the infinite resistive and reactive components are transferred to an area inside a circle. As the resistive and reactive components are in circular form, the smith chart is also known as **circular chart.**

The smith chart is basically a polar plot of the reflection coefficient K expressed in terms of the normalized impedance.

The input impedance of the line can be expressed in terms of the characteristics impedance of the dissipationless line is given by,

$$Z_{in} = Z_S = R_0\left[\frac{1+|K|\angle\phi-2\beta s}{1-|K|\angle\phi-2\beta s}\right] \quad \text{... (1)}$$

The **normalized impedance** is defined as,

$$\frac{Z_{in}}{R_0} = \frac{1+|K|\angle\phi-2\beta s}{1-|K|\angle\phi-2\beta s} \quad \text{... (2)}$$

Basically the normalized impedance in a complex quantity which can be represented in rectangular from as r_i+jx_i.

$$\therefore \quad \frac{Z_{in}}{R_0} = r_i + jx_i = \frac{1+|K|\phi-2\beta s}{1-|K|\phi-2\beta s} \quad \text{...(3)}$$

Let $|K|\angle\phi-2\beta s$ be represented in rectangular form as u + jv. Then equation (3) can be written as

$$\frac{Z_{in}}{R_0} = r_i + jx_i = \frac{1+(u+jv)}{1-(u+jv)} \quad \text{... (4)}$$

Rationalizing denominator term R.H.S. of equation (4), we get,

$$r_i + jx_i = \frac{(1+u)+jv}{(1+u)-jv}\cdot\frac{(1-u)+jv}{(1-u)+jv}$$

$$\therefore \quad r_i + jx_i = \frac{(1-u^2)+jv[1'-u+1+u]-v^2}{(1-u^2)+v^2}$$

$$\therefore \quad r_i + jx_i = \frac{(1-u^2-v^2)+j2v}{(1-u)^2+v^2}$$

$$\therefore \quad \boxed{r_i + jx_i = \frac{1-u^2-v^2}{(1-u)^2+v^2}+j\frac{2v}{(1-u)^2+v^2}} \quad \text{... (5)}$$

Equating real and imaginary terms , we get,

$$\boxed{r_i = \frac{1-u^2-v^2}{(1-u)^2+v^2}} \quad \text{... (6)}$$

and

$$\boxed{x_i = \frac{2v}{(1-u)^2 + v^2}} \quad \text{... (7)}$$

A. Constant Resistance Circles

From equation (6) we can write,

$$r_i\left[(1-u)^2 + v^2\right] = (1-u^2-v^2)$$

$$\therefore \quad r_i\left[1-2u+u^2+v^2\right] = (1-u^2-v^2)$$

$$\therefore \quad u^2(r_i+1) - 2ur_i + v^2(r_i+1) = 1 - r_i$$

Dividing both the sides of the equation by factor $(r_i + 1)$

$$\therefore \quad u^2 - 2u\frac{r_i}{r_i+1} + v^2 = \frac{1-r_i}{r_i+1}$$

i.e.

$$u^2 - 2u\left(\frac{r_i}{1+r_i}\right) + v^2 = \frac{1-r_i}{1+r_i} \quad \text{... (8)}$$

To complete the square for the term on L.H.S., we get,

$$u^2 - 2u\left(\frac{r_i}{1+r_i}\right) + \left(\frac{r_i}{1+r_i}\right)^2 + v^2 = \frac{1-r_i}{1+r_i} + \left(\frac{r_i}{1+r_i}\right)^2$$

$$\therefore \quad \boxed{\left[u - \frac{r_i}{1+r_i}\right]^2 + v^2 = \frac{1-r_i^2+r_i^2}{(1+r_i)^2} = \frac{1}{(1+r_i)^2}} \quad \text{... (9)}$$

Equation (9) represents a family of circles having equation of the form $x^2 + y^2 = r^2$. The actual radius for the circles is given by,

$$\boxed{r = \frac{1}{1+r_i}} \quad \text{... (10)}$$

and the centre in located at the point given by,

$$\boxed{C \equiv \left(\frac{r_i}{1+r_i}, 0\right)} \quad \text{... (11)}$$

The constant (normalized) resistance circles are as shown in the fig. 2.28(a).

B. Constant Reactance Circles

Form equation (7), we can write

$$x_i\left[(1-u)^2+v^2\right] = 2v$$

$$\therefore \quad x_i\left[1-2u+u^2+v^2\right]-2v = 0$$

$$\therefore \quad \left(u^2-2u+1\right)x_i+v^2x_i-2v = 0$$

Dividing all the terms by x_i, we get,

$$\left(u^2-2u+1\right)+v^2-\frac{2v}{x_i} = 0$$

$$\therefore \quad (u-1)^2+v^2-\frac{2v}{x_i} = 0 \qquad \text{... (12)}$$

To complete the square term on L.H.S., adding term $\frac{1}{x_i^2}$ on both the sides of the equation.

$$\therefore \quad (u-1)^2+v^2-\frac{2v}{x_i}+\frac{1}{x_i^2} = \frac{1}{x_i^2}$$

$$\therefore \quad \boxed{(u-1)^2+\left(v-\frac{1}{x_i}\right)^2 = \left(\frac{1}{x_i}\right)^2} \qquad \text{... (13)}$$

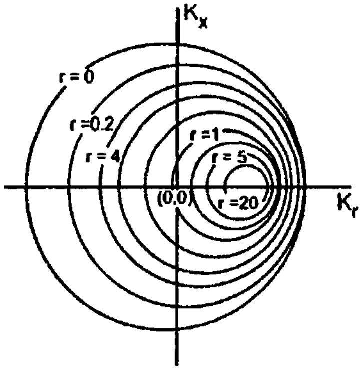
Equation (13) also represents a family of circles called **constant reactance circles.** The actual radius of the circle is given as,

$$\boxed{r = \frac{1}{x_i}} \qquad \text{... (14)}$$

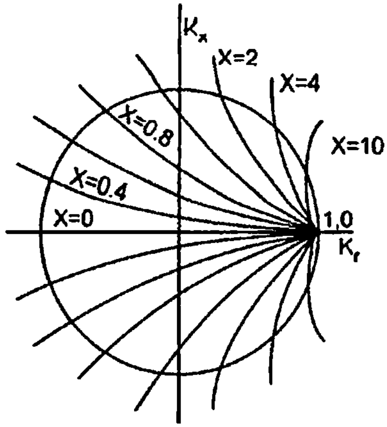
and the center is located at the point given by

$$\boxed{C \equiv \left(1, \frac{1}{x_i}\right)} \qquad \text{... (15)}$$

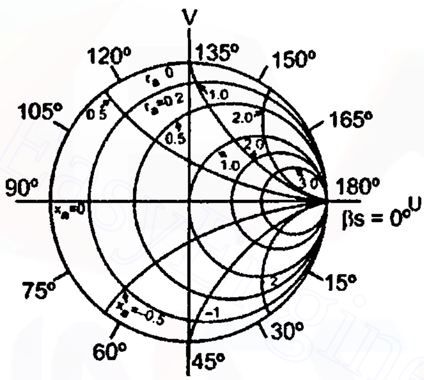
The constant reactance circles are as shown in the Fig. 2.28(b). Practically, the presently in use Smith chart is the superposition of the constant resistance circles and the constant reactance circles as shown in the Fig. 2.28 (c).



(a) Family of constant r-circles



(b) Family of constant X-circles



(c) Basis of the Smith circle diagram

Fig. 2.28 Smith chart circle diagrams

C. Constant S-circles

Consider equation (2)

$$\left(u-\frac{r_i}{1+r_i}\right)^2+v^2 = \left[\frac{1}{1+r_i}\right]^2$$

Assume v = o, such that the centers of all S-circles lie on the horizontal real axis of the Smith chart. Then we can modify above equation as,

$$\left(u-\frac{r_i}{1+r_i}\right)^2 = \left[\frac{1}{1+r_i}\right]^2$$

$$\therefore \qquad u-\frac{r_i}{1+r_i} = \pm\frac{1}{1+r_i}$$

$$\therefore \qquad u = \frac{r_i}{1+r_i}\pm\frac{1}{1+r_i} = \frac{r_i \pm 1}{1+r_i}$$

$$\therefore \quad u = \frac{r_i + 1}{r_i + 1} \quad \text{or} \quad \frac{r_i - 1}{r_i + 1}$$

$$\text{i.e.} \quad u = 1 \quad \text{or} \quad \frac{r_i - 1}{r_i + 1}$$

$$\text{consider} \quad u = \frac{r_i - 1}{r_i + 1}$$

$$\therefore \quad \frac{1+u}{1-u} = \frac{r_i + 1 + r_i - 1}{r_i + 1 - (r_i - 1)} \quad \text{... by componendo-dividendo}$$

$$\therefore \quad \frac{1+u}{1-u} = \frac{2r_i}{2} = r_i$$

$$\text{But} \quad u + jv = |K| \angle \phi - 2\beta s \quad \text{We have assumed } v = 0.$$

$$\therefore \quad u = |K|$$

Substituting value in above equation, we get,

$$\boxed{r_i = \frac{1+|K|}{1-|K|} = S}$$

Thus the constant S-circle can be drawn with radius equal to distance between center of the Smith chart and point $r_i = s$. The constant S-circle cut horizontal real axis at point $r_i = s$ to the right hand side of the center while at point $r_i = \frac{1}{s}$ to the left hand side of the center.

2.14.2 Properties of the Smith Chart

1. The Smith chart may be used for impedances as well as for admittances.
2. The Smith chart consists constant r_i-circles and constant x_i-circles superpositioned on one chart. The values of r_i and x_i are **normalized** and they are given by.

$$r_i = \frac{R}{R_0} \quad \text{and} \quad x_i = j\frac{x_R}{R_0} \quad \text{where } Z_R = R_R + jx_R$$

   The constant r_i-circles have their centres on the horizontal axis i.e. u-axis and constant x_i-circles have their centers on the vertical axis i.e v-axis.

3. The Smith chart is based on the assumption that,

$$|K| \angle \phi - 2\beta s = u + jv$$

The maximum magnitude of u + jv is the maximum value of |K| i.e. unity. Thus, in the chart, it is possible to locate all possible values of impedances inside the outer circle of unit radius.

4. The impedance of a line of any length can be read at any point on the given S-circle. For properly terminated line and any length, the impedance is represented by the point (1, 0) which acts as the centre of the Smith chart as shown in the Fig. 2.29.

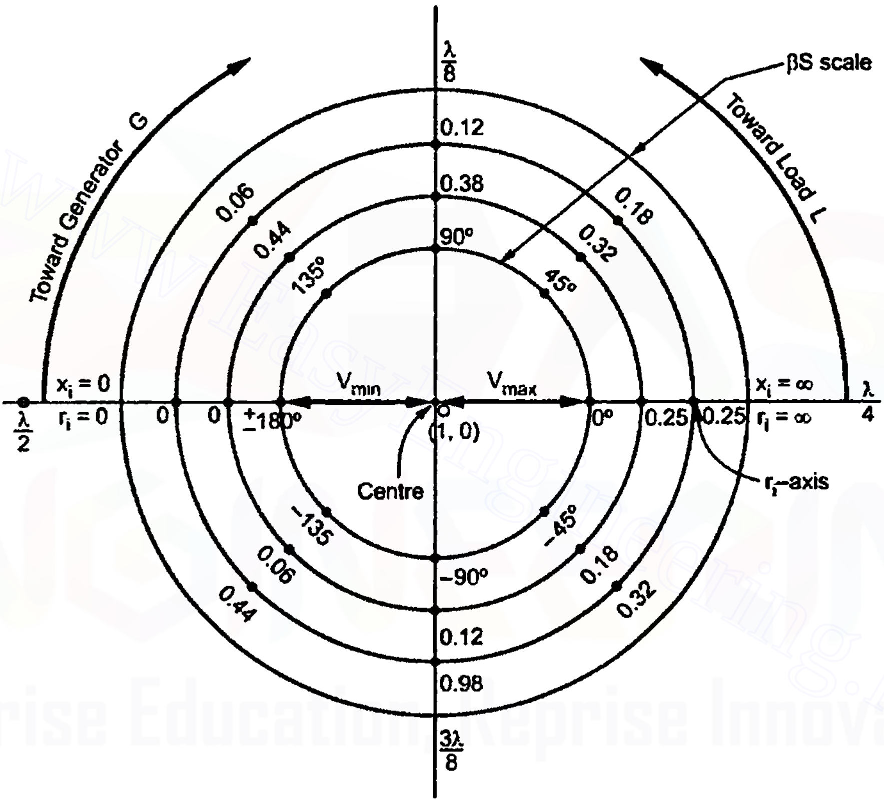


Fig. 2.29 The Smith Chart properties

5. The horizontal line passing through the centre of the smith chart represents real axis or r_i-axis for impedance plot or g_i-axis for admittance plot.

To the extreme left of the chart, along the r_i-axis, $r_i = 0$ and $x_i = 0$ i.e. $Z_L = 0 + j0$. This indicates **zero impedances at load point** or the **short circuit condition**. Similarly, to the extreme right of the chart, along r_i-axis, $r_i = \infty$ and $x_i = \infty$ i.e $Z_L = \infty + j\infty$. This indicates **infinite impedance at load point** or the **open circuit condition.**

6. The outer rim of the chart is scaled into either degrees or wavelengths with an arrow. This arrow indicates the direction of travel along the line. The outer circle is called βs **scale** of the chart which indicates the electrical length of the line.

7. A complete revolution of 360° around the center of the chart represents a distance of $\frac{\lambda}{2}$ on the line. The **clockwise movement** along the outer rim indicates the **travel towards the generator from load** along the line. Vice a versa, **anticlockwise movements** along outer rim indicates the **travel towards the load from the generator**. Both these directions are indicated by the arrows G and L in the Fig.2.29.

   The distance $\frac{\lambda}{2}$ on the line corresponds to a movement of 360° on the chart. Thus the distance λ corresponds to a 720° movement on the chart.

   $$\boxed{\lambda = 720^\circ}$$

8. On the periphery of the Smith chart, three scales are provided. Eventhough there are three scales, they serve the same purpose. These scales are useful in determining the distance from the load or generator in degrees or wavelengths. The three scales are as shown in the Fig.2.29.

   The outermost scale may be used to calculate the distance from the generator in wavelengths, along the line. The next scale is used to determine the distance from the load in wavelengths, along the line. The innermost scale is used to determine the angle of the reflection coefficients. The innermost scale is in degrees.

9. If the Smith chart is used for impedances, the inductive reactance are above r_i-axis or u-axis while the capacitive reactance are below u-axis.

   Similarly if the Smith chart is used for admittances, the r_i axis becomes g_i axis while x_i axis becomes b_i axis. Then the extreme left of g_i axis represents **zero conductances** or **open circuit**, while the extreme right of g_i-axis represents **infinite conductance or short circuit.**

10. The voltage maxima (V_{max}) occur where $z_{in\ (max)}$ is located while the voltage minima(V_{min}) occur where z_{in} is located. Thus voltage minima occur to the left of the centre of the chart, along r_i-axis while the voltage maxima occur to the right of the centre of the chart, along r_i-axis as shown in the Fig. 2.29.

2.15 Applications of the Smith Chart

Let us consider few applications of the Smith chart through an example.

Consider a 30 m long lossless transmission line with the characteristic impedance of $50\,\Omega$ operating at 2 MHz. If the line is terminated in impedance $(60+j40)\,\Omega$ calculate the reflection coefficient, the standing wave ratio, the input impedance, if the velocity on the line is $v = 0.6c\left(c = 3\times10^{8}\ \text{m/s}\right)$.

The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{Z_R - R_0}{Z_R + R_0} = \frac{(60+j\,40)-50}{60+j\,40+50} = \frac{10+j40}{110+j40}$$

$$= \frac{41.231\angle 75.96^{\circ}}{117.0469\angle 19.98^{\circ}}$$

$$\therefore \quad K = 0.3523\angle 56^{\circ}$$

The standing wave ratio is given by,

$$S = \frac{1+|K|}{1-|K|} = \frac{1+0.3523}{1-0.3523} = \frac{1.3523}{0.6477} = 2.088$$

The velocity on the line given by

$$v = \frac{\omega}{\beta} \qquad \text{i.e. } \beta = \frac{\omega}{v}$$

But the electrical length of the line is βs where s = 30 m.

$$\therefore \quad \beta s = \frac{\omega}{v}\,s = \frac{2\times\pi\times f}{v}.s = \frac{2\times\pi\times 2\times10^{6}\times 30}{0.6\times 3\times 10^{8}} = (0.6666)\pi = 120^{\circ}$$

Then the input impedance is given by,

$$Z_{in} = R_0\left[\frac{Z_R + jR_0\tan\beta s}{R_o - jZ_R\tan\beta s}\right]$$

$$\therefore \quad z_{in} = 50\left[\frac{(60+j40)+j50\tan120^{\circ}}{50-j(60+j240)\cdot\tan120^{\circ}}\right]$$

$$\therefore \quad z_{in} = 24.01\angle 3.22^{\circ}\Omega = 23.97 + j\,1.35\ \Omega$$

Applications

1. Plotting an Impedance

Any complex impedance can be represented by a single point on the Smith chart. This point is nothing but the intersection of constant r_i circle i.e. $r_i = \frac{R_R}{R_0}$ circle and x_i circle i.e. $x_i = j\,\frac{X_R}{R_0}$ circle..

Consider above example

$$Z_R = R_R + j\,X_R = (60 + j40)\Omega\,,\quad R_0 = 50\,\Omega$$

$$\therefore \text{The normalized impedance } Z_R = \frac{Z_R}{R_0} = \frac{60 + j40}{50} = 1.2 + j\,0.8$$

Locate a point P on the the Smith chart as shown in the Fig.2.30, where r_i = 1.2 circle and x_i = 0.8 circle meet together. The intersection of the two circles is represented by the dotted lines and the point P indicates the normalized impedance on the chart.

2. Measurement of VSWR

After plotting the normalized impedance, we can determine the value of VSWR by drawing constant S circle with center of the chart [i.e. point (1,0) on the u-axis] and radius equal to distance between centre O and point indicating the normalized impedance. Then the point of intersection of S-circle with the real axis at the right side of the centre indicates a VSWR for given line.

Consider above example. Select a centre of the circle as point O(1,0) Take a distance from O to the point P indicating normalized impedance and then draw a circle. The circle cuts the horizontal real axis at point Q(2.1, 0). This indicates value of the VSWR for the line considered as shown in the Fig. 2.30.

From the Fig. 2.30 the value of VSWR is 2.1. approximately.

3. Measurement of reflection coefficient K [magnitude and phase]

The angle of the reflection coefficient K can be obtained by extending a line from center to the outer rim of the chart through the point which indicates the normalized impedance. The point at which the extended line cuts the outer rim gives directly the value of angle of the reflection coefficient K.

In the commercial Smith chart, the K-scale is provided at the bottom of the chart. Then by selecting point center on this Scale draw an arc just intersecting the straight line of voltage reflection coefficient, with radius equal to distance between the centre of the chart and a point indicating normalized impedance. Then the distance from center to the point of intersection on the horizontal K-scale gives directly the magnitude of K.

For the transmission line considered in above example, draw a straight line starting form center and extending upto the outer rim through point P. This line intersects outer rim at point M, which indicates angle of k, as shown in the Fig. 2.30. At point M, angle of k = ϕ = 55.5 (approx.)

Similar on the K-scale, at the bottom of the chart as shown in the Fig.2.30, the arc draw with radius equal to OP, intersects with horizontal line at N. At N, $|K| = 0.35$.

4. Measurement of Input Impedance of the Line

To find the input impedance of the line, first locate the load point by plotting normalized load impedance and extending upto the outer rim of the chart. As we have to find the impedance at the generator, move along the outer rim in clockwise direction, towards generator, with a distance equal to the length of the transmission line. This is the point which indicates the generator side. Mark the point on outer rim and draw a line from point O to the generator point. This line cuts the circle drawn corresponding to the SWR of the line. This point of intersection indicates the generator point. This point gives the normalized input impedance and the actual input impedance can be obtained by multiplying this normalized impedance by R_0.

For the transmission line in above example, the total length is 30 m. Let us first calculate length, interms of λ or degrees.

$$\lambda = \frac{v}{f} = \frac{0.6c}{f} = \frac{0.6 \times 3 \times 10^8}{2 \times 10^6} = 90\text{m}$$

But $$S = 30 \text{ m} = \frac{30}{90}\lambda = \frac{\lambda}{3} \text{ m or } \frac{720}{3} = 240^\circ \qquad \ldots (\lambda = 720^\circ)$$

Now in the the chart, the line OP extended upto outermost scale cuts at point E say. The distance corresponding to point E is 0.173λ. Then move from point E, a distance equal to 0.3333λ in clockwise direction to reach generator point F as shown in the Fig. 2.30. Now the total distance to be travelled is 0.3333λ. From point E to the extreme right point corresponding to 0.25λ is given by $0.25\lambda - 0.173\lambda = 0.077\lambda$. Then from extreme right point on the chart to the extreme left point the distance of travel equals 0.25λ. Then to reach point F from extreme left point corresponding to 0 or 0.3λ is given by $0.333\lambda - (0.077 + 0.25\lambda) = 0.0063\lambda$. The point of intersection of line 0T and constant S circle is represented by point T, which is the intersection of $r_i = 0.48$ circle and $x_i = 0.035$ circle. As point T is above horizontal axis, the reactance is positive. Hence the normalized input impedance represented by point T is given by

$$z_{in} = 0.48 + j\,0.035$$

Hence the actual value of the input impedance is given by,

$$Z_{in} = R_0\,(z_{in}) = 50\,(0.48 + j\,0.035) = 24 + j\,1.75\ \Omega$$

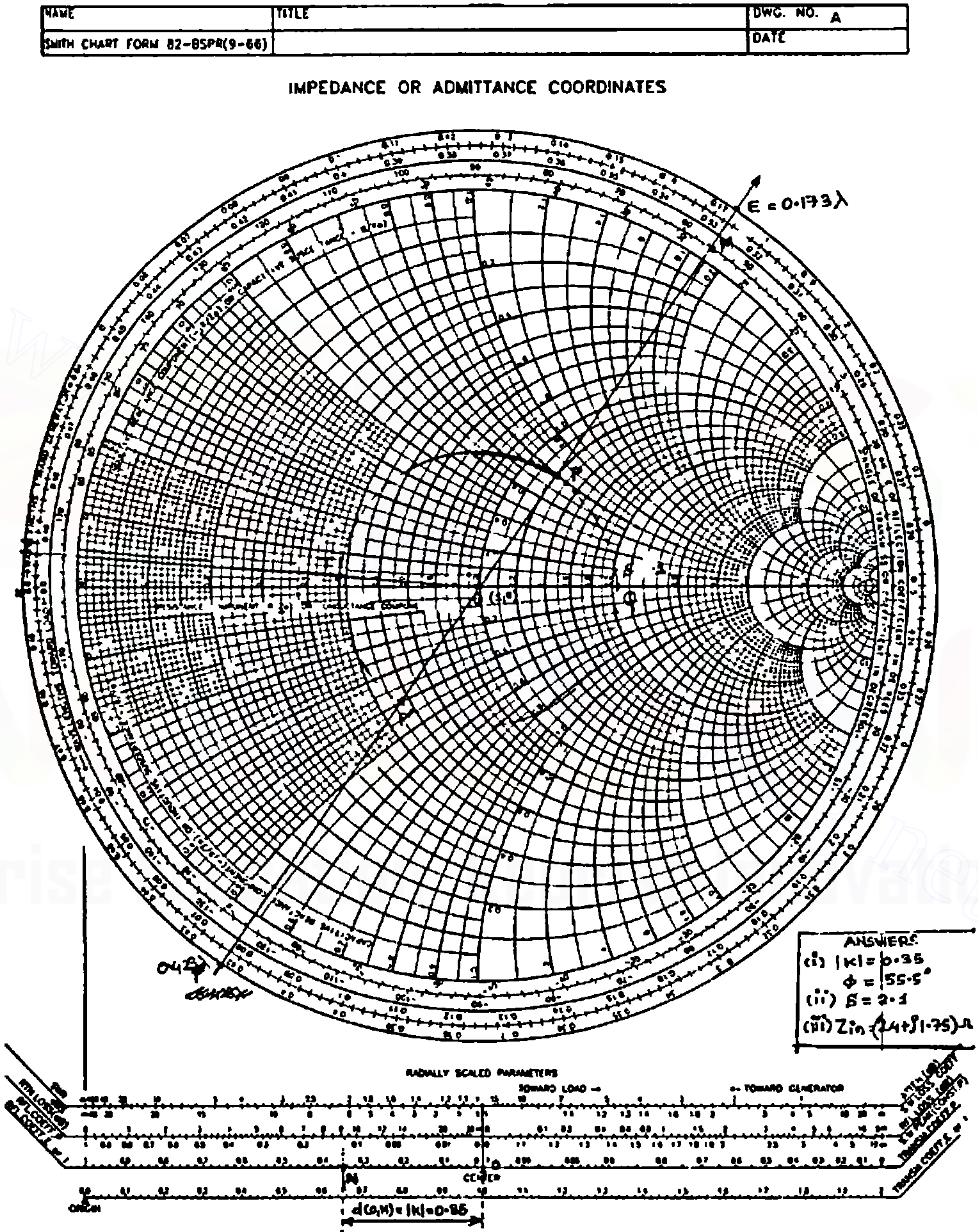

NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
E = 0·173λ
ANSWERS
(i) |K| = 0·35
ϕ = 55·5°
(ii) S = 2·1
(iii) Zin = (24+j1·75)Ω
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN
d(O,M) = |K| = 0·35

Fig. 2.30

5. Impedance to Admittance Conversion

Consider that the impedance is represented by point P on the Smith chart as shown in the Fig. 2.30. To find the admittance of the impedance located at point A, it is rotated through the constant S-circle by a distance equal to 0.25 λ. Let the point p' is located on the diameter exactly opposite to P.

For above example, the terminating impedance is given by,

$$Z_R = 60 + j\,40$$

Hence the terminating admittance, by calculation is given by,

$$Y_R = \frac{1}{Z_R} = \frac{1}{60+j40} = \frac{1}{72.111\angle 33.69} = 0.01386\angle -33.69^{\circ}$$

$$= 0.0115 - j\,7.688 \times 10^{-3}\ \mho$$

Now from the chart point P' placed diametrically opposite to P can be obtained by moving point P exactly a distance equal to 0.25λ.

Point P' is the intersection of g_r= 0.58 circle and $-b_i$ = – 0.4 circle. Hence the normalized at pint P' is given by,

$$y_R = 0.58 - j\,0.4$$

Hence the actual value of the admittance is given by,

$$Y_R = G_0[y_R] = \frac{1}{R_0}[y_R] = \frac{0.58 - j0.4}{50} = 0.016 - j\,8 \times 10^{-3}\ \mho$$

2.16 Single Stub Matching using Smith Chart

When the high frequency line is terminated in its characteristic impedance R_0, it operates as a smooth line. Under such conditions reflections are absent. But practically, the load connected to the line may not be equal to R_0 giving rise to reflections. In order to overcome this difficulty, impedance matching technique is used. The section used for the impedance matching can be realized by using a open or closed stub lines.

In single stub matching technique, a stub of suitable length is connected in parallel with a line at certain distance from load. Using such stub line, antiresonance is achieved providing impedance at antiresonance equal to R_0. The stub to be connected provides susceptance equal in magnitude but opposite in phase as compared with that of the load.

We have discussed the numerical method for the analysis of single stub matching of a line. Let us now study the same analysis using the Smith chart.

Firstly, the stub is connected in parallel with a line hence it is convenient to work with admittances. Following are the steps to be followed while using Smith chart of single stub matching analysis.

Step 1 : First locate the normalized load impedance point on the Smith chart, say point p.

Step 2 : Draw a constant S - circle with centre of the chart as the centre of circle and distance from centre to point P as radius. This circle cuts the horizontal axis at the right of the centre. This indicates SWR value before the use of stub.

Step 3 : Locate a point Q on the constant S-circle drawn, exactly diametrically opposite to the point P. This indicates normalized load admittance.

Step 4 : Locate the point of intersection of the constant S-circle and the chart circle corresponding to $\frac{Y}{G_0}=1$. The circle for $\frac{Y}{G_0}=1$ is the locus of points with real part of line admittance i.e. line conductance is unit. Let the point is denoted by R.

Step 5 : Measure the distance along βs scale from point Q to R by moving clockwise i.e from load to generator. This distance corresponds to the distance from the load at which the stub is to be connected.

Step 6 : The susceptance at point R i.e. at point of intersection of constant S-circle and $g_i = 1 \left(\text{i.e.} \frac{Y}{G_0}=1\right)$ circle indicates the susceptance of the line at the point of stub connection. When such point R is located below the horizontal axis, the susceptance is negative indicating inductive susceptance. And if the point R is located above the horizontal axis, the susceptance is positive indicating capacitive susceptance. Assume that point R is located above the horizontal axis indicating capacitive susceptacne of positive value is jb. This susceptance must be resonated by the stub line having negative inductive susceptance of value –jb.

Step 7 : The input admittance of the short-circuited stub line with inductive susceptance equal to –jb say can be marked at the intersection of K = 1 i.e. $S=\infty$ circle (outer rim of the chart) and -jb susceptance circle. Let the point be denoted by T.

Step 8 : The length of the short circuited stub can be calculated by moving from extreme right point on the horizontal point (i.e. short circuit point) to the point T.

The analysis of the single stub matching to the line using the Smith chart is illustred in the Fig. 2.31.

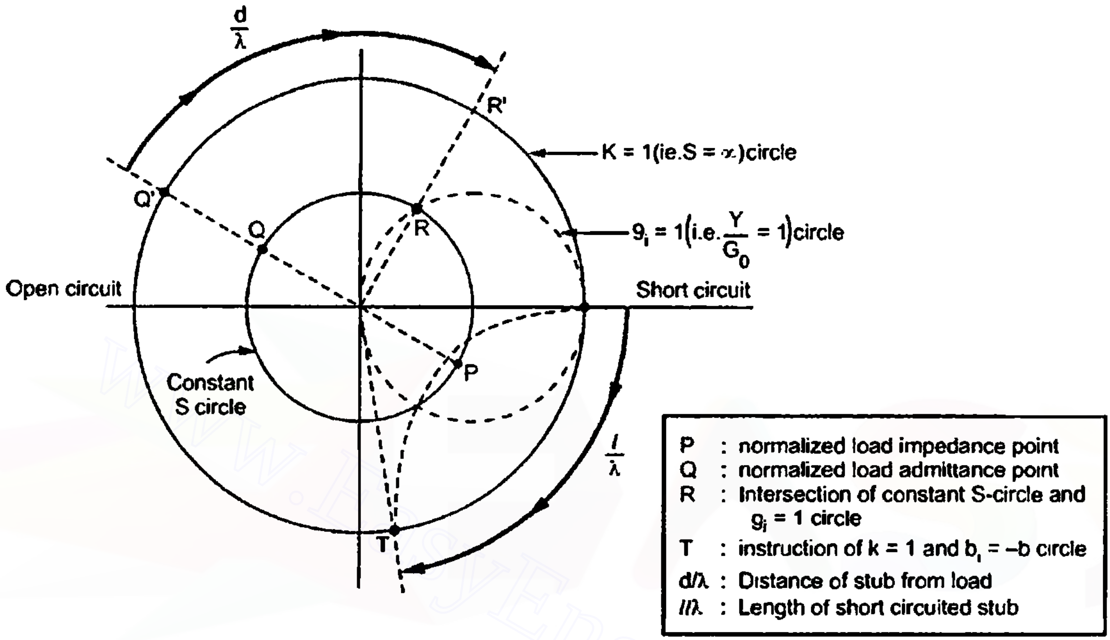


Fig. 2.31 Illustration of single stub matching to the line using Smith Chart

2.17 Double Stub Impedance Matching of a Line

By using single stub impedance matching technique, the reflection losses are reduced considerably. But the main disadvantage is that this technique is suitable for fixed frequency only. So as frequency changes, the location of the stub will have to be changed. Anotehr disadvantages is that, for adjusting the stub for final position, along the line, the stub has to be moved or repositioned. This is possible for open wire conductor transmission line. But in case of co-axial cable it is difficult to locate V_{min} point without a slotted section.

To overcome these disadvantages, a double stub impedance matching technique is used. In this technique, two different short circuited stubs of lengths l_1 and l_2 are used for impedance matching. The practical set up for the double stub impedance matching of a line is as shown in the Fig. 2.32.

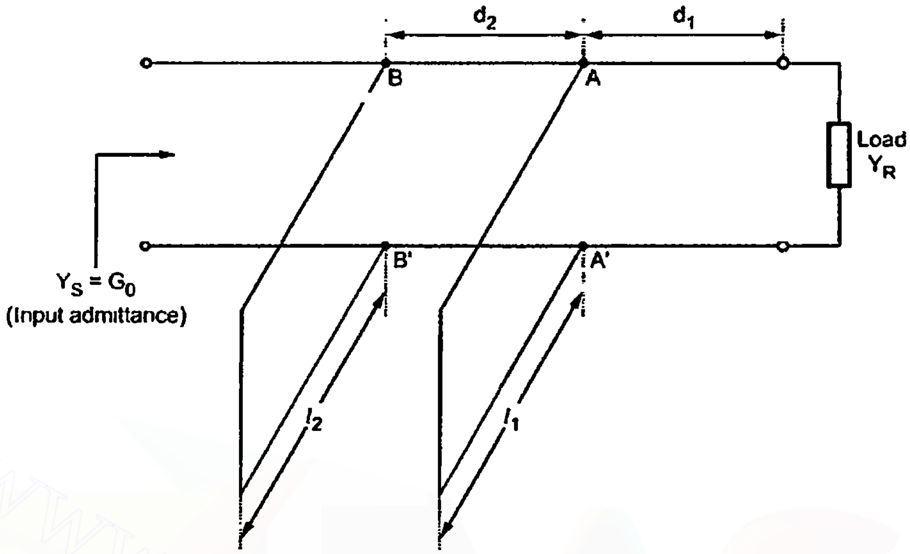


Fig. 2.32 Double stub impedance to a line with termination Y_R

Let stub1 be located at point A – A′, a distance d_1 from the load. Let the length of the stub-1 be l_1. Similarly let stub–2 be located at point B – B′, a distance d_2 away from the stub-1. Let the length of the stub-2 be l_2.

The input impedance of a dissipationless line at any point distance s away from load is given by,

$$\boxed{Z_S = Z_0\left[\frac{Z_R + jZ_0 \tan \beta s}{Z_0 + jZ_r \tan \beta s}\right]} \qquad \text{... (1)}$$

But $Z_S = \frac{1}{Y_S}$, $Z_0 = \frac{1}{Y_0}$, $Z_R = \frac{1}{Y_R}$.

Hence equation (1) can be written as,

$$\frac{1}{Y_S} = \frac{1}{Y_0}\left[\frac{\frac{1}{Y_R} + j\frac{1}{Y_0} \tan \beta s}{\frac{1}{Y_0} + j\frac{1}{Y_R} \tan \beta s}\right]$$

$$\therefore \qquad \frac{Y_0}{Y_S} = \left[\frac{1 + j\frac{Y_R}{Y_0} \tan \beta s}{\frac{Y_R}{Y_0} + j \tan \beta s}\right]$$

Taking reciprocal on both the sides, we get,

$$\therefore \quad \frac{Y_S}{Y_0} = \left[\frac{\frac{Y_R}{Y_0} + j\tan\beta s}{1 + j\frac{Y_R}{Y_0}\tan\beta s}\right] \quad \text{...(2)}$$

Let $\frac{Y_S}{Y_0} = y_S$ = normalized input admittance and

$\frac{Y_R}{Y_0} = y_R$ = normalized load admittance

Substituting values of normalized admittances in equation (2), we get,

$$\boxed{y_S = \frac{y_R + j\tan\beta s}{1 + j\, y_R \tan\beta s}} \quad \text{... (3)}$$

Rationalizing R.H.S. of equation (3), we get,

$$y_S = \frac{(y_R + j\tan\beta s)(1 - jy_R\tan\beta s)}{1 + y_R^2\tan^2\beta s}$$

$$\therefore \quad y_S = \frac{y_R + j\tan\beta s - j\, y_R^2 \tan\beta s + y_R \tan^2\beta s}{1 + y_R^2 \tan^2\beta s}$$

$$\therefore \quad y_S = \frac{y_R + y_R \tan^2\beta s - j\tan\beta ys - j\, y_R^2 \tan\beta s}{1 + y_R^2 \tan^2\beta s}$$

$$\therefore \quad \boxed{y_S = \frac{y_R(1 + \tan^2\beta s)}{(1 + y_R^2\tan^2\beta s)} + j\frac{(1 - y_R^2)\tan\beta s}{(1 + y_R^2\tan^2\beta s)}} \quad \text{... (4)}$$

Now the stub–1 is located at point A – A′, at a distance $s = d_1$ from the load. Hence substituting value of s as d, in equation (4), we get,

$$y_S = \frac{y_R(1 + \tan^2\beta d_1)}{(1 + y_R^2\tan^2\beta d_2)} + j\frac{(1 - y_R^2)\tan\beta d_1}{(1 + y_R^2\tan^2\beta d_1)} = g_i + jb_i \quad \text{...(5)}$$

When a stub-1 having a susceptance $\pm jb_1$ is added at this point, the new admittance value will be

$$y'_S = g_i + jb'_i \quad \text{...(6)}$$

Since the input admittance of a short circuited stub is purely imaginary, the conductance i.e. real part of the new admittance y'_S will remain unchanged. Here $b'_i = b_i \pm b_1$. Now the input admittance of line at points B – B′ should be equal to G_0

so that line appears to be terminated into its characteristic impedance. This point B – B' should be located such that the normalized admittance at this point is given by,

$$y'_B = \frac{Y_S}{G_0} = 1 \pm jb_2 \quad \text{...(7)}$$

Then finally the length of stub-2 is adjusted such that susceptance of the stub–2 $\mp jb_2$ resonates with susceptance jb_2 at point 2-2' and the desired admittance is 1 at B – B'.

Generally there is restriction on the spacing between the two stubs. Practically the two stubs must be separated by a distance not move than or equal to $\frac{\lambda}{2}$. Because the input admittance repeats after every $\frac{\lambda}{2}$ distance. Typically the two stubs are separated by fixed distances $\frac{\lambda}{16}, \frac{\lambda}{8}, \frac{3\lambda}{16}, \frac{3\lambda}{8}$ etc. The most commonly $\frac{\lambda}{4}$ ro $\frac{3\lambda}{8}$ separation between two stubs is preferred.

The important point to be noted is that the matching takes place between the points B-B' and the generator. So there are chances of having reflection loss after points B-B'. In order to minimize these losses, the stubs are located very close to the load. Sometimes the first stub is located at load itself. But the common practice is to keep distance of 0.1λ to 0.15λ between load and the first stub.

2.17.1 Double Stub Impedance Matching with Spacing between Two Stubs Equal to $\frac{\lambda}{4}$

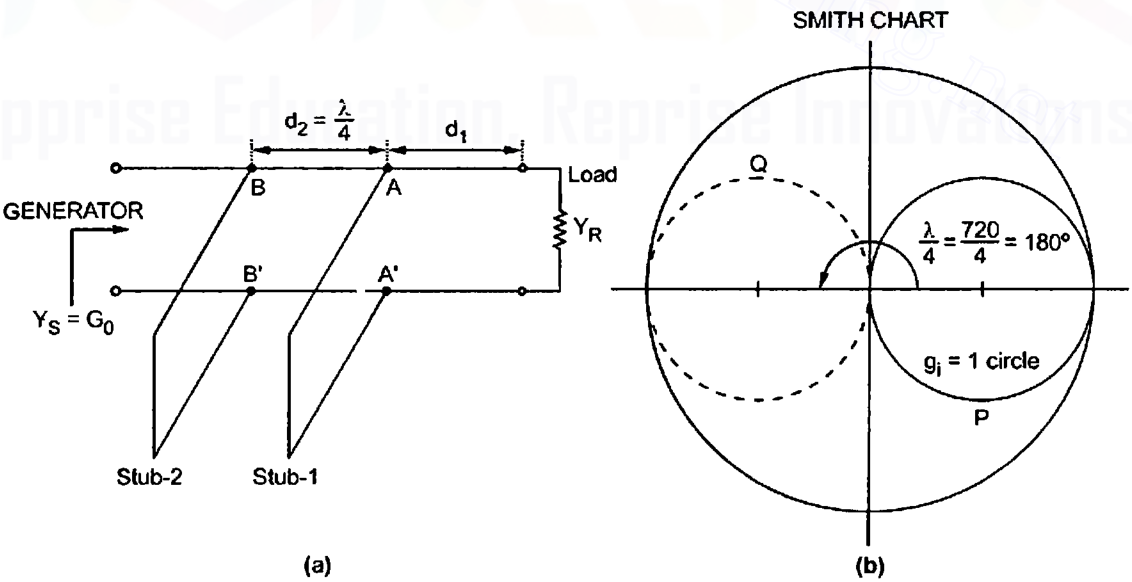


Fig.2.33 Double stub impedance matching with $\frac{\lambda}{4}$ separation between stubs

Consider that the two stubs are separated by a distance $\frac{\lambda}{4}$. The main requirement is that the input admittance looking toward load at point B-B' must be $Y_s = G_0$. Thus at this location the normalized admittance should be

$$\frac{Y_S}{G_0} = 1 + jb_i$$

The locus of all such points on the chart is given by a $\frac{Y_S}{G_0}$ circle passing through g_i=1. This locus is nothing but a circle P as shown in the Fig. 2.33 (b). All the points on this circle resonates with the stub-2 of susceptance $\pm jb_i$ so as to give normalized input admittance at $B - B'$ equal to $\frac{Y}{G_0} = 1$.

The portion of the line between B - B' and A - A' is of length $\frac{\lambda}{4}$. Hence it serves a quarter wave line as transformer. This portion transforms all the admittances on the locus circle P into the new admittances an another locus circle Q. This locus circle Q can be obtained by shifting each point on circle P in anticlockwise direction by 180° rotation or quarter wavelength on the chart.

When stub-1 also transforms the input admittance of the line to the right of A-A' into the admittances represented on the locus circle Q, the transformer between two stubs further the admittances in anticlockwise direction on the locus circle P at B-B'. This gives the admittance at B-B' as $\frac{Y_S}{G_0} = 1 + j\,b_i$. Then the susceptance of the the stub-2 can resonate with the line susceptance at point B-B' representating the line to the left of B-B' with a proper termination indication $\frac{Y_S}{G_0} = 1 + j0$.

2.17.2 Double Stub Impedance Matching with Spacing between Two Stubs Equal to $\frac{3\lambda}{8}$

When the spacing between the two stubs equal to $\frac{\lambda}{4}$ is not suitable for particular application, stub spacing equal to $\frac{3\lambda}{8}$ is preferred. The procedure used for $\frac{\lambda}{4}$ spacing between the stubs can be used for $\frac{3\lambda}{8}$ spacing but then we get a different locus circle Q. As the distance between the two stubs has increased to $\frac{3\lambda}{8}$, the locus circle Q can

be obtained by rotating the locus circle P in anticlockwise direction through $\frac{3\lambda}{8}$ wavelength i.e. $\frac{3(720)}{8} = 270°$ as shown in the Fig. 2.34.

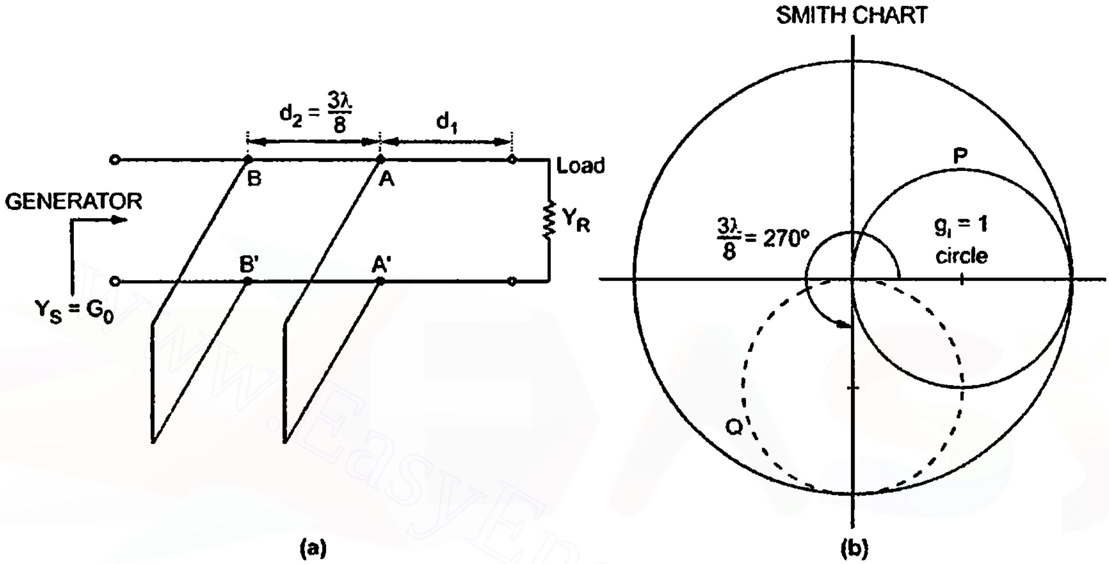


Fig. 2.34 Double Stub impedance matching with $\frac{3\lambda}{8}$ separation between stubs

Examples with Solutions

Example 2.7 *A line having characteristic impedance of 50 Ω is terminated in load impedance (75 + j75) Ω. Determine the reflection coefficient and voltage standing wave ratio.*

Solution : The reflection coefficient is given by

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{(75 + j\,75) - 50}{(75 + j\,75) + 50} = \frac{25 + j\,75}{125 + j\,75}$$

$$\therefore \quad K = \frac{79.056 \angle 71.56°}{145.7738 \angle 30.96°} = 0.5423 \angle 40.6$$

The voltage standing wave ratio is given by,

$$VSWR = SWR = S = \frac{1 + |K|}{1 - |K|} = \frac{1 + 0.5423}{1 - 0.5423}$$

$$\therefore \quad S = 3.369$$

Example 2.8 *A line with characteristic impedance of $692\angle -12°$ is terminated in $200\ \Omega$ resistor. Determine K and S.*

Solution : Given

$$Z_0 = 692\angle -12°\ \Omega = 678.878 - j\,143.87$$

$$Z_R = 200\ \Omega$$

The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{200-(678.878 - j\,143.87)}{200+(678.878-j\,143.87)} = \frac{-467.878 + j\,143.87}{878.878 - j\,143.87}$$

$$\therefore \quad K = \frac{498.1\ \angle 163.21}{890.57\angle -9.29}$$

$$\therefore \quad K = 0.559\ \angle 172.5°$$

Then the standing wave ratio is given by,

$$S = \frac{1+|K|}{1-|K|} = \frac{1+0.559}{1-0.559} = 3.535$$

Example 2.9 *A load of admittance $\frac{Y_R}{G_0} = 1.25 + j\,0.25$. Find the length and location of single stub tuner short circuited.* **(April-98)**

Solution : The normalized load admittance is given as,

$$\frac{Y_R}{G_0} = 1.25 + j\,0.25$$

$$\therefore \quad \frac{1/Z_R}{1/R_0} = (1.25 + j\,0.25)$$

$$\therefore \quad \frac{R_0}{Z_R} = (1.25 + j\,0.25)$$

The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{Z_R - R_0}{Z_R + R_0} = \frac{1 - \frac{R_0}{Z_R}}{1 + \frac{R_0}{Z_R}}$$

$$\therefore \quad K = \frac{1-(1.25+j\,0.25)}{1+(1.25+j\,0.25)}$$

$$= \frac{-0.25 - j\,0.25}{2.25 + j\,0.25}$$

$$= \frac{0.3535 \angle -135°}{2.2638 \angle 6.34°}$$

$$= 0.1561 \ \angle -141.34°$$

$$= 0.1561 \ \angle -2.466°$$

Calculating value of $\cos^{-1}(|K|)$,

$$\therefore \qquad \cos^{-1}(|K|) = \cos^{-1}(0.1561) = 1.414$$

Calculation for the length and location of the stub :

Case (1) :

$$s_1 = \frac{\phi + \pi - \cos^{-1}(|K|)}{2\beta}$$

$$= \frac{\phi + \pi - \cos^{-1}(|K|)}{2\left(\frac{2\pi}{\lambda}\right)}$$

$$= \frac{-2.466 + \pi - 1.414}{4\pi} \cdot \lambda$$

$$= -0.0587\ \lambda$$

Length of the stub,

$$L = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-|K|^2}}{2\,|K|}\right]$$

$$= \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(0.3535)^2}}{2(0.3535)}\right]$$

$$= 0.1469\ \lambda$$

Case (2) :

$$s_1 = \frac{\phi + \pi + \cos^{-1}(|K|)}{4\pi} \cdot \lambda$$

$$= \frac{-2.466 + 3.142 + 1.414}{4\pi} \cdot \lambda$$

$$= 0.1662\ \lambda$$

Length of the stub,

$$L = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-|K|^2}}{-2\,|K|}\right]$$

$$= \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(0.3535)^2}}{-2(0.3535)}\right]$$

$$= \frac{\lambda}{2\pi} \tan^{-1}(-1.3231)$$

$$= \frac{\lambda}{2\pi} \left[\pi - \tan^{-1}(1.013231)\right]$$

$$= 0.353\,\lambda$$

Example 2.10 *A lossless transmission line with $Z_0 = 75\,\Omega$ and of electrical length $l = 0.3\,\lambda$ is terminated with load impedance of $Z_R = (40 + j\,20)\Omega$. Determine the reflection coefficient at load, SWR of line, input impedance of the line.* **[Oct - 96]**

Solution : Given

$$Z_0 = R_0 = 75\,\Omega$$

$$Z_R = (40 + j\,20)\,\Omega$$

The reflection coefficient is given by,

$$K = \frac{Z_R - R_0}{Z_R + R_0}$$

$$= \frac{(40 + j\,20) - 75}{(40 + j\,20) + 75}$$

$$= \frac{-35 + j\,20}{115 + j\,20}$$

$$= \frac{40.311\,\angle 29.74^\circ}{116.726\,\angle 9.86^\circ}$$

$$= 0.3453\,\angle 19.88^\circ$$

The standing wave ratio is given by,

$$S = \frac{1 + |K|}{1 - |K|}$$

$$= \frac{1 + 0.3453}{1 - 0.3453}$$

$$= 2.0548$$

The input impedance of the line is given by,

$$Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan\left(\frac{2\pi s}{\lambda}\right)}{R_0 + j Z_R \tan\left(\frac{2\pi s}{\lambda}\right)} \right]$$

$$= 75 \left[\frac{(40 + j\,20) + j\,75 \tan\left(\frac{2\pi \times 0.3\lambda}{\lambda}\right)}{75 + j(40 + j20) \tan\left(\frac{2\pi \times 0.3\lambda}{\lambda}\right)} \right]$$

$$= 75 \left[\frac{40 + j\,20 + j(-230.82)}{75 + (-j\,123.1) + (+61.55)} \right]$$

$$= 75 \left[\frac{40 - j\,210.82}{136.55 - j\,123.1} \right]$$

$$= 75 \left[\frac{214.581 \angle -79.25°}{183.84 \angle -42.03°} \right]$$

$$= 75\,(1.167 \angle -37.222°)$$

$$= (69.7 - j\,52.95)\ \Omega$$

Example 2.11 *A lossless RF line has Z_0 of 600 Ω and is connected to a resistive load of 75 Ω. Find the position and length of short circuited stub of same construction as line which would enable the main length of a line to be correctly terminated at 150 MHz.* **[April - 98]**

Solution : Given

$$f = 150 \text{ MHz}$$

$$R_0 = 600\ \Omega$$

$$Z_R = 75\ \Omega$$

Calculating λ first,

$$f.\lambda = c$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{150 \times 10^6} = 2 \text{ m}$$

The reflection coefficient is given by

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{Z_R - R_0}{Z_R + R_0}$$

$$\therefore \quad K = \frac{75-600}{75+600} = \frac{-525}{675} = -0.7777$$

$$\therefore \quad K = 0.7777 \angle \pi^c = 0.7777 \angle 180°$$

The two possible locations of stub are as follows.

Case (1) :

$$s_1 = \frac{\phi + \pi - \cos^{-1}(|K|)}{2\beta}$$

But

$$\beta = \frac{2\pi}{\lambda}$$

$$\therefore \quad s_1 = \frac{\phi + \pi - \cos^{-1}(|K|)}{4\pi} \cdot \lambda$$

$$= \frac{\pi + \pi - \cos^{-1}(0.7777)}{4\pi}(2)$$

$$= 0.8918 \text{ m}$$

The length of the stub is given by

$$L = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(|K|)^2}}{2|K|}\right]$$

$$\therefore \quad L = \frac{2}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(0.7777)^2}}{2(0.7777)}\right]$$

$$= 0.1222 \text{ m}$$

Case (2) :

$$s_1 = \frac{\phi + \pi + \cos^{-1}(|K|)}{4\pi} \cdot \lambda = \frac{\pi + \pi + \cos^{-1}(0.7777)}{4 \times \pi} \times 2$$

$$= 1.108 \text{ m}$$

The length of the stub is given by

$$L = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(|K|)^2}}{-2|K|}\right] = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(0.7777)^2}}{-2(0.7777)}\right]$$

$$= \frac{2}{2\pi} \tan^{-1}(-0.4041) = \frac{1}{\pi}(\pi - 0.384)$$

$$= 0.8777 \text{ m}$$

Selecting a point located nearest to the load. Hence the stub location nearest to the load is calculated in case (1).

The stub must be located at a distance 0.8918 m from the load and the length of the stub required is 0.1222 m.

Example 2.12 *Design a quarter wave transformer to match a load of 200 Ω to a source resistance of 500 Ω. Operating frequency is 200 MHz.* **[Oct - 97]**

Solution : For a quarter wave transformer, the input impedance is given by,

$$Z_{in} = Z_S = \frac{R_0^2}{Z_R}$$

The source impedance $Z_S = 500\Omega$

load impedance = $Z_R = 200\ \Omega$

$$\therefore \quad 500 = \frac{R_0^2}{200}$$

$$\therefore \quad R_0^2 = (500)(200)$$

$$\therefore \quad R_0^2 = 100000$$

$$\therefore \quad R_0 = 316.22\ \Omega$$

The frequency of operation is

$$f = 200 \text{ MHz}$$

Hence the wavelength is given by,

$$f.\lambda = c$$

$$\therefore \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{200 \times 10^6} = 1.5 \text{ m}$$

∴ The length of quarter wave line is given by,

$$s = \frac{\lambda}{4} = \frac{1.5}{4} = 0.375 \text{ m}$$

Hence the quarter wave line is as shown below in the Fig. 2.35

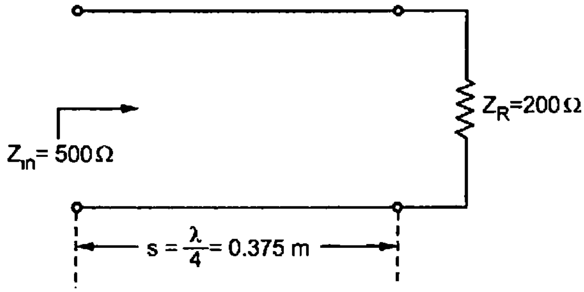


Fig. 2.35

Example 2.13 *Determine a length and impedance of a quarter wave transformer that will match a 150Ω load to a 75Ω line at a frequency of 12 GHz. Derive formula used.* **[April-97]**

Solution : Given : $Z_R = 150\ \Omega$, $R_0 = 75\ \Omega$, f = 12 GHz

For a quarter wave transformer, the input impedance is given by,

$$Z_{in} = Z_S = \frac{R_0^2}{Z_R} \left(\text{or } R_0' = \sqrt{Z_S \cdot Z_R}\right)$$

$$\therefore \quad Z_{in} = Z_S = \frac{(75)^2}{150} = 37.5\ \Omega$$

Thus R_0' of the matching section is 37.5 Ω

The operating frequency is 12 GHz.

∴ The wavelength can be calculated as,

$$f.\lambda = c$$

$$\therefore \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{12 \times 10^9} = 0.025 \text{ m} = 25 \text{ cm}$$

Hence the length of quarter wave line is given by

$$s = \frac{\lambda}{4} = \frac{0.025}{4} = 6.25 \text{ cm}$$

Thus the quarter wave transformer is as shown in the Fig. 2.36.

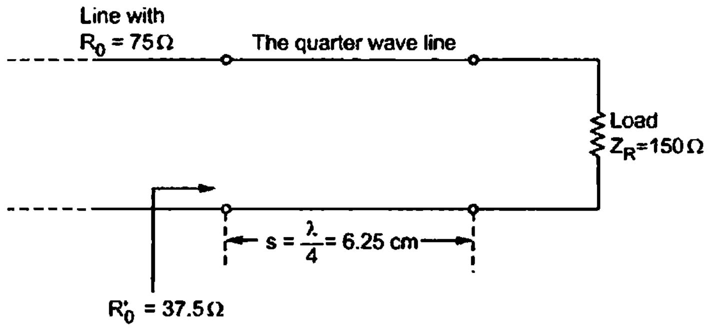


Fig. 2.36 A quarter wave matching section

Example 2.14 *A slotted line experiment performed with the following results : Distance between successive minima is 21 cm, distance of first voltage minimum from load is 0.9 cm. SWR of line is 2.5. If Z_0 = 50 Ω, determine load impedance.* **[Oct - 96]**

Solution : The distance between two successive voltage minima is given by,

$$\frac{\lambda}{2} = 21$$

$$\therefore \quad \lambda = 42 \text{ cm} = 42 \times 10^{-2} \text{ m} = 0.42 \text{ m}$$

The frequency of operation is given by,

$$f\lambda = c$$

$$\therefore \quad f = \frac{c}{\lambda} = \frac{3 \times 10^8}{42 \times 10^{-2}} = 0.7142 \text{ GHz}$$

The standing wave ratio S = 2.5

The load impedance interms of S and s' (i.e. distance of first voltage minimum from load) is given by

$$Z_L = R_0 \left[\frac{1 - j\, S\tan\left(\frac{2\pi s'}{\lambda}\right)}{S - j\tan\left(\frac{2\pi s'}{\lambda}\right)}\right] = 50\left[\frac{1 - j(2.5)\cdot\tan\left(\frac{2\pi \times 0.9}{42}\right)}{2.5 - j\tan\left(\frac{2\pi \times 0.9}{42}\right)}\right]$$

$$= 50\left[\frac{1 - j(0.25)(0.1355)}{2.5 - j(0.1355)}\right] = 50\left[\frac{1 - j\,0.033875}{2.5 - j\,0.1355}\right]$$

$$= 50\left[\frac{1 \angle -1.9401}{2.5 \angle -3.102}\right] = 50\,[0.4 \angle 1.1622°]$$

$$= 20 \angle 1.1622° \ \Omega$$

Example 2.15 *VSWR on a lossless line is found to be 5 and successive voltage minima are 40cm apart. The first voltage minima is observed to be 15 cm from load. The length of a line is 160 cm and the characteristic impedance is 300 Ω. Using Smith chart determine - (i) Load impedance (ii) Sending end impedance.* **[Oct-97]**

Solution : Given $S = 5$ and $\frac{\lambda}{2} = 40$ $\therefore \lambda = 80$

Then first draw a S-circle with radius equal to 5 and chart centre (1, 0) as centre of it.

This circle cuts real axis in two points one at right hand side of centre while other at left hand side.

The point on the left hand side of the centre, where S-circle with S = 5 cuts the axis, is the point of voltage minimum. Let us denote this point as A. The co-ordinates are (0.2, 0).

The last minimum is 15 cm apart from load. But the total length is 160 cm.

Hence the distance between the voltage minimum nearest to the load and the load is $= \frac{15}{80}\lambda = 0.1875\,\lambda$.

So move from point A, from generator to load i.e. in anticlockwise direction, a distance of $0.1875\,\lambda$ to get point B which is nothing but load point.

The co-ordinates of B are (1.05, – 1.9)

Hence normalized load impedance is given by,

$$\frac{Z_L}{Z_0} = \frac{Z_L}{R_0} = 1.05 - j\,1.9$$

$$\therefore \quad Z_L = R_0(1.05 - j\,1.9)$$

$$= 300(1.05 - j\,1.9)$$

$$= (315 - j\,570)\ \Omega$$

Now the total length of the line is 160 cm

Please refer Fig. 2.37 on next page.

So if we travel a distance $\frac{160}{80}\lambda = 2\lambda$ from a point B (i.e. load point) we reach a point C which is a generator or sending end point. But it is clear that the distance 2λ travelled from point B will reach the same point after 4 revolutions each of $0.5\,\lambda$.

$$\therefore \quad Z_{in} = Z_S = Z_R = (315 - j\,570)\ \Omega$$

Hence the line repeats its terminating impedance at the input.

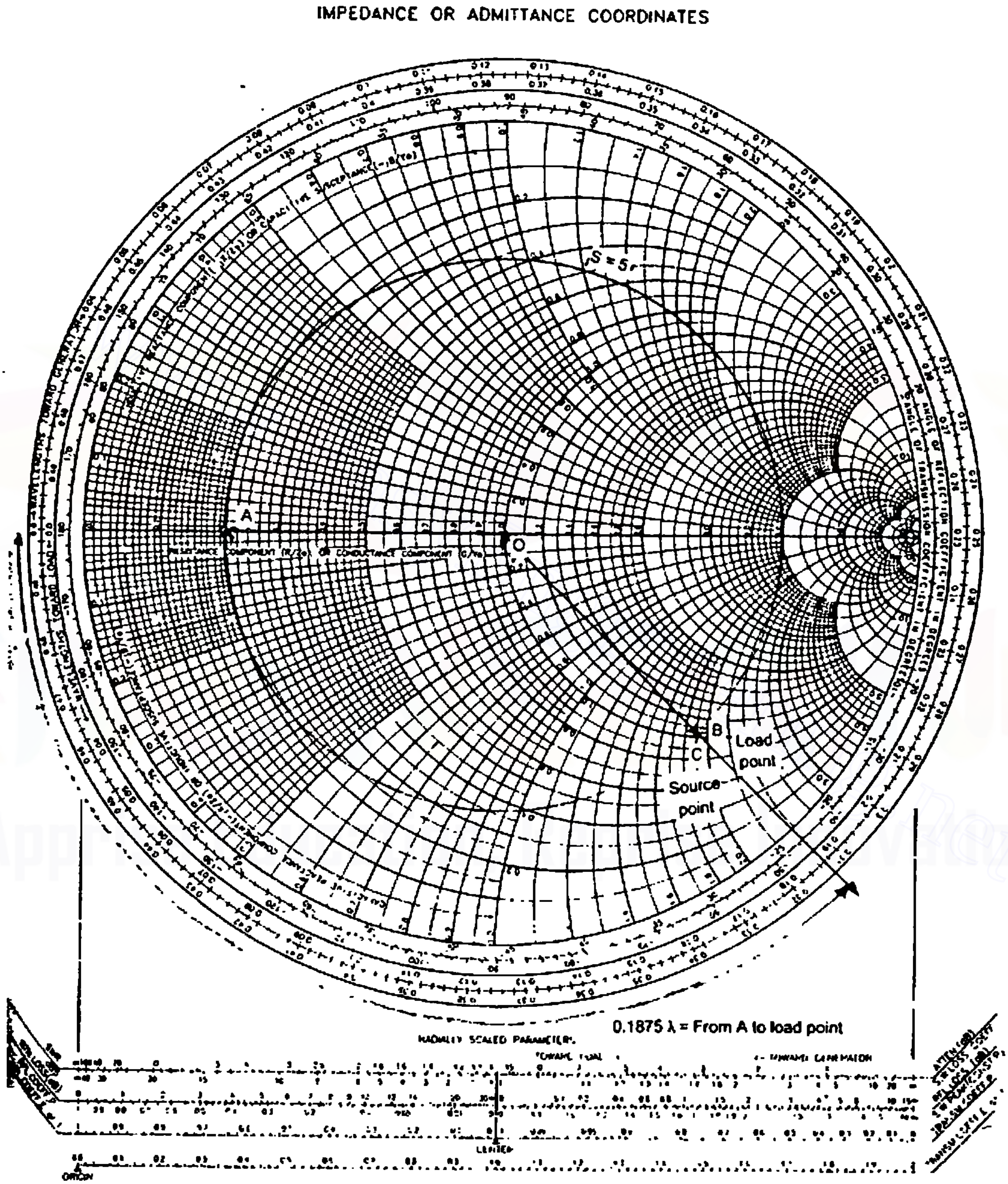

IMPEDANCE OR ADMITTANCE COORDINATES
S = 5r
A
O
B
C
Load point
Source point
0.1875 λ = From A to load point
RADIALLY SCALED PARAMETERS
ORIGIN

Fig. 2.37

Example 2.16 *Find the input impedance of a co-axial line having $R_0 = 95\ \Omega$. The line is 20 m long short circuited at far end and operated at 10 MHz. Neglect line dissipation. Verify answer by solving the problem using Smith chart.*

Solution :

[A] Let us first calculate λ.

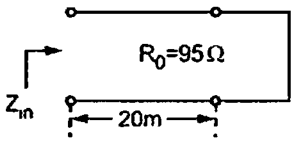


Fig. 2.38

$$f.\lambda = c$$

$$\therefore \quad \lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^6} = 30\ m$$

Input impedance of a line which is shorted at the end is given by,

$$Z_{in} = j R_0 \tan\beta s$$

$$\therefore \quad Z_{in} = j R_0 \tan\left(\frac{2\pi s}{\lambda}\right)$$

$$\therefore \quad Z_{in} = j\,95 \tan\left(\frac{2\times\pi\times 20}{30}\right)$$

$$\therefore \quad Z_{in} = j\,95 \tan(4.178^c)$$

$$\therefore \quad Z_{in} = j\,95\cdot(1.732)$$

$$\therefore \quad Z_{in} = j\,164.54\ \Omega$$

Thus input impedance is inductive.

[B] Let us solve problem by using Smith chart

The line is short circuited.

$$\therefore \quad Z_R = Z_L = 0$$

Hence the normalized value of $\frac{Z_R}{Z_0} = 0$

Locate this point as A point on Smith chart. This will be on U-axis (real axis) at extreme left hand side.

The line is 20 m long and wavelength is 30 m.

$\therefore$ Length of line in terms of wavelength $= \frac{20}{30}\lambda = 0.667\ \lambda$

Please refer Fig. 2.38 on next page.

Hence from point A travel a distance 0.667 λ along the extreme circle (for which R = 0) in clockwise direction (from load to generator).

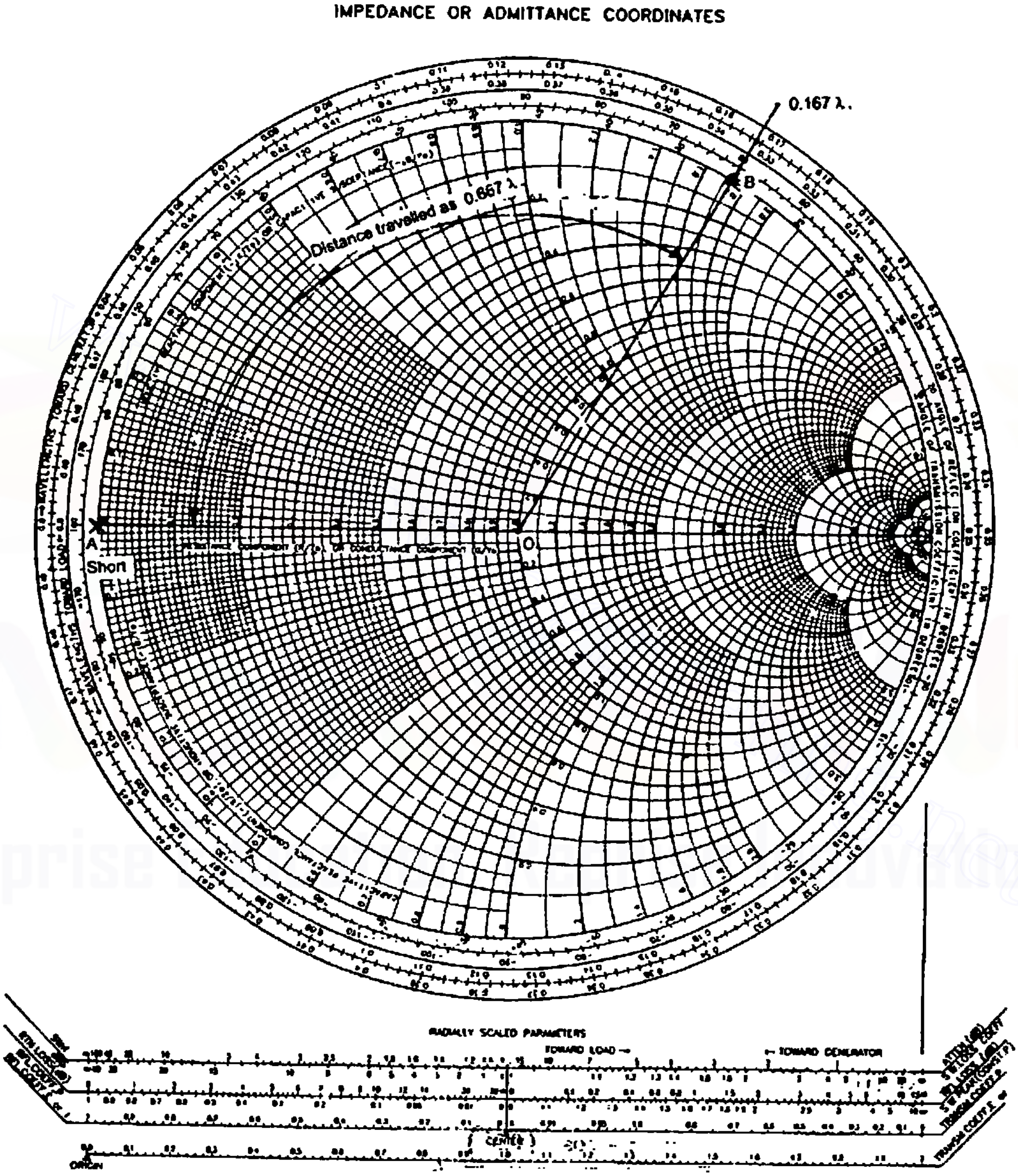

IMPEDANCE OR ADMITTANCE COORDINATES
0.167 λ.
B
Distance travelled as 0.667 λ
A
Short
O
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.39

If we travel a distance $= \frac{\lambda}{2} = 0.5\,\lambda$, we will be again at point A. So move in clockwise direction from point A a distance $= 0.667\,\lambda - 0.5\,\lambda = 0.167\lambda$ to get point B which represents input impedance.

The co-ordinates of point B are (0, j 1.73)

Hence input impedance is given by,

$$Z_{in} = R_0 (0 + j\,1.73)$$

$$= 95\,(0 + j\,1.73)$$

$$= j\,164.35\ \Omega \quad \text{which is purely inductive.}$$

Note : As input impedance is inductive in nature for the line, we got the point in the upper half of the real axis.

Example 2.17 *Determine the length and location of the stub to produce an impedance match on a line of $R_0 = 600\ \Omega$ terminated in $200\angle 0°\ \Omega$. The stub is short circuited at the other end. Verify result by using Smith chart.*

Solution : [A] The reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{200-600}{200+600} = \frac{-400}{800} = -0.5 = 0.5\angle \pi^c$$

Then the location of stub can be calculated as follows.

$$s_1 = \frac{\phi + \pi \pm \cos^{-1}(|K|)}{2\beta} = \frac{\phi + \pi \pm \cos^{-1}(|K|)}{4\pi} \cdot \lambda$$

Case (1) :

$$s_1 = \frac{\phi + \pi - \cos^{-1}(|K|)}{4\pi} \cdot \lambda$$

$$= \frac{\pi + \pi - \cos^{-1}(0.5^c)}{4\pi} \cdot \lambda$$

$$= 0.4166\,\lambda$$

$$\therefore \text{ Length of the stub, } L = \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-|K|^2}}{2\,|K|}\right]^c$$

$$= \frac{\lambda}{2\pi} \tan^{-1}\left[\frac{\sqrt{1-(0.5)^2}}{2(0.5)}\right]$$

$$= 0.1135\,\lambda$$

Case (2) :

$$s_1 = \frac{\phi + \pi + \cos^{-1}(|K|)}{4\pi} \cdot \lambda$$

$$= \frac{\pi + \pi + \cos^{-1}(0.5^c)}{4\pi} \cdot \lambda$$

$$= 0.0833\ \lambda$$

Here $2\pi^c$ are added to $\cos^{-1}(0.5)$; so angle is not changed.

Hence we can write,

$$s_1 = \frac{\cos^{-1}(0.5)}{4\pi}\lambda = 0.08333\ \lambda$$

The length of the stub $L = \frac{\lambda}{2\pi}\tan^{-1}\left[\frac{\sqrt{1-|K|^2}}{-2|K|}\right]$

$$= \frac{\lambda}{2\pi}\tan^{-1}\left[\frac{\sqrt{1-(0.5)^2}}{-2(0.5)}\right]$$

$$= \frac{\lambda}{2\pi}\left(\pi - \tan^{-1} 0.866\right)$$

$$= 0.386\ \lambda$$

[B] Using Smith chart :

Normalized load impedance is given by,

$$\frac{Z_L}{R_0} = \frac{Z_R}{R_0} = \frac{600}{200} = 3$$

$$\therefore \quad S = 3$$

Locate load point A (3, 0) on real axis as shown in the chart. Then draw S-circle with radius 3 and centre of the chart as centre of the circle.

This S-circle cuts unity conductance circle at points B and C as shown in the chart.

First Case : Consider point B

1) Draw radial line OB.

To get location of stub, travel along S-circle towards generator from OA to OB

Hence location of stub = s_1

$$= \text{Radial distance between OB and OA}$$

$$= 0.333\,\lambda - 0.25\,\lambda$$

$$= 0.083\,\lambda$$

2) At point B, the line susceptance is $-j$ 1.2. Hence to cancel this, the stub should offer susceptance of $+ j$ 1.2. Find the point on the outer rim where the $+ j$ 1.2 circle cuts.

From the chart, the circle of j 1.2 cuts outer rim at 0.14 λ.

3) Now starting from SHORT CIRCUIT END move towards generator (in clockwise direction)

$$\therefore \text{Length of the stub} = [0.5 - 0.25]\,\lambda + 0.14\,\lambda$$

$$= 0.39\,\lambda$$

Second Case : Consider point C

1) Draw radial line OC.

2) To get location of stub, travel along S-circle towards generator from OA to OC.

Hence location of stub = s_1

$$= 0.25\,\lambda + 0.168\,\lambda$$

$$= 0.418\,\lambda$$

3) At point C, the susceptance of line is $+j\,1.2$. To cancel this susceptance the stub should offer $-j\,1.2$

Please refer Fig. 2.40 on next page.

Find point on the outer rim where $-j\,1.2$ circle cuts.

From the chart, the circle of $-j\,1.2$ cuts outer rim at 0.36 λ.

4) Now starting from SHORT CIRCUIT END move towards generator (in clockwise direction) to get length of the stub.

$$\therefore \text{Length of the stub} = [0.36\lambda - 0.25\lambda] = 0.11\lambda$$

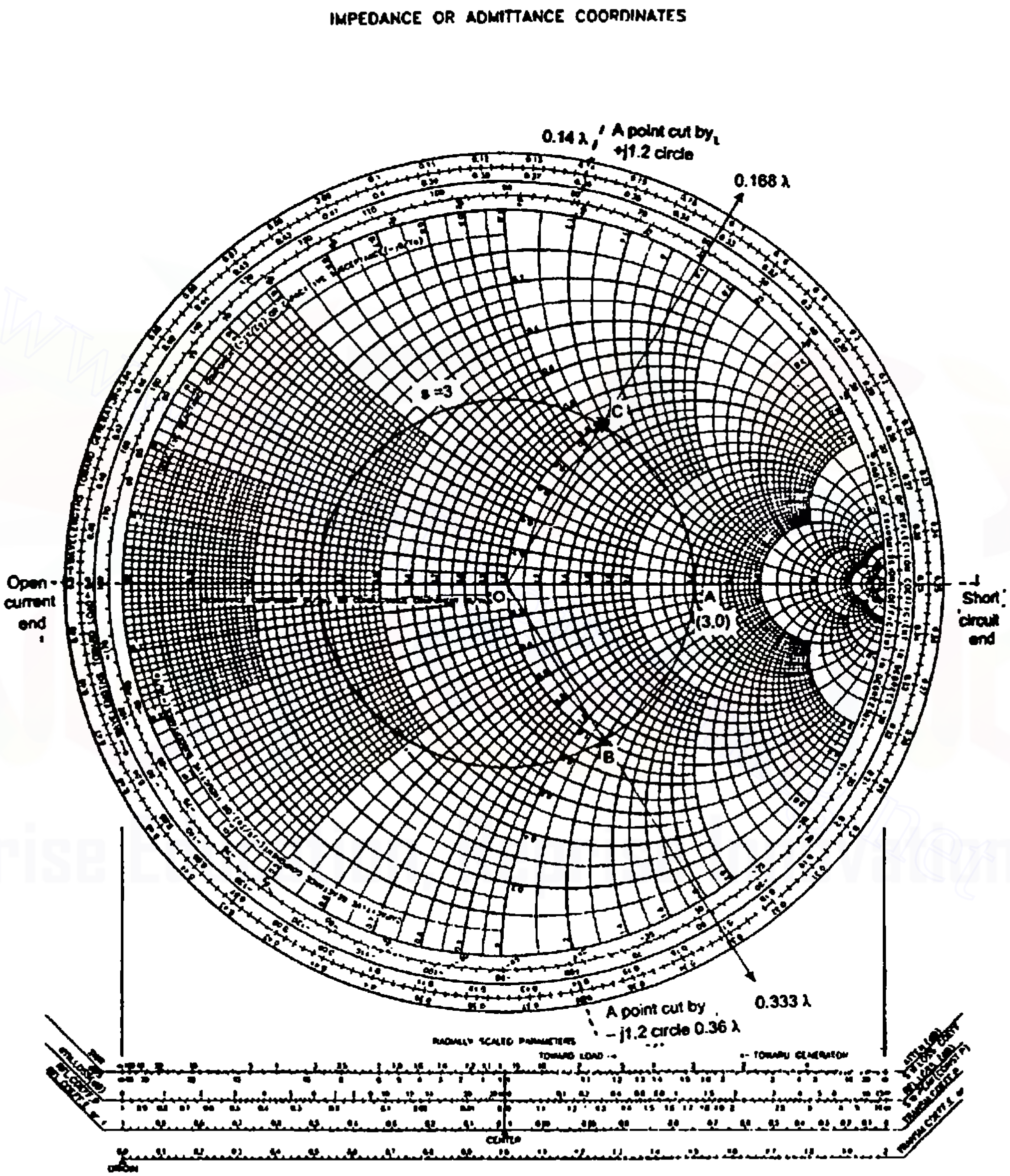

IMPEDANCE OR ADMITTANCE COORDINATES
0.14 λ
A point cut by +j1.2 circle
0.168 λ
s = 3
C
O
A
(3,0)
B
Open - circuit end
Short circuit end
A point cut by - j1.2 circle 0.36 λ
0.333 λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.40

Example 2.18 *A transmission line is terminated in Z_L. Measurements indicate that standing wave minima are 102 cm apart and the last minimum is 35 cm from the load side. The value of S = 2.4 and the characteristic impedance $R_0 = 250\ \Omega$.*

Find : (i) Z_L (ii) Frequency of operation.

Using Smith chart verify the results.

Solution : [A] The successive voltage minima or maxima are separated by distance $\frac{\lambda}{2}$ always.

$$\therefore \quad \frac{\lambda}{2} = 102 \text{ cm}$$

$$\therefore \quad \lambda = 204 \text{ cm} = 2.04 \text{ m}$$

Hence frequency of operation can be calculated as,

$$f.\,\lambda = c$$

$$\therefore \quad f = \frac{c}{\lambda} = \frac{3\times 10^8 \text{ m/s}}{2.04 \text{ m}} = 147.058 \text{ MHz}$$

The load impedance in terms of standing wave ratio S and distance of first minima s′ is given by

$$Z_L = R_0\left[\frac{1 - jS\tan\left(\frac{2\pi s'}{\lambda}\right)}{S - j\tan\left(\frac{2\pi s'}{\lambda}\right)}\right]$$

Here $R_0 = 250$, $S = 2.4$, $s' = 35$ cm, $\lambda = 204$ cm

$$\therefore \quad Z_L = 250\left[\frac{1 - j(2.4)\cdot\tan\left(\frac{2\times\pi\times 35}{204}\right)}{2.4 - j\tan\left(\frac{2\times\pi\times 35}{204}\right)}\right]$$

$$\therefore \quad Z_L = 250\left[\frac{1 - j(2.4)\tan(1.078^c)}{2.4 - j\tan(1.078^c)}\right]$$

$$\therefore \quad Z_L = 250\left[\frac{1-j\,4.469}{2.4-j\,1.8622}\right]$$

$$\therefore \quad Z_L = 250\left[\frac{4.5795\ \angle -77.38°}{3.0377\ \angle -37.8°}\right]$$

$$\therefore \quad Z_L = 376.88\ \angle -39.58°$$

$$\therefore \quad Z_l = (290.47 - j\,240.13)\,\Omega$$

Please refer Fig. 2.41 on next page.

[B] Let us solve using Smith chart.

$$S = 2.4$$

First draw a circle with chart centre (1, 0) as centre of circle and 2.4 as radius. This is S-circle with S = 2.4.

This circle cuts real axis on right hand side of the centre at a point whose co-ordinates are (2.4, 0). This circle cuts at point A on left hand side of centre on the real axis. This point A is the point of voltage minima. Co-ordinates of the point A are (0.42, 0).

The last minimum is 35 cm away from the load.

The total length is 204 cm.

Hence the load is $\frac{35\lambda}{204} = 0.1716\lambda$ away from last minimum.

So if we travel a distance 0.1716 λ from the point A, we will get a point B which is nothing but the load point. Note that we are moving from a voltage minima to load i.e. from generator to load, hence move in counterclockwise direction.

The coordinates of B are, (1.15, – 1.0)

Hence normalized load impedance is given by,

$$\frac{Z_L}{R_0} = (1.15 - j\,1.0)$$

$$\therefore \quad Z_L = R_0\,(1.15 - j\,1.0)$$

$$\therefore \quad Z_L = 250(1.15 - j\,1.0)$$

$$\therefore \quad Z_L = (287.5 - j\,250)\ \Omega$$

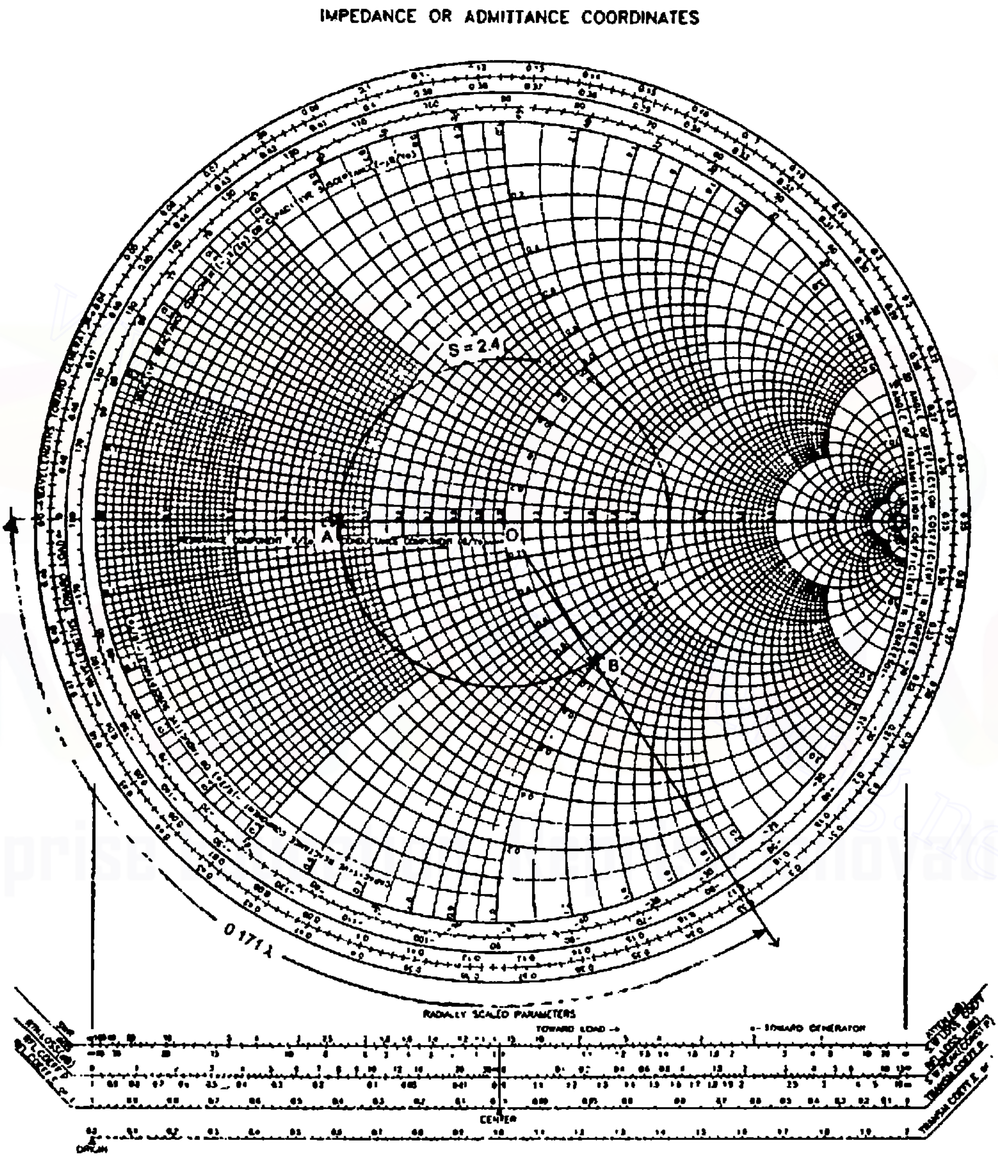

IMPEDANCE OR ADMITTANCE COORDINATES
S = 2.4
A
O
B
0.171 λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.41

The input impedance is given by,

$$Z_S = R_0 \left[\frac{Z_R + j R_0 \tan \beta s}{R_0 + j Z_R \tan \beta s}\right]$$

$$s = \text{total length of line} = 1.183 \lambda$$

$$\therefore \quad Z_S = 55 \left[\frac{(115 + j\,75) + j\,(55) \tan\left(\frac{2\pi \times 1.183\lambda}{\lambda}\right)}{55 + j(115 + j\,75) \tan\left(\frac{2\pi \times 1.183\lambda}{\lambda}\right)}\right]$$

$$= 55\left[\frac{115 + j\,75 + j(55)(2.233)}{55 + (115 + j\,75)(j\,2.233)}\right]$$

$$= 55\left[\frac{115 + j\,75 + j\,122.815}{55 + j\,256.795 - j\,167.475}\right]$$

$$= 55\left[\frac{115 + j\,197.815}{-112.475 + j\,256.795}\right]$$

$$= 55\left[\frac{228.8138 \angle 59.82°}{280.346 \angle 113.653°}\right]$$

$$= 55\,[0.8162 \angle -53.83°]$$

$$= 55\,[0.4816 - j\,0.6589]$$

$$= (26.488 - j\,36.24)\ \Omega$$

[B] By Smith chart :

Normalized load impedance $= \dfrac{Z_R}{R_0} = \dfrac{115 + j\,75}{55}$

$$\therefore \quad \frac{Z_R}{R_0} = 2.09 + j\,1.3636$$

First locate point with co-ordinates $(2.09 + j\,1.3636)$ as load point A as shown in Smith chart.

Then joint point A and centre of the chart O. And with centre as origin draw a circle with radius equal to distance OA. The circle cuts real axis at two points namely B(3.1, 0) and C (0.33, 0)

Please refer Fig. 2.42 on next page.

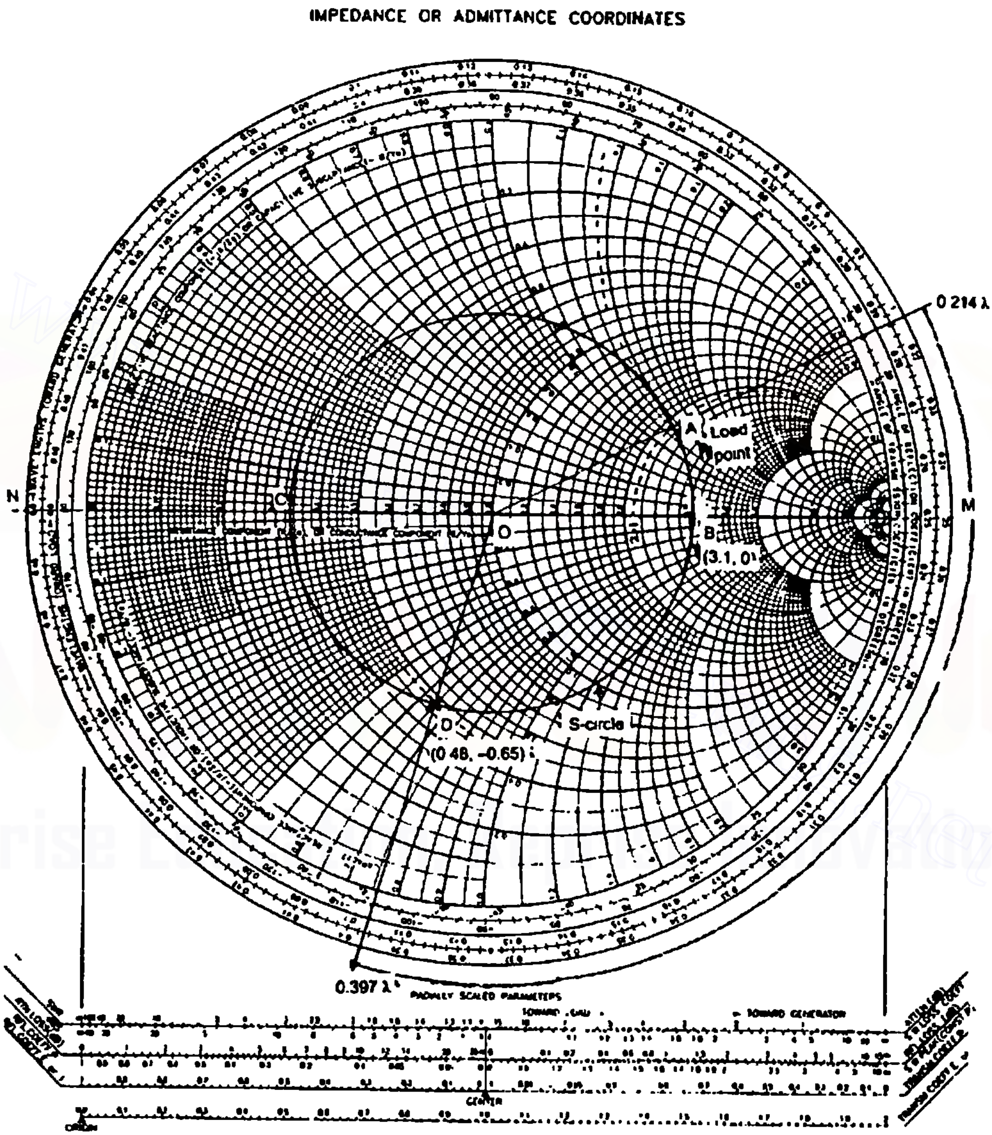

IMPEDANCE OR ADMITTANCE COORDINATES
0.214 λ
A
Load point
N
M
C
O
B
(3.1, 0)
D
S-circle
(0.48, -0.65)
0.397 λ
RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.42

The standing wave ratio is given by the co-ordinate of point B on real axis, i.e. $S = 3.1$.

At the point C we get the voltage minimum nearest to the load.

To get distance of the nearest voltage minimum from the load, let us calculate distance from point A to C moving from load to generator in clockwise direction. For this extend lines OA and OC upto the extreme rim on βs scale. Distance between extreme right point M to extreme left point N is $\frac{\lambda}{4}$ i.e. 0.25 λ. Distance from a line OA to point M on βs scale is given by

$$d = (0.25)\lambda - (0.214)\lambda = 0.036\lambda$$

Adding both distances together, the distance of first voltage minimum from load is given by

$$s' = (0.036\lambda) + (0.25\lambda) = 0.286\lambda$$

Now the distance between the voltage minimum and voltage maximum is $\lambda / 4 = 0.25\,\lambda$.

$\therefore$ s = distance of voltage maximum from the load

$$= s' - \frac{\lambda}{4} = (0.286\,\lambda) - (0.25\,\lambda) = 0.036\,\lambda$$

The distance between the generator and load is nothing but the length of the line which is given as 1.183 λ. To get source point travel distance 1.183 λ along the extreme rim in clockwise direction from load to generator or source. From point A if we complete one revolution to come again at point A, the distance travelled is 1 λ. Now travel remaining distance of 0.183 λ as shown in the chart. Mark the point on the extreme rim and join centre to this point drawing a line.

The line cuts S-circle at point D which gives source or generator point and its coordinates are $(0.48, -0.65)$.

Hence the normalized value of the input impedance is

$$\frac{Z_{in}}{R_0} = \frac{Z_S}{R_0} = (0.48 - j\,0.65)$$

$$\therefore \quad Z_S = Z_{in} = R_0\,(0.48 - j\,0.65)$$

$$= 55(0.48 - j\,0.65)$$

$$= 55\,(0.808\ \angle -53.55°)$$

$$= 44.441\ \angle -53.55°$$

$$= (26.4 - j\,35.75)\ \Omega$$

Example 2.20 *A load impedance of $70\angle 30°$ is connected to a line of 50 Ω. Calculate resulting standing wave ratio and location of the voltage minimum nearest to the load. Verify the result solving the same problem using Smith chart.*

Soution : [A] $Z_R = Z_L = 70\angle 30° = (60.62 + j\,35)\ \Omega$

$$Z_0 = R_0 = 50\Omega$$

Hence the reflection coefficient is given by,

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{(60.62 + j\,35) - 50}{(60.62 + j\,35) + 50}$$

$$= \frac{10.62 + j\,35}{110.62 + j\,35}$$

$$= \frac{36.5757\angle 73.12°}{116.0249\angle 17.55°}$$

$$= 0.3152\ \angle 55.57°$$

Hence the standing wave ratio S is given by,

$$S = \frac{1 + |K|}{1 - |K|} = \frac{1 + 0.3152}{1 - 0.3152} = 1.92$$

The maximum imput impedance occurs at

$$s = \frac{\phi}{2\beta}$$

where ϕ is the angle of the reflection coefficient.

$$\therefore \quad s = \frac{\phi}{2\left(\frac{2\pi}{\lambda}\right)} = \frac{\phi\cdot\lambda}{4\pi} = \frac{\phi}{4\pi}\cdot\lambda$$

$$\therefore \quad s = \frac{55.57°}{4\pi} = \frac{(0.96987)^c}{4\pi}\lambda = (0.0771)\,\lambda$$

Now the minimum impedance occurs at $\lambda / 4$ distance after maximum impedance point. This is the point of voltage minimum i.e. s'.

$$\therefore \quad s' = s + \frac{\lambda}{4} = (0.0771)\,\lambda + (0.25)\,\lambda = 0.3271\,\lambda$$

Please refer Fig. 2.43 on next page.

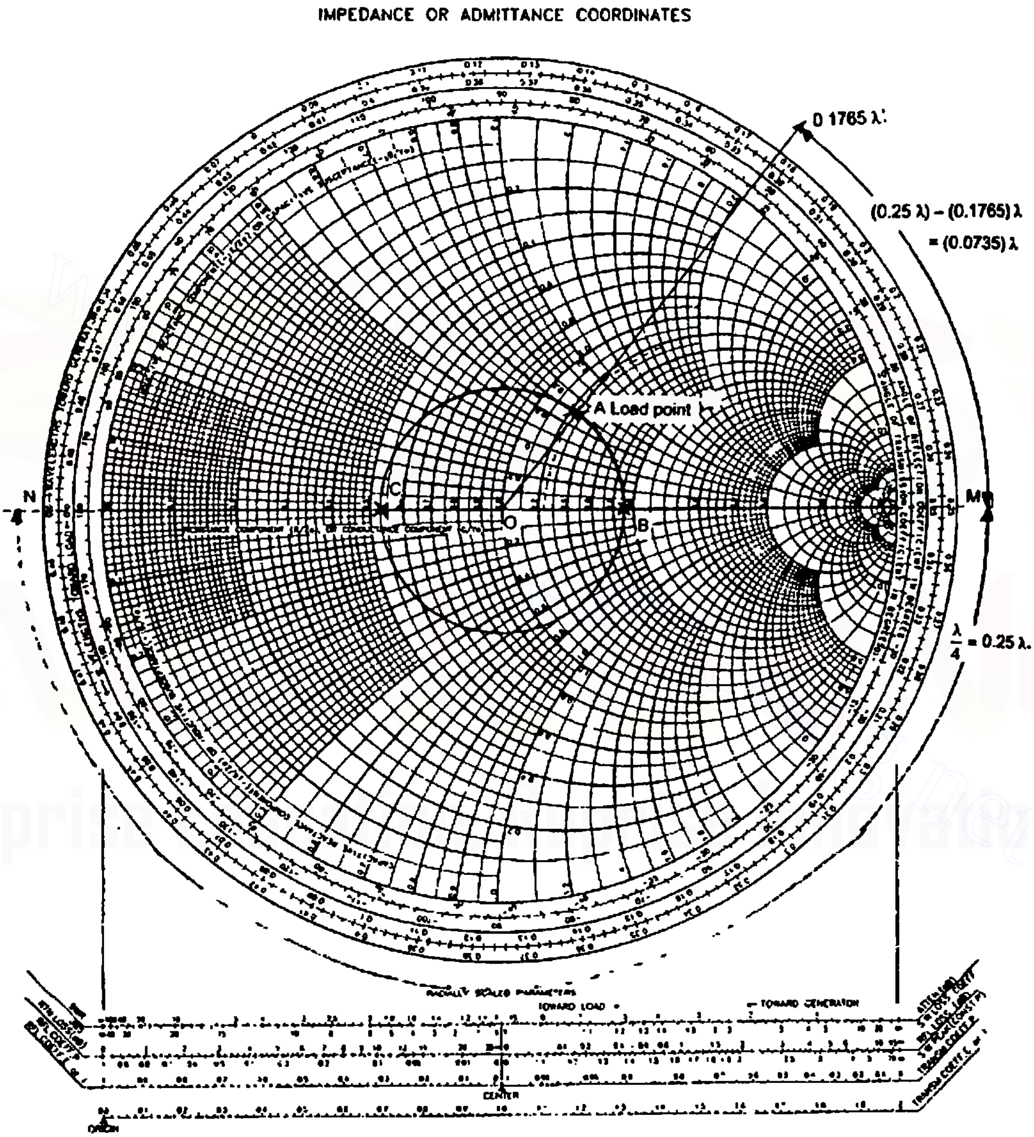

IMPEDANCE OR ADMITTANCE COORDINATES
0 1765 λ
(0.25 λ) – (0.1765) λ
= (0.0735) λ
A Load point
N
M
C
O
B
λ/4 = 0.25 λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.43

[B] Using Smith chart :

From given impedances first find normalized impedance as follows,

$$\frac{Z_L}{Z_0} = \frac{Z_R}{R_0} = \frac{70\angle 30^\circ}{50} = 1.4\angle 30^\circ = (1.212 + j\,0.7)$$

As imaginary part of the normalized impedance is positive, it indicates inductive reactance. Thus load point must be located in the upper half of the real-axis.

First locate point A as an intersection of two circles as R = 1.212 and X = 0.7 on Smith chart. So the coordinates of A will be (1.212, 0.7).

Then draw a line from centre of the chart, O to point A. Then draw a circle with O as centre and radius equal to distance OA.

The circle is actually S-circle which cuts the real axis in two points B and C.

Then the distance between O and B gives the standing wave ratio S.

From Smith chart, S = 1.9

The point C, the intersection of S-circle and the real axis, is on the left hand side of the centre and represents a voltage minimum nearest to the load. To get the distance of voltage minimum from the load, measure distance (on βs scale) from point A to point C.

So the distance from A to C on extreme rim is the distance from extended point A on extreme rim to M plus distance between M and N.

The distance from M to N is the half revolution is equal to $\frac{\lambda}{4}$ i.e. (0.25) λ.

Then distance from A to M is given by location of

$$(M - A) = (0.25)\lambda - (0.1765)\lambda = (0.0735)\lambda$$

Hence distance of nearest voltage minimum from the load is

$$= (0.25)\lambda + (0.0735)\lambda = (0.3235)\lambda$$

Example 2.21 : *A transmission line 52 cm long has a characteristic impedance of 50 Ω. It is terminated in a load of Z_R. The VSWR is 2.5. The voltage minima occur at 18cm and 38cm from the load end. Using Smith chart, determine load Z_R and input impedance of the line.*

Solution : VSWR = 2.5

First draw a circle of radius 2.5 and centre (1, 0) of the chart as the centre of the circle. This S-circle cuts real axis on left hand side of the centre of the chart at point A. The co-ordinates of point A are (0.4, 0). This represents a voltage minimum nearest to the load.

The distance between successive voltage minima is given by

$$\frac{\lambda}{2} = (38-18) = 20 \text{ cm}$$

$$\therefore \qquad \lambda = 40 \text{ cm}$$

Now the first voltage minimum is 18 cm apart from load. So from point A moving in anticlockwise direction (i.e. from generator or source to load) the total distance of 18 cm or $\frac{18\,\lambda}{40} = 0.45\,\lambda$.

The load point is indicated as a point B (0.43, 0.28).

Hence normalized load impedance is given by

$$\frac{Z_L}{Z_0} = \frac{Z_R}{R_0} = (0.43 + j\,02.8)$$

Total length of the line is 52 cm = $\frac{52}{40}\lambda = 1.3\,\lambda$.

This is the distance between load and sending end. Hence travel a distance of 1.3 λ from point B in clockwise direction (i.e. from load to source) along S-circle and mark point C as the sending end point.

While travelling two complete revolutions, from point B, total distance travelled will be 1λ. So move the remaining distance of 0.3 λ as shown in the chart. The co-ordinates of the point C are (0.9, −0.9). Hence normalized input impedance is given by,

$$\frac{Z_S}{R_0} = (0.9 - j\,09)\ \Omega$$

Please refer Fig. 2.44 on next page.

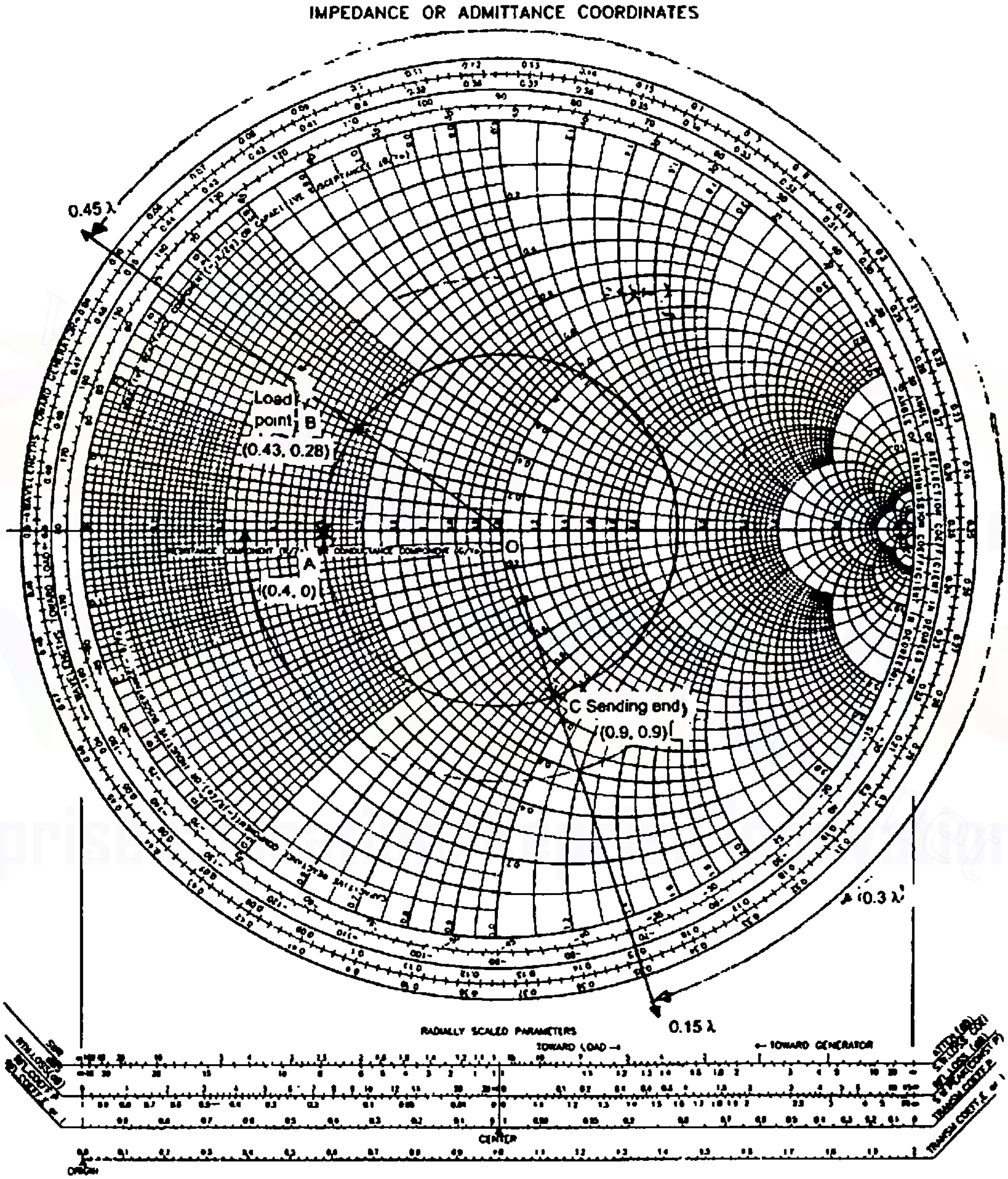

IMPEDANCE OR ADMITTANCE COORDINATES
0.45 λ
Load point
B
(0.43, 0.28)
A
(0.4, 0)
O
C Sending end
(0.9, 0.9)
0.3 λ
0.15 λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.44

Examples from Univesity Question Papers

Example 2.22 *For a load of* $\frac{Z_S}{Z_0} = 0.8 + j1.2$ *design a double stub tuner making the distance between the stubs* $\frac{3\lambda}{8}$. *Specify the stub length and distance from the load to the first stub. The stubs are short circuited. Verify using Smitch chart.* **(Dec.-2005)**

Solution : Let us first find normalized load admittance

$$\frac{Z_R}{Z_0} = 0.8 + j\,1.2$$

$$\therefore \quad Z_R \cdot \frac{1}{R_0} = 0.8 + j\,1.2$$

$$\therefore \quad Z_R \cdot G_0 = 0.8 + j\,1.2$$

$$\therefore \quad \frac{1}{Z_R\, G_0} = \frac{1}{0.8 + j\,1.2}$$

$$\therefore \quad \frac{Y_R}{G_0} = \frac{0.8 - j\,1.2}{2.08} = 0.4 - j\,0.6$$

Hence locate point P having conductance of 0.4 and susceptance of – 0.6.

Circle A is the locus of all points for which $\frac{Y_2}{G_0} = 1$.

The distance between stubs = $\frac{3}{8}\lambda = 0.375\,\lambda$, the circle B is obtained by displacing each point on circle A counterclockwise by 0.375 λ or the diameter of circle B projects downwards from centre of the chart.

Stub-1 should change $\frac{Y_R}{G_0}$ to $\frac{Y_1}{G_0}$ which will lie on circle B. Stub-1 is shorted. Hence it can not add any conductance. Hence travelling along constant conductance circle, from P towards generator we meet circle B at point 1.

At 1 : $\frac{Y_1}{G_0} = 0.4 - j\,0.2$

Hence stub-1 should add + j 0.4. The constant susceptance circle of 0.4 cuts outer rim of chart at 0.06 λ.

Hence length of shorted first stub is given by

L_1 = Distance between extreme right hand end to the point at which + 0.4 susceptance circle cuts outer rim in clockwise direction.

Please refer Fig. 2.45 on next page.

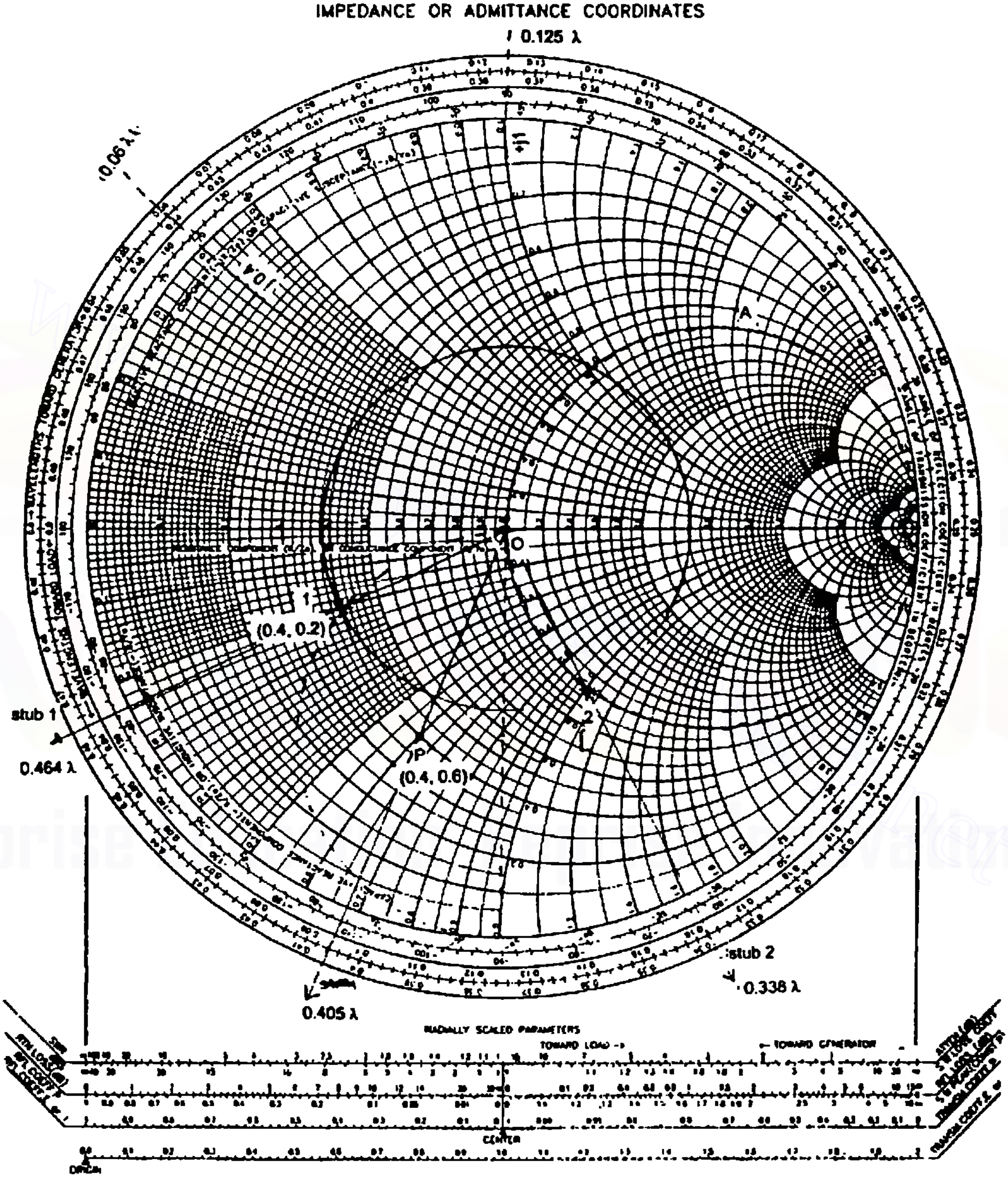

IMPEDANCE OR ADMITTANCE COORDINATES
0.125 λ
0.06 λ
A
1
(0.4, 0.2)
stub 1
0.464 λ
O
2
P
(0.4, 0.6)
stub 2
0.338 λ
0.405 λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.45

$$\therefore \quad L_1 = (0.5-0.25)\lambda + 0.06\,\lambda$$
$$= 0.31\,\lambda$$

As the distance between two stubs is $\frac{3}{8}\lambda = 0.375\,\lambda$. Hence starting from point 1 and going a distance equal to 0.375 λ along S-circle, we get point 2 on circle-A.

At 2 : $\frac{Y_2}{G_0} = 1 - j\,1$

The stub-2 should provide + j 1 susceptance.

The constant susceptance circle cuts the outer rim of chart at 0.125 λ

∴ Length L_2 of stub-2 is given by

L_2 = Distance measured from extreme right hand side end upto the point on the outer rim when + j1 constant susceptance circle cuts the outer rim, in clockwise direction.

$$= [0.5-0.25]\,\lambda + 0.125\lambda$$
$$= 0.375\,\lambda$$

Example 2.23 *An R.F. transmission line with a characteristic impedance of $300\angle 0°\,\Omega$ is terminated in an impedance of $100\angle -45°\,\Omega$. The load is to be matched to the tranmission line by using a short circuited stub. Wtih the help of Smith chart determine the length of the stub and the distance from the load.* **(May-2005)**

Solution : Given : $Z_R = 100\angle -45°\,\Omega = (70.71 - j70.71)\,\Omega$

$$Z_0 = R_0 = 300\angle 0° = 300\,\Omega$$

1. The normalized impedance is given by,

$$z_R = \frac{Z_R}{Z_0} = \frac{Z_R}{R_0} = \frac{70.71 - j70.71}{300} = 0.2357 - j\,0.2357$$

2. Locate point A on the Fig. 2.46 as the intersection of r = 0.2357 circle and x = – 0.2357 circle. As the imaginary component i.e. reactive component of the impedance is negative, the point A is located below the horizontal axis.

3. Draw a circle with 0 (i.e. centre of the chart) as origin and OA as radius. This is constant S-circle. It cuts real axis at 4.6 to the right of point O. This is the value of s before stub connection.

4. Draw line from point A to O and extend to reach other end on constant S circle. In the chart-A, this point is represented as B at which the normalized admittance is $y_R = 2.1 + j\ 2.1$. Extend line AOB to the outer rim upto point B'.

6. Travel along the constant S-circle in the clockwise direction from load to generator to reach a point C at which constant S-circle intersects $\frac{Y}{G_0} = 1$ circle.

   Draw line from O to C and extend line to point C' on the outer rim.

7. Now are B'C' gives the distance of the stub from the load.

$$\therefore \quad d = \text{arc B'C'} = 0.32\lambda - 0.213\lambda = \mathbf{0.107\lambda}$$

8. At the point C, the normalized admittance value is 1 + j 1.65. This is the point at which stub is to be connected. Thus the stub must provided susceptance of – j 1.65.

9. The point D' represents a susceptance of – j 1.65 and is located above real axis as shown.

10. Now the movement from extreme right point on the real axis upto the point D' (on the outer rim) in clockwise direction (from load to generator) indicates the total length of the stub required.

See Fig. 2.46 on next page.

So point D' is at 0.163λ from extreme left point on the real axis. And the distance along the outer rim from extreme right point to the extreme left point corresponds to $\frac{\lambda}{4}$ i.e. 0.25 λ. Hence the length of the stub is given by,

$$l = \frac{\lambda}{4} + 0.163\lambda = 0.25\lambda + 0.163\lambda = \mathbf{0.413\lambda}$$

Example 2.24 *The charcterisitic impedance of a high frequency line is 100 Ω. It is terminated in an impedance of 100 + j 100 Ω. Using Smith chart find the impedance at one eighth wavelength away from the load end.* **(May-2005)**

Solution : Given $Z_R = 100 + j\ 100\ \Omega$, $Z_0 = R_0 = 100\ \Omega$.

1. The normalized impedance is given by,

$$Z_R = \frac{Z_R}{Z_0} = \frac{Z_R}{R_0} = \frac{100 + j100}{100} = 1 + j1$$

   Plot this impedacne on the Fig. 2.47 as point A which is the intersection of r = 1 circle and x = j 1 circle..

2. Draw a constant circle with O as a centre and OA as radius. This circle cuts real axis i.e. horizontal axis at two points. The point of intersection to the right of the centre indicates VSWR value.

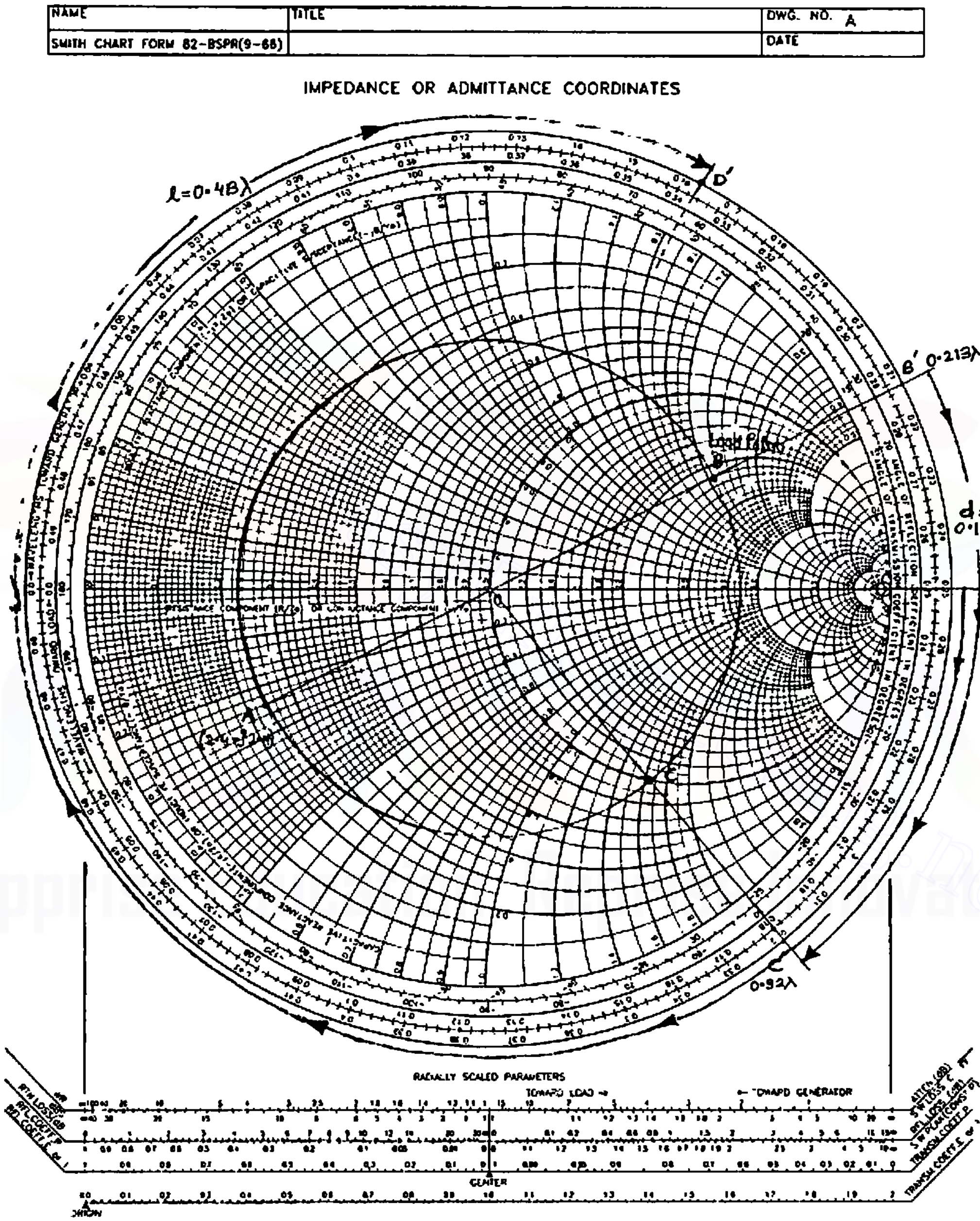

NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
l=0.48λ
D'
B' 0.213λ
d=0.107
Load Point
C
0.92λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER

Fig. 2.46

3. Draw line OA and extend it upto the periphery of the chart at point A'. The point A' is located at 0.164λ on the outer rim.
4. To find impedance of the line $\frac{1}{8}\lambda$ away from load, move clockwise direction from point A' through $0.125\lambda\left(\text{i.e. } \frac{1}{8}\lambda\right)$ distance on the outer rim and locate point as B'.
5. Join O and B'. This line OB' cuts constant -S circle at point B.
6. From the chart, point B is the intersection of r = 2 circle and x = –j 1 circle approximately. Hence the normalized impedance at point B is given by,

$$z = 2 - j\ 1 = \frac{Z}{R_0}$$

7. Hence the actual impedance at $\frac{1}{8}\lambda$ distance is given by,

$$Z = R_0\ (z) = 100\ (2 - j1) = 200 - j100\ \Omega$$

See Fig. 2.47 on next page.

Example 2.25 *Determine the input impedance of the transmission line of electrical length 28° with terminated load of* $\frac{Z_R}{R_o} = 2.6 + j\,1$. *Use Smith chart.*

(Dec.-2004)

Solution : Given : Z_R = normalized load impedacne = 2.6 + j 1 electrical length of line = 28°

1. Mark point A as the intersection of r = 2.6 circle and x = +1 circle. This represents load point on the chart-C.
2. Draw a circle with centre of the chart i.e. O(1,0) as the centre and radius equal to distance OA.
3. Draw a line from O to A and extend it upto the point A' on outer scale as shown in Fig. 2.48.
4. Now length of line is 28° which corresponds to length expressed in wavelength as $\frac{28^\circ}{360^\circ}\ \lambda = 0.07777\ \lambda \approx 0.078\lambda$.
5. Next move along βs scale, 0.078λ distance from A' to locate point B' on outer rim at $(0.227 + 0.078)\lambda = 0.305\lambda$.
6. Draw a line from point O to B'. This line intersects constant S-circle at point B which is nothing but the source point.

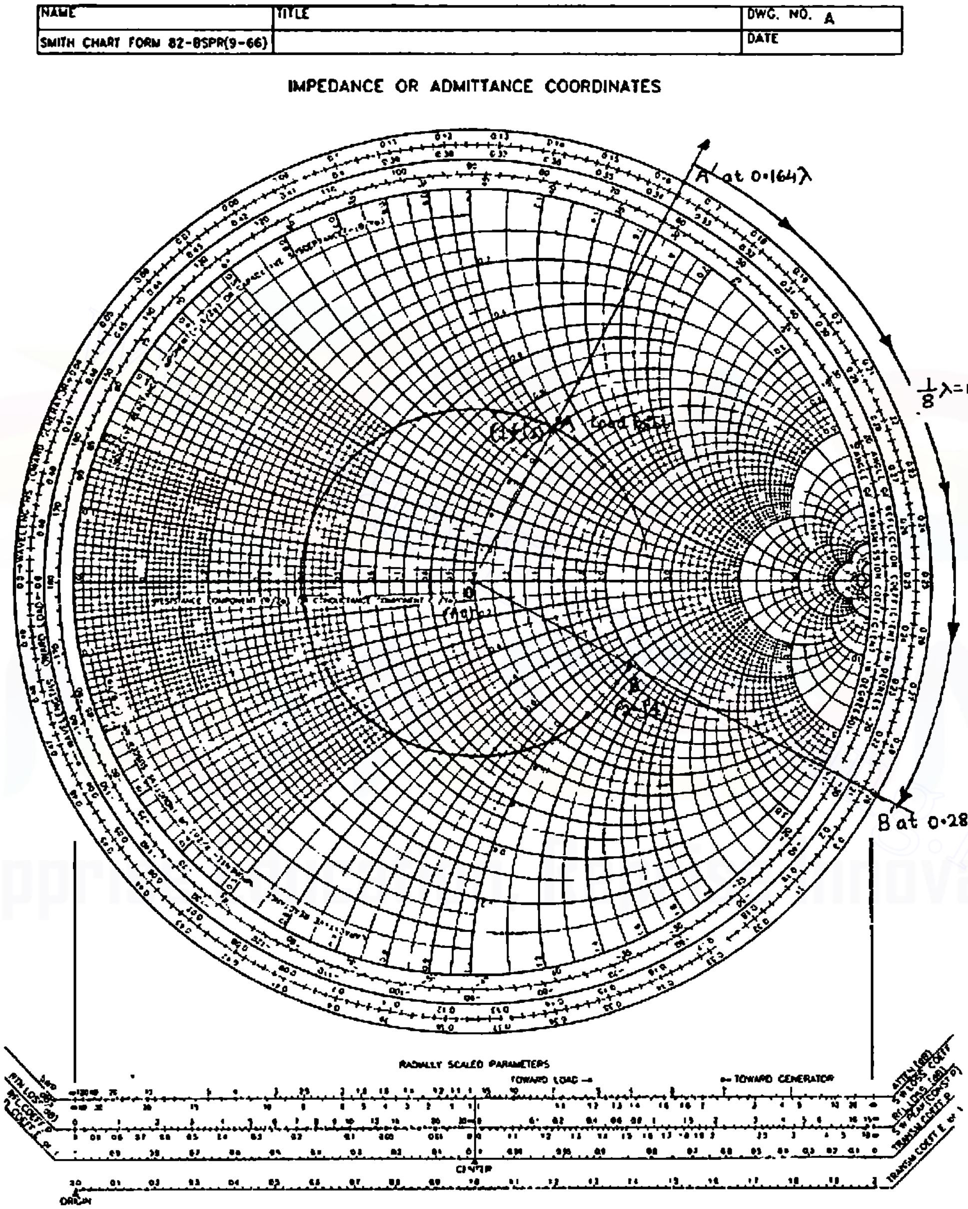

NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
A at 0·164λ
$\frac{1}{8}\lambda = 0{\cdot}125$
Load Point
B at 0·289λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.47

7. The source point or generator point B is the intersection of r = 1.58 circle and x = – 1.4 circle . Note that we are working with impedances. Also point B is not located on real axis which indicates input impedances consists real as well as imaginary part. Moreover point B is located below the real axis. That means the imagingary part or reactive part of input impedance is negative capacitive reactance. Hence the value of normalized input impedance is given by,

$$\frac{Z_S}{R_0} = 1.58 - j\,1.4$$

See Fig. 2.48 on next page.

Example 2.26 *The transmission line has standing wave ratio S = 2.5 and voltage minima exists at 0.15 λ from the load. Find the load and input impedance for a line of 0.35 λ length. Use Smith chart.* **(Dec.-2004)**

Solution : Given $S = 2.5$

V_{min} at 0.15 λ from load

l = length of the line = 0.35 λ

1. Draw a constant S-circle passing through point 2.5 on the real axis, with centre as O(1,0) as shwon in the chart -D.
2. According to the property, the voltage minima occurs at a point on the real axis to the left of the origin at which the constant S-circle (s=2.5) cuts. Let it be denoted by point A. The point A representing voltage minima is at 0.4 $\left(\text{i.e. } \frac{1}{s} = \frac{1}{2.5} = 0.4\right)$
3. Now the voltage minima exists at 0.15 λ from load. Hence to obtain load point from the voltage minima, we must travel toward load 0.15 λ distance in anticlockwise direction.
4. Move from point A' in anticlockwise direction to reach point B' through 0.15 λ as shown. Draw a line from O to B'. This line intersects constant - S circle at point B which is nothing but load point. The point B is the intersection of r = 0.89 circle and x = – j 0.9 circle. As point B is below the real axis, reactance is capacitive hence negative. Thus the normalized load impedance is,

$$z_R = \frac{Z_R}{R_0} = 0.89 - j\,0.9$$

See Fig. 2.49 on next page no.

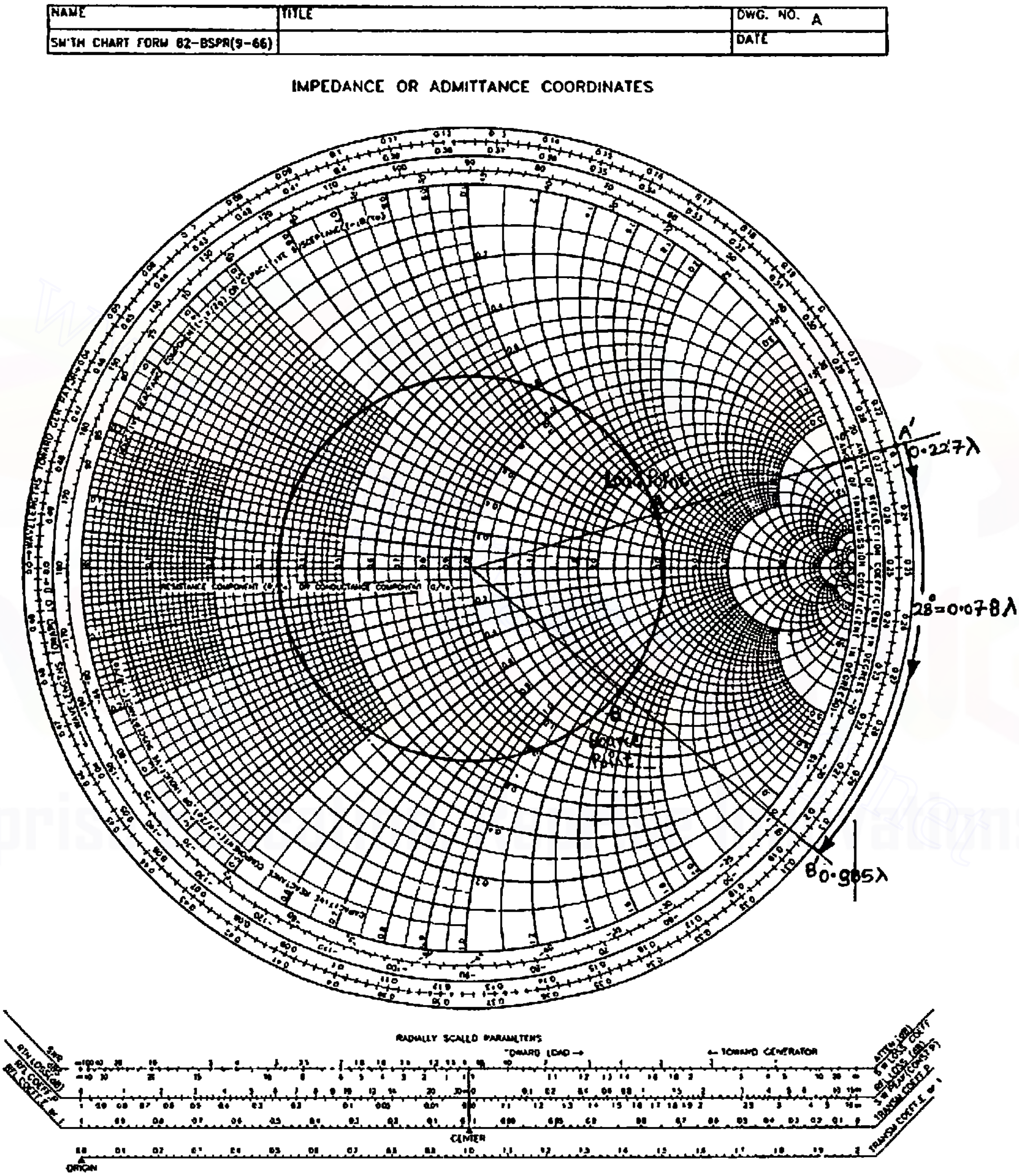

NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
LOAD POINT
A'
0·227λ
28°=0·078λ
B
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.48

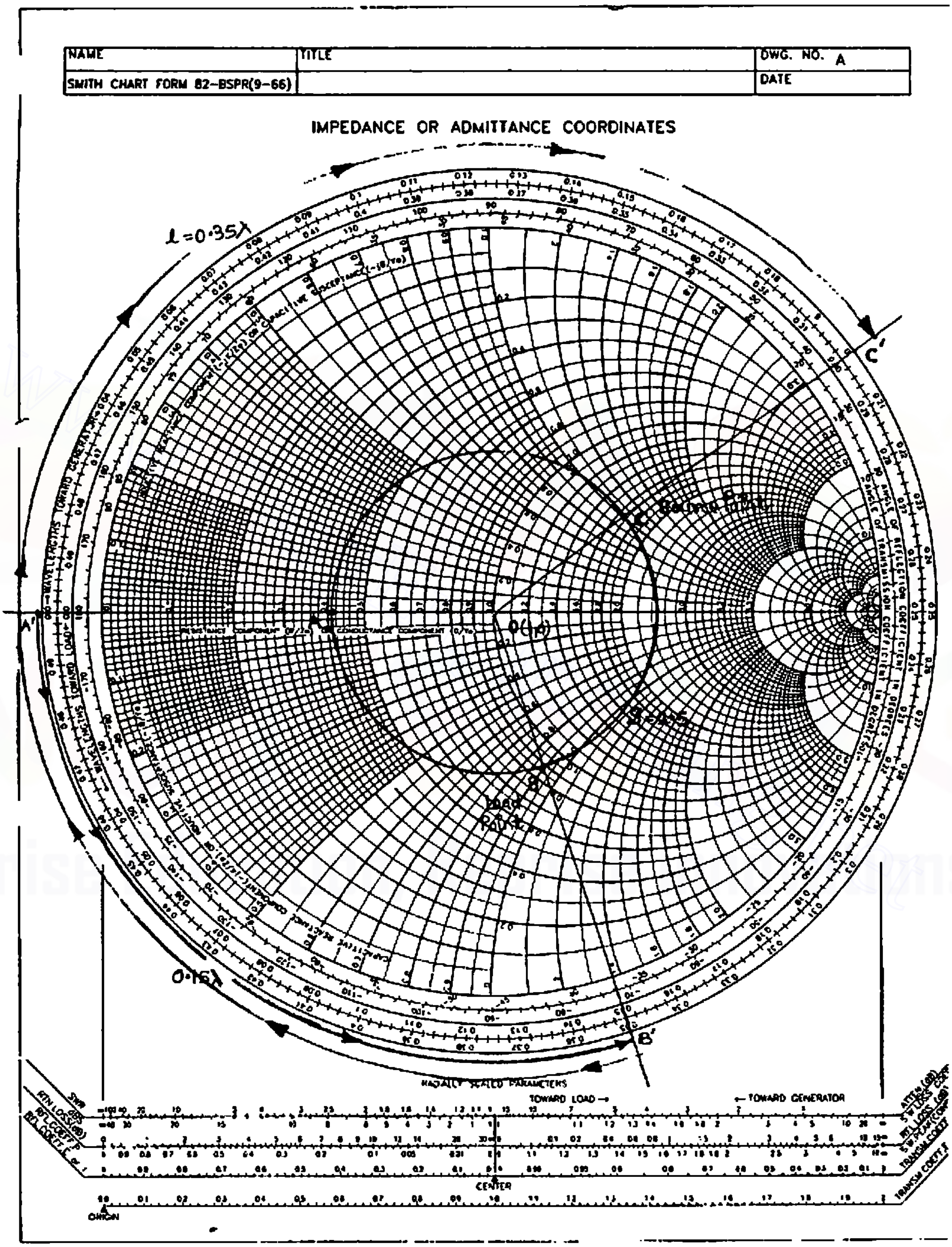

NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
$l = 0.35\lambda$
0.16λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.49

5. Total length of the line is 0.35 λ. Hence to obtain source or generator point, we have to travel 0.35 λ from B' on the outer rim in clockwise direction (toward generator).
6. The point on the outer rim 0.35 λ away from load is represented by pioint C' Draw a line from O to C'. This line intersects the constant S-circel (S=2.5) at point C which is nothing but a source or generator point.
7. Point C is the intersection of r = 1.68 circle and x= j 1 circle. Hence the normalized input impedance is given by,

$$z_S = \frac{Z_S}{R_0} = \mathbf{1.68 + j\ 1.0}$$

Note that the generator point (i..e source point) C is located above the real axis which indicates the reactive part is positive and hence it is inductive reactance.

Example 2.27 *A transmission line has a characteristic impedance of 300 Ω and terminated in a load Z_L = 150 + j150 Ω. Find the following using Smith chart.*

i) VSWR ii) Reflection coefficient (iii) input impedance at distance 0.1 λ from the load (iv) input admittance from 0.1 λ from load (v) position of first voltage minimum and maximum from the load. **(May-2004)**

Solution :

Given $Z_0 = R_0 = 300\ \Omega$

$Z_L = 150 + j\ 150\ \Omega$

1. The normalized load impedance is given by,

$$z_L = \frac{Z_L}{Z_0} = \frac{Z_L}{R_0} = \frac{150 + j150}{300} = 0.5 + j\ 0.5$$

Hence locate point A at the intersection of r = 0.5 circle and x = +0.5 circle as shown in the Fig. 2.50. Point A represents the load point.

2. Draw a circle with point O(0,1) as center and radius equal to distance OA. This is constant S - circle. This circle cuts the real axis at 2.6. Hence the value of VSWR is given by ,

$$\mathbf{S = 2.6}$$

3 Draw a line OA and extend it upto point A' on the outer rim. This line intersects the scale for the angle of reflection coefficient at 118°. This is the angle of reflection coefficient.

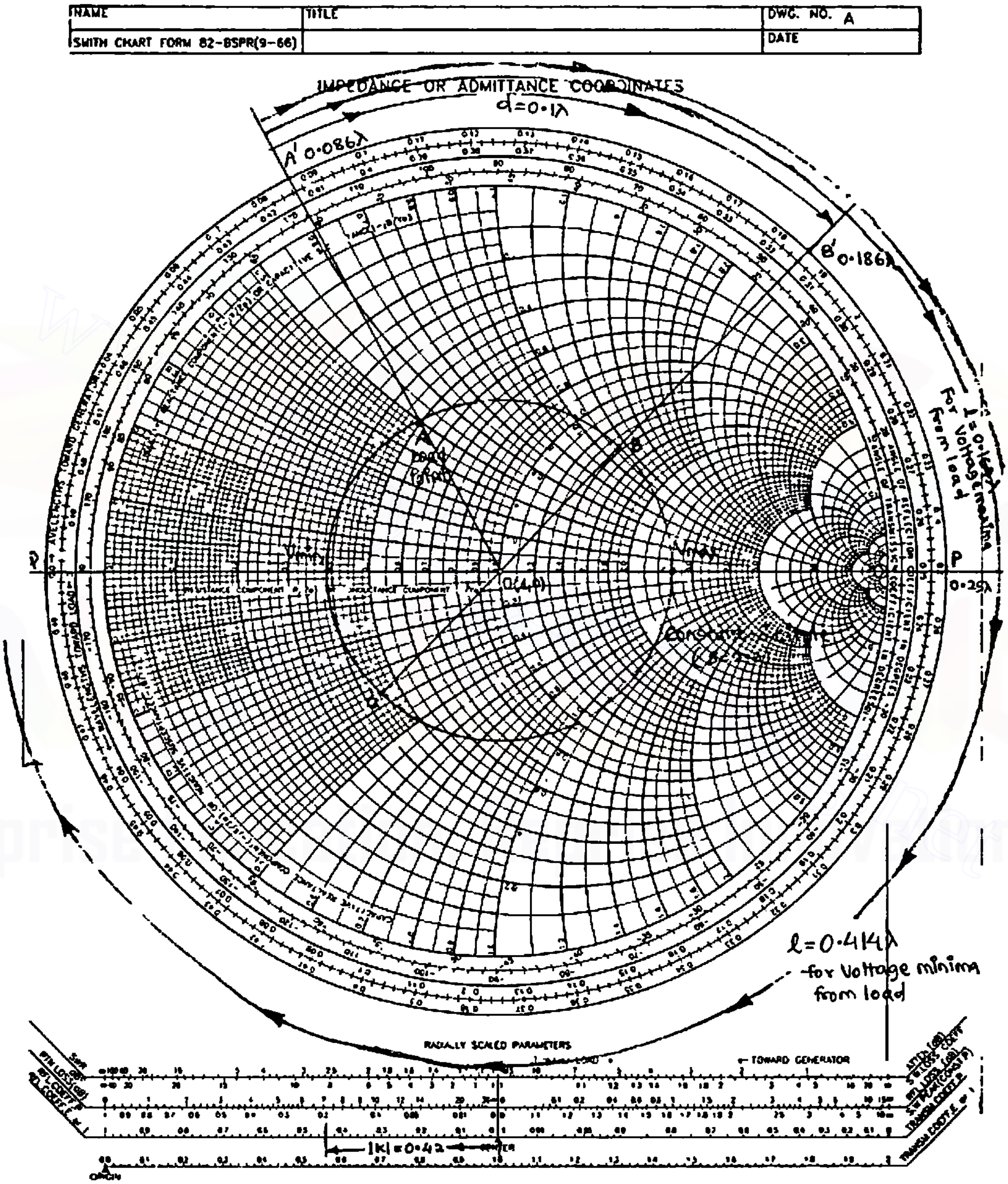

NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
d=0·1λ
A' 0·086λ
B' 0·186λ
l=0·164λ
For voltage maxima
from load
Load
point
0(1,0)
P
0·25λ
Vmax
Vmin
Constant
l=0·414λ
for voltage minima
from load
RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
|K|=0·42
ORIGIN

Fig. 2.50

To get magnitude of $\overline{K}$, select the linear scale shown at the bottom of the chart - E. Marks a point from certre at a distance equal to distance OA.This gives value of $|K|$ as 0.42. Hence the reflection coefficient is given by,

$$K = 0.42 \angle 118°$$

4. To find input impedance at 0.1λ from load, move in clockwise direction from point A' a distance equal to reach point B'. A' is at 0.086λ. Hence B' will be located at $(0.086 + 0.1)\lambda = 0.186\lambda$ as shown in the chart E

5. Draw a line OB' which cuts S-circle (S=2.6) at point B. The point B is intersection of r = 1.38 circle and x = 1.12 circle. Hence input impedacne 0.1λ away from the load is given by,

$$Z_d = \frac{Z_d}{R_0} = 1.38 + j1.12$$

Hence actual impedance at 0.1λ away from load is given by,

$$Z_d = R_0(1.38 + j\,1.12) = 300\,(1.38 + j\,1.12) = (414 + j336)\Omega$$

6. To obtain input admittance 0.1λ away from load, draw a diameter through O and B. This will cut constant - S circle at point C. As this point is the intersection of g = 0.44 circle and b = −0.38 circle.

Hence input admittance is

$$y_d = 0.44 - j\,0.38 = \frac{Y_d}{G_0}$$

Hence actual admittance is given by

$$\therefore \quad Y_d = G_0(0.44 - j\,0.38) = \frac{1}{R_0}(0.44 - j\,0.38) = \frac{0.44 - j\,0.38}{300}$$

$$\therefore \quad Y_d = (1.4667 - j\,1.2667) \times 10^{-3}\ \mho$$

7. The voltage minima occurs to the left of centre on real axis at 0.39 while the voltage maxima occurs to the right of centre on the real axis at S = 2.6.

The find location of the votlage maxima from load, move in clockwise direction from point A' (load point) to point P (voltgae maxima point). Total travel is $(0.2S - 0.086)\lambda = 0.164\lambda$. Hence **the first voltage maxima is located at 0.164λ from the load.** Now to find location of voltage minima from load move in clockwise to find location of voltage minima from load move in clockwise direction from point A' to Q (voltage minima point). The total travel is $0.25\lambda + 0.164\lambda = 0.0414\lambda$. Thus **the first voltage minima exists at 0.414λ from the load.**

Example 2.28 *A transmission line of 100 m long is terminated in load of* $(100 - j\,200)\,\Omega$. *Determine the line impedance at 25 m from the load end at a frequecny of 10 MHz. Assume line impedance* $Z_0 = 100\ \Omega$. *Determine the input impedance and admittance using Smith chart.* **(Dec.-2003)**

Solution : (A) Calculate first normalized impedance.

$$\frac{Z_L}{Z_0} = \frac{Z_R}{R_0}$$

$$= \frac{100 - j200}{100} = 1 - j\,2$$

Locate point A at the intersection of constant resistance circle of 1 and capacitive reactance circle of –2.

Draw a circle passing through point A which represents standing wave ratio.

For given line, $\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^6} = 30$ m

Now from load point A, move towards generator in clockwise direction. The distance to be covered is given by

$$s_1 = 25 \text{ m} = \frac{25}{30}\lambda = 0.8333\,\lambda$$

Thus locating point B at a distance 0.8333 λ from point A.

At point B susceptance is + 1.2 and resistance is 0.44

Thus the impedance of a line at a distance 25 m from load is,

$$Z_S = R_0\,(0.44 + j\,1.2) = 100\,(0.44 + j\,1.2)$$

$$\therefore \quad Z_S = (44 + j\,120)\ \Omega$$

To calculate input impedance, travelling 100 m ≡ 3.3333 λ distance from load point A. It is observed that we reach at same point. Thus the input impedance is again same.

$$\therefore \quad Z_{in} = (44 + j\,120)\ \Omega$$

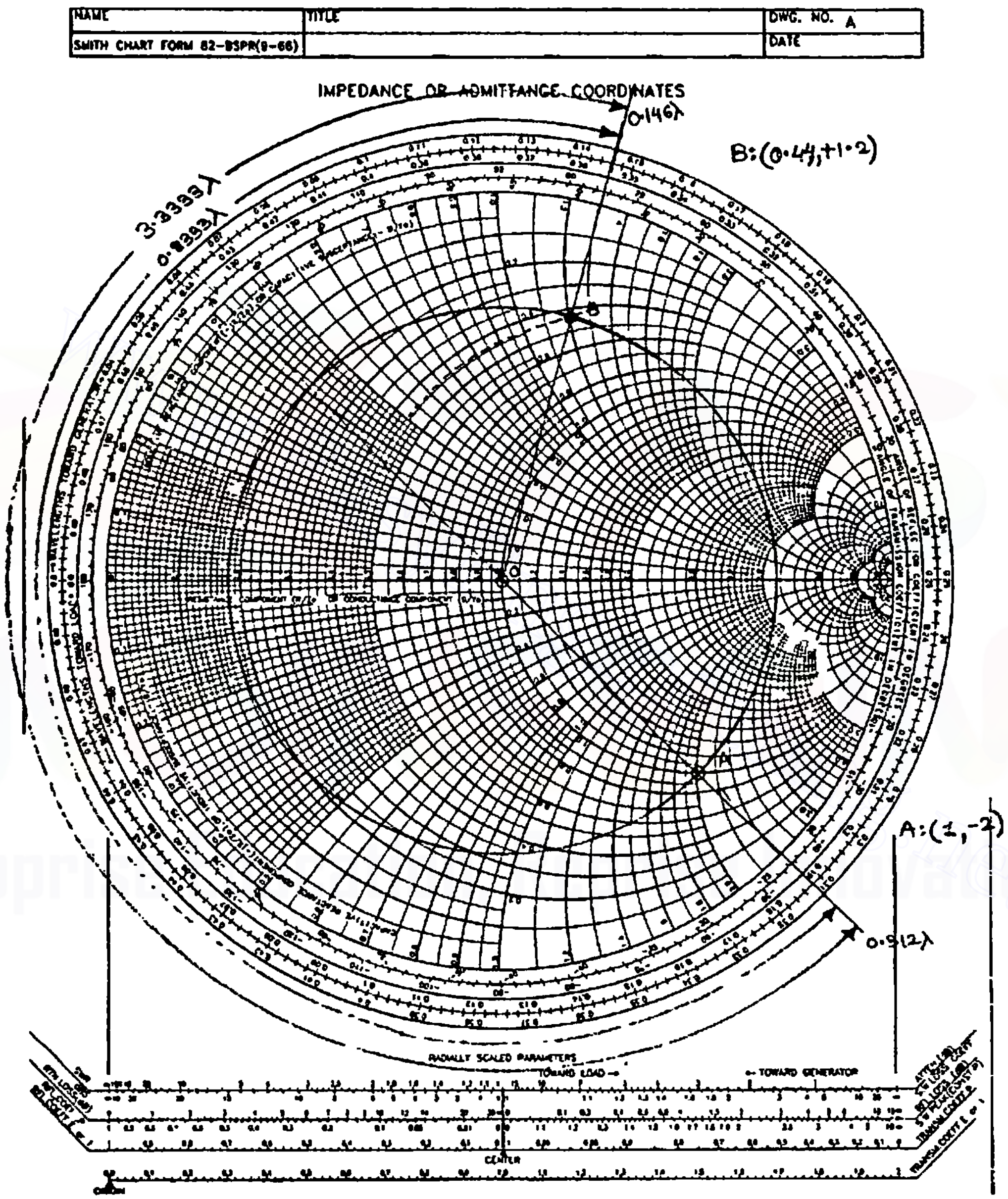

NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
0·146λ
B:(0·44,+1·2)
3·3333λ
0·3333λ
A:(1,-2)
0·312λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.51

[B] To calculate input admittance, let us first calculate normalized admittance.

$$\frac{Y_R}{G_0} = \frac{\frac{1}{Z_R}}{\frac{1}{Z_0}} = \frac{Z_0}{Z_R} = \frac{100}{100 - j\,200} = \frac{1}{1-j2}$$

$$\therefore \quad \frac{Y_R}{G_0} = \frac{1+j2}{5} = 0.2 + j\,0.4$$

Locate point L as shown on the Smith chart. It is a load point. Travelling 100 m i.e. 3.3333 λ distance from point L we reach at input point M.

At point M conductance is 0.28 while susceptance is – 0.75. Hence the admittance at the input is given by

$$Y_{in} = G_0\,(0.28 - j\,0.75) = (0.28 - j\,0.75)$$

$$Y_{in} = 2.8 \times 10^{-3} - j\,7.5 \times 10^{-3}\ \mho$$

(Fig. 2.52 See on next page)

Example 2.29 *A load (50 – j 100) Ω is connected across a 50 Ω line. Design a short circuited stub to provide matching between the two at a signal frequecny of 30 MHz using Smith chart.* **(Dec.-2003)**

Solution : $Z_R = 50 - j\,100$, $Z_0 = R_0 = 50\ \Omega$

The normalized load admittance is given by,

$$\frac{Y_R}{G_0} = \frac{\frac{1}{Z_R}}{\frac{1}{Z_0}} = \frac{Z_0}{Z_R} = \frac{50}{50 - j\,100}$$

$$= \frac{1}{1-j2}$$

$$= \frac{1+j2}{5}$$

$$= 0.2 + j\,0.4$$

Locate point A on the chart indicating the normalized admittance.

Draw constant S circle through point A. It cuts the r_a axis at 5.8 which indicates standing wave ratio before the use of stub is 5.8.

Locate point B at the intersection of S circle and unity conductance circle. Now this point is nearest to the load, At B the susceptance is + 2.0 which is capacitive in nature. This point gives the position at which the stub is to be connected.

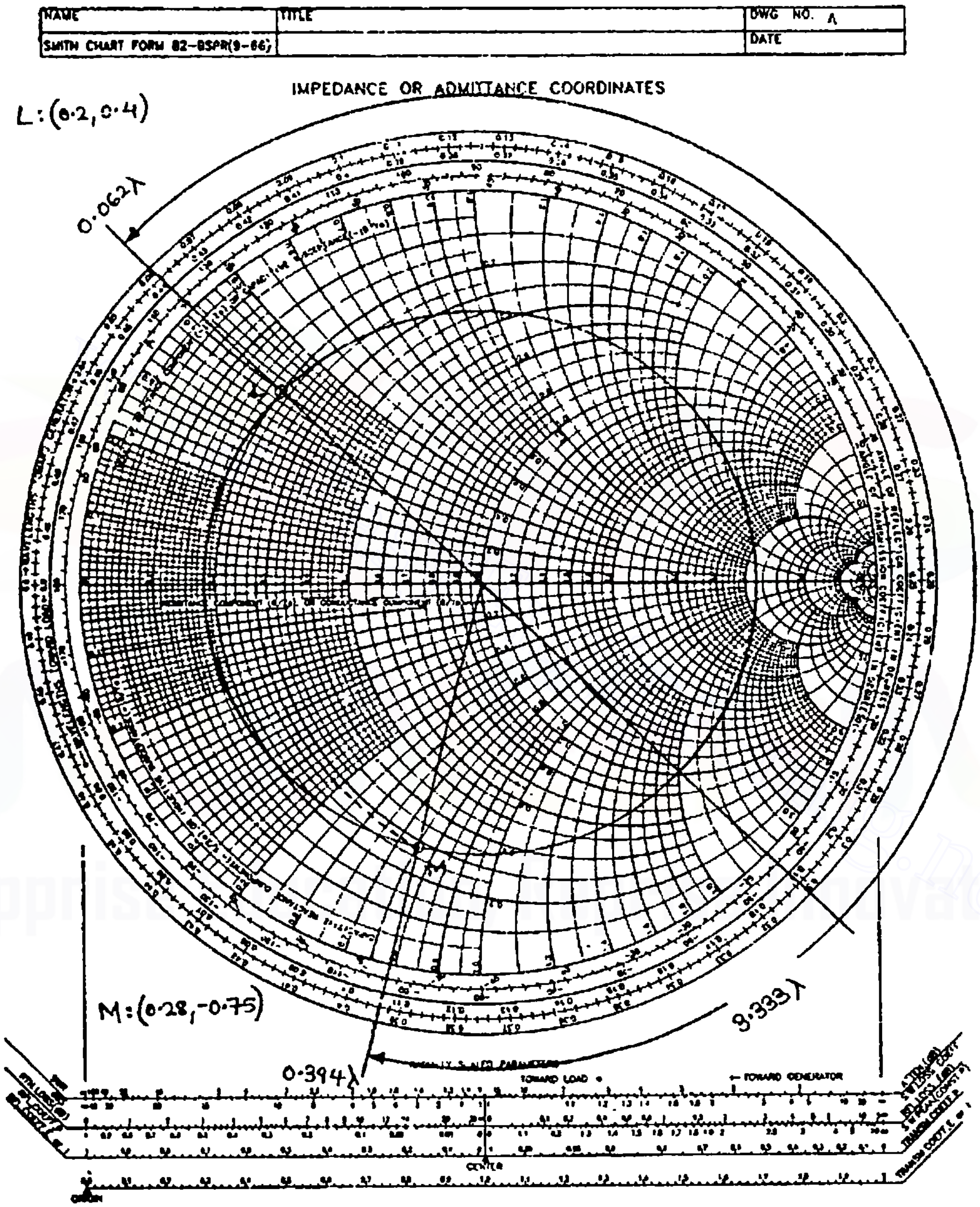

NAME
TITLE
DWG NO.
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
L:(0·2,0·4)
0·062λ
M:(0·28,-0·75)
0·394λ
9·333λ
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.52

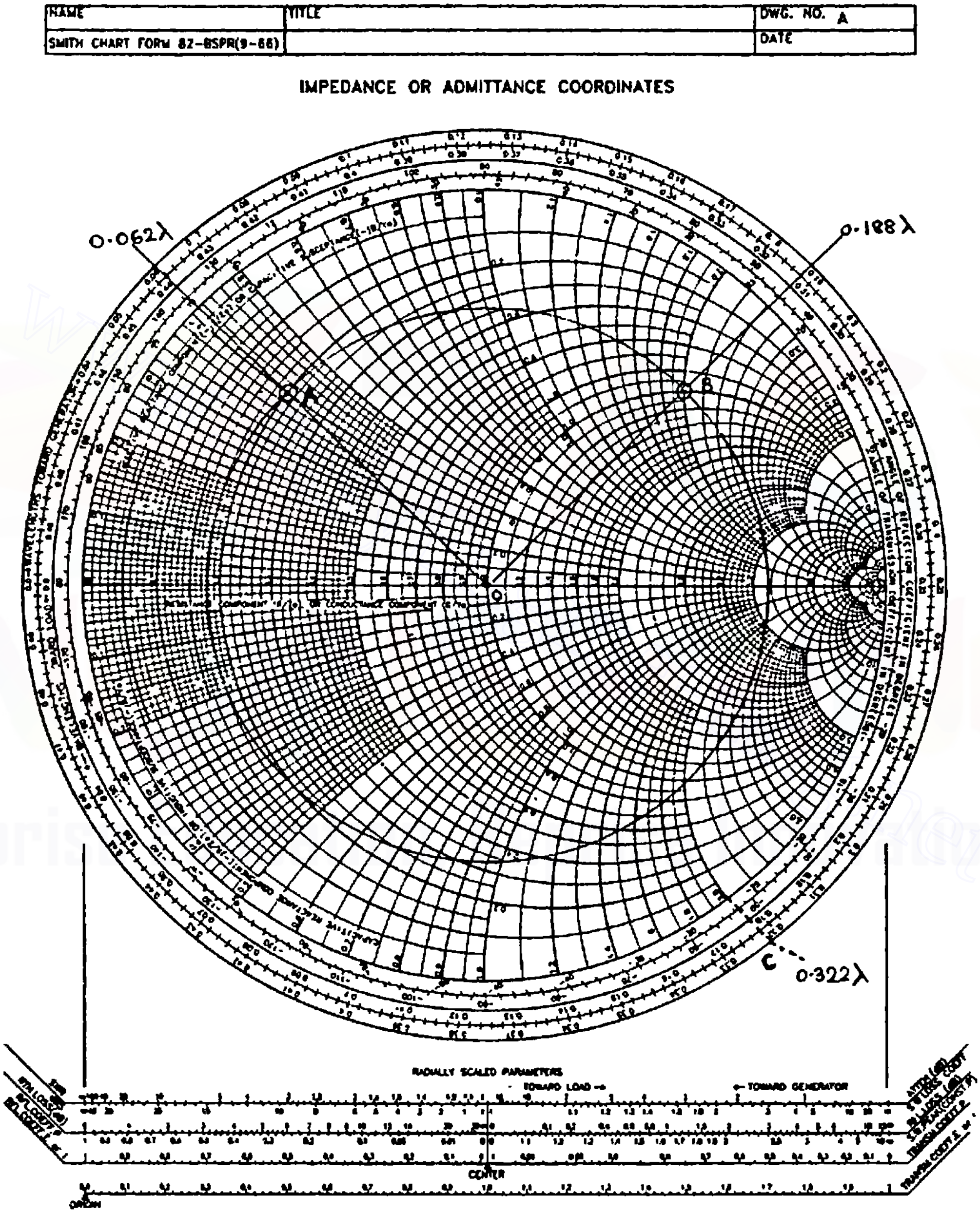

NAME
TITLE
DWG. NO. A
SMITH CHART FORM 82-BSPR(9-66)
DATE
IMPEDANCE OR ADMITTANCE COORDINATES
0·062λ
0·188λ
A
B
O
C
0·322λ
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN

Fig. 2.53

Extending line through point A and B from the centre of the chart upto the outermost rim of the chart, we get $\beta s = 0.062\ \lambda$ for point A and $\beta s = 0.188\ \lambda$ for point B. Travelling from load to generator in clockwise direction, the distance from the load at which stub is to be connected is given by,

$$s_1 = [0.188\ \lambda - 0.062\ \lambda] = 0.126\ \lambda$$

As the line susceptance is capacitive, the line must provide inductive susceptance of –2.0. Locate point C at the intersection of – 2.0 susceptance circle with the outer rim of the chart. Thus at C, $\beta S = 0.322\ \lambda$. As the stub is short circuited at other end, measuring distance from extreme right hand point on the real r_a axis i.e. from short circuit point ($\beta S = 0.25\ \lambda$). Thus the length of the stub required is given by,

$$L = (0.322\ \lambda - 0.25\ \lambda] = 0.072\ \lambda$$

For matching load of (50 – j 100) Ω with 50 Ω line, a short circuited stub must be located at distance 0.126 λ from the load. The length of the stub required is 0.072 λ.

(Fig. 2.53 See on next page)

Review Questions

1. *What are the standard assumptions made for radio frequency line ? What is small dissipation line and zero dissipation line ?*
2. *Derive the expressions for voltage and current at any point on the radio frequency line terminated in Z_R. Obtain the expressions for the same for different receiving end conditions. Support with the graph of voltage and current on a line for all conditions.*
3. *What are the standing waves ? Define node and antinode.*
4. *Explain briefly standing wave ratio.*
5. *Derive relationship between standing wave ratio and the magnitude of reflection coefficient* ($|K|$).
6. *Derive the relationship between standing wave ratio and the reflection coefficient.*
7. *For a radio frequency line terminated in Z_R prove that*

$$Z_{in} = R_0 \left[\frac{Z_R + j R_0 \tan\beta s}{R_0 + j Z_R \tan\beta s} \right]$$

8. *Derive the input impedance of open circuited and short circuited line and sketch the variations of the normalized value of reactance with distance s.*
9. *Explain method of power and impedance measurement on the line.*
10. *Find the expression for input impedance of the eighth-wave line.*
11. *Write a note on quarter wave line.*
12. *Write applications of quarter wave line.*
13. *Show that "the half-wave line repeats its terminating impedance".*
14. *Explain in detail single stub matching on a line.*
15. *Derive the conditions for the location and the length of the stub used for matching on a line.*

16. *Write a note on Smith chart.*
17. *Explain briefly properties of Smith chart.*
18. *Explain applications of the Smith chart.*
19. *Write a detailed note on a double stub matching on a line using Smith chart.*
20. *A transmission line has characteristic impedance of 50 Ω and terminated in load impedance of $(75 + j40)\,\Omega$. Calculate reflection coefficient and VSWR.* **[Ans. : $K = 0.359\angle 40.25°$ VSWR = 2.12]**
21. *A lossless line of 300 Ω characteristic impedance is terminated in a pure resistance of 200 Ω. Find value of standing wave ratio.* **[Ans. : SWR = S = 1.5]**
22. *Write a note on circle diagram.*
23. *Explain the significance of circle diagram using appropriate diagram.*